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An EOQ model for multiple products with varying degrees of substitutability

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Abstract
In this paper, the authors present an EOQ model with substitutions between products and a dynamic inventory replenishment policy. Their key assumption is that many products in the market are substitutable at different levels, and that, in most cases, a customer who discovers that a desired product is unavailable will choose to consume a product with similar attributes or functionality, rather than not purchase at all. Therefore, given a firm that stocks multiple substitutable products, the authors assume that a stock out of one product has a direct impact on other products' demand. The main purpose of our model is to enable inventory managers to develop ordering policies that ensures that, in the event that a specific product runs out and cannot be replenished due to unforeseen circumstances, the consequent increase in demand for related products will not cause further stock out incidents. To this end, the authors introduce a dependency factor, a variable that indicates the level of dependency, or correlation, between one product and another. The dependencies among the various products offered by the firm are embedded into the EOQ formula and assumptions, enabling managers to update their ordering schedules as needed. This approach has the potential to generate more practical and realistic purchasing and inventory optimization policies.

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Keywords Economic order quantity (EOQ); optimization; substitute products; dependency factor; reorder point

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1 Introduction

Inventory management is a key aspect of fulfilling customer demand in a satisfactory manner: Specifically, a firm must ensure that it holds the appropriate range of products, and at the appropriate inventory levels. When making inventory decisions, it is important for a firm to take into account the fact that the various products that it offers are likely to be interconnected in some way. Relationships among products include substitution or complementarity relationships, in addition to hierarchical relationships. In the presence of such relationships, decisions made with regard to one product are likely to affect the demand and inventory of other products. Consider an example when a retailer offers substitutable product, the customer is likely to choose and consume an alternative product with similar attributes or functionality, if such a product is available, rather than not purchase at all. This scenario occurs frequently in pharmaceutical markets: If a patient wishes to purchase a specific brand of medication that is not available, he or she might choose a different brand (or generic form) of the same type of medication, or select a different type of medication with a similar effect. The firm incurs substitution cost, when a customer is forced to suffice with a substitute for a desired product. Such costs can arise from the need to rework an item to make it substitutable for another, and from the loss of a customer's goodwill due to substitution, etc.

Notably, despite its clear practical importance, the influence of product substitution relationships on inventory management decisions has received little attention in the operations literature (see McGillivray and Silver 1978, Parlar and Goyal 1984, Noonan 1995, Parlar 1988, Wang and Parlar 1994, Rajaram and Tang 2001, Ernst and Kouvelis 1999). Herein, we seek to bridge this gap and propose a model of inventory management that takes these relationships into account, in terms of their effects on products' actual demand rate. This model incorporates a dependency factor, a variable that indicates the dependency between pairs of multiple substitutable products, and uses it to compute the products' future demand rate in the event that a specific related product should stock out. More specifically, the model, which is based on EOQ principles, comprises a five-step process in which, after identifying products that are substitutable for one another, the decision maker (probabilistically) computes the dependency factor for each pair of products. Then, our model uses this information to compute future demand and to generate a dynamic reorder point in the case of an unexpected stock out of a particular product.

The purchasing model we propose aims to assist inventory managers in dealing with the reality in which, even if the demand rate is purely constant—such that a standard EOQ model could presumably be easily applied—the supply side is highly complex and characterized by many uncertainties such as order delays, force majeure, disasters, quality issues, regulation and many others. In the presence of such circumstances, the use of a dynamic reorder point rather than a static one can enable the inventory manager to adapt to unforeseen circumstances and to replenish stock as needed and not even a day too late, regardless of the original ordering schedule—thereby decreasing the probability that a stock out of a given item will be followed by stock outs among substitutable items. This proactive approach has the potential to improve firms' performance in terms of demand sensing, purchasing, inventory optimization, and more importantly, higher product availability and customer satisfaction.

2 Literature Review

Harris (1915) developed the first inventory model, and this model was further generalized by Wilson (1934), who derived the formula to obtain the economic order quantity (EOQ). McGillivray and Silver (1978) studied the first inventory model of substitutable items. In their model, all substitutable items were assumed to have the same unit variable cost and shortage penalty. Parlar and Goyal (1984) developed a similar model for optimal ordering decisions under stochastic demand. Pasternack and Drezner (1991) proved numerically that if the products are not substitutable then the associated optimal order quantities can be larger or smaller. They considered a stochastic model for two products having single period inventory structure and which can be used as substitutes for each other should the need arise. Considering the case of a single substitution the results reported increase or decrease in total optimum order quantities with the substitution revenue.

Drezner et al. (1995) developed an EOQ model with substitution for two substitutable products and then compared the results with those obtained in the case of no substitution. Gurnani and Drezner (2000) extended the model of Drezner et al. (1995) to consider multiple products in an inventory system. Ernst and Kouvelis (1999) proposed an efficient numerical search algorithm to determine the optimal stocking levels for three partially substitutable products.

Porras and Dekker (2008) provided a complete analysis of an inventory system with a joint replenishment policy (JRP) and presented a new inventory model that incorporates a correction for empty replenishment. Hong and Kim (2009) later developed a genetic algorithm for JRP and devised an unbiased estimator to find out the exact cost. Schulz and Telha (2011) theoretically showed that it might not be possible to devise a polynomial-time algorithm to optimize a JRP with constant demand.

Taleizadeh et al. (2015) developed a model that jointly optimizes price, replenishment frequency, and replenishment cycle and production rate in a vendor-managed inventory system with deteriorating items. In recent years, Krommyda et al. (2015), Salameh et al. (2014), Rasouli and Nakhai Kamalabadi (2014), and Gerchak and Grosfeld-Nir (1999) developed inventory models that consider two substitutable items with deterministic demand, constant holding cost and fixed ordering cost. None of these studies considered the role of deterioration in inventory decisions regarding substitutable items. Zhao et al. (2014) considered pricing decisions for two substitutable products with price-dependent probabilistic demand, fixed ordering costs, and constant holding costs. Additional studies have examined inventory policies for multiple substitutable items with stochastic demand, fixed ordering costs and constant holding costs (Ye 2014, Huang and Ke 2017, Li et al. 2013, Li and You 2012, Hsieh 2011).

Past research focuses on the substitute product in case on stock out and formalizes the process for maintaining optimal inventory. The research was carried out for two or more substitutable products using stochastic approach. The research did not consider the actual demand rate for the products and kept it as constant. Seldom has it happened in realistic scenario, that the demand rate is constant as mentioned economic order quantity models. This research considers a dynamic reorder point as compared to a static one, which then enables the inventory manager to adapt as per the demand fluctuations and replenish stock as required. The research addresses the gap by considering dynamic demand rate thereby decreasing the probability of stock out for the items as well as for the substitutable items.

Several inventory models of substitutable products have distinguished between specific types of product substitution relationships: Tang and Yin (2007), for example, considered stock-out-based substitution, price-based substitution and assortment-based substitution. Recently, Kim and Bell (2011) distinguished between symmetrical substitution and asymmetrical substitution. Our model builds on this idea by introducing a general variable called the dependency factor, which reflects the dependency between the demand rates of the two products, and effectively reflects the degree of substitutability between them.

3 Model Development

3.1 Problem Description

From a retailer perspective, one of the main goals is to have an efficient purchasing system (e.g. policy) so that at any given moment the availability of the consumer's desired product will be achieved in order to both keep high customer satisfaction and avoid lost sales. Hence, our model addresses the problem of the difficulty of having efficient and effective inventory management system which can result in high inventory level to avoid stockouts or poor service level which leads to poor reputation, loss of market share and lost sales.

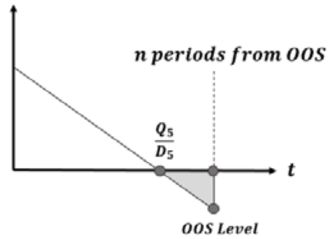
This model provides a holistic inventory planning solution for a group of substitute products, based on the understanding that the products are all substitutes for one another to some degree, such that their demand rates are correlated with a certain probability.

3.2 Illustrative Example: What Happens in the Event of a Stockout

For clarity, before diving into the details of the model, we will first provide an example of the model's basic concept. In this example, we consider a set of 5 products, with 5 different demand rates, where the demand rate for product *i* is denoted D_i. Assume that product 5 (with demand rate $\boldsymbol{D_5}$) goes out of stock (OOS).

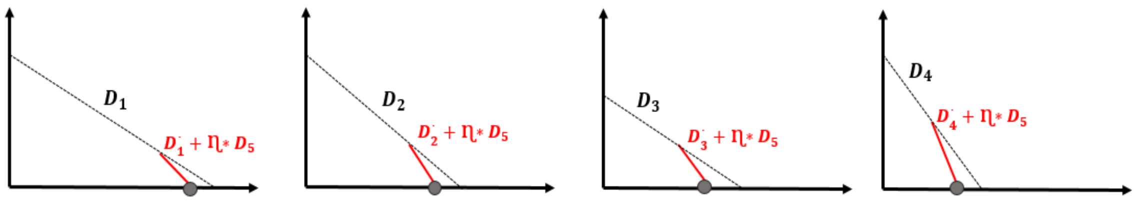
In the below graph we can see the demand rate of product 5, in a certain moment product 5 goes OOS and a backorder/ lost sales are consequently created. The consumer will probably choose an available alternative (e.g. product 1, 2, 3, 4) in order to fulfil his needs.

$$\textbf{OOS Level} = \sum_{t}^{n} \mathbf{D_{i_t}}$$



When product 5 is OOS, the other 4 substitute products will be consumed based on their dependency matrix, known as the dependency factor, ɳ. The magnitude of that additional consumption is a change in the demand curve of the other 4 substitute products as demonstrated below in the red color:

3.3 Model



3.4 Assumptions

I. The demand rate D_i for each product i is linear and constant (Deterministic).
II. Lead time (LT) is known and deterministic.
III. Penalty cost is known.
IV. The dependency factor (DF) is probabilistic (stochastic) – based on historical data.
V. When there is a dependency between products, demand is likely to pass over from one product to its substitute.

3.5 Notations

G_i – Group of products that can serve as substitutes for one another where i={1,2,3,…..,n}

n – Number of products in group G_i

P_i– A product in group G, i={1,2,3,…..,n}

Pij^b – A binary variable denoting the presence or absence of a dependency between product i and product j.

$A[G_i]$ – A matrix comprising the binary values Pij^b, denoting the presence or absence of a dependency between each pair of products in G

$ɳ_{ij}$ – Dependency between product i and product j

A[ɳ] – Matrix of dependency factors between each pair of products in G

LT – Lead time from purchase order to actual arrival at destination.

D_{i_t} – Demand over time t of product i

$D_{i_t}^N$– The new demand of product i due to shortage of product j

ROP_i – Reorder point for product i.

ROP_i^N – New reorder point for product i

T - Time periods (buckets)

CG_{ij} – Consumption gap between the typical consumption of product i and the consumption of level of product i following a shortage of product j

Q_{A_i} – Available stock of product i

OOS – Out of stock item

PO – Purchase order

3.6 Process Flow: The Five Steps of the Model

Figure 1 presents a flow chart of the model process. The model is made up of five steps, summarized in Figure 2 and described in detail in what follows.

Figure 1: Process Flow Chart

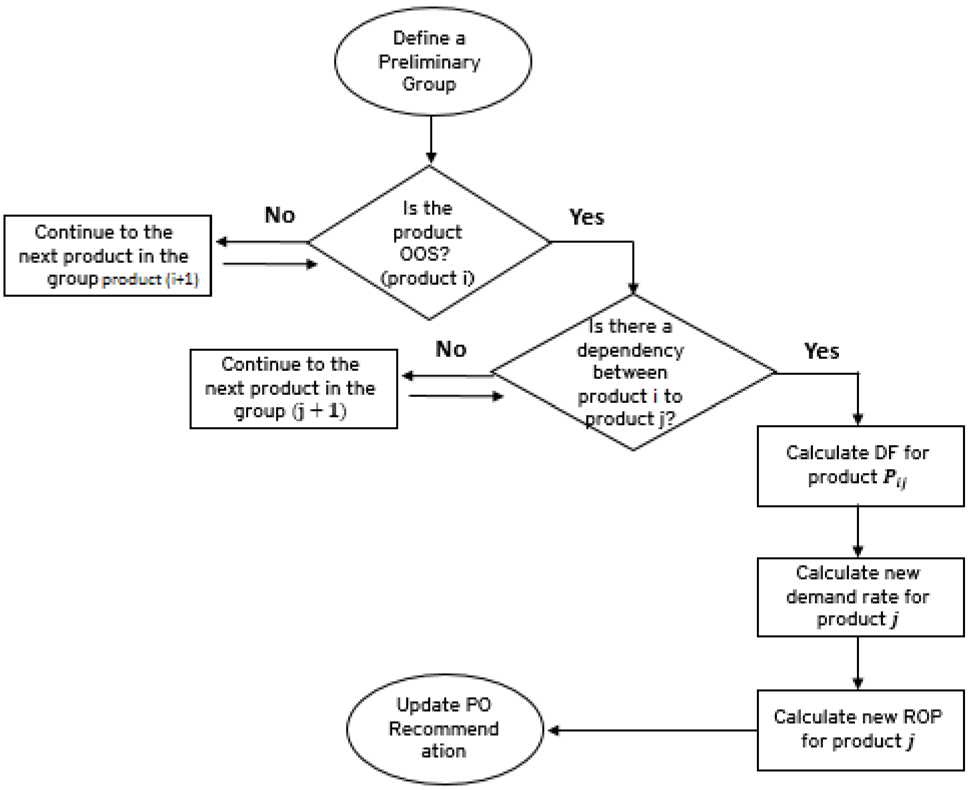
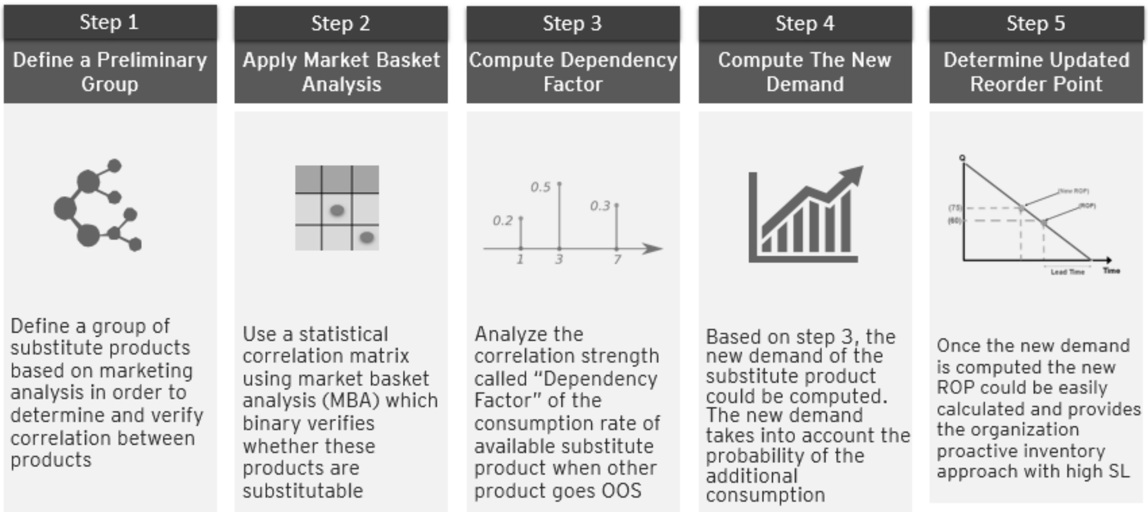


Figure 2: The Five Steps of the Model



STEP 1: *Identify a preliminary group G of potentially substitutable products*

Identify a group of potentially substitutable products based on qualitative marketing analysis, performed by marketing professionals. This group at that point is only preliminary and constitutes a starting point for advanced statistical models that will be applied in step 2.

$$\mathbf{P_i \in G} \qquad\qquad \mathbf{i=\{1,2,3,\ldots..,n\}}$$

STEP 2: *Apply market basket analysis*

After completing step 1, use market basket analysis in order to determine and verify dependency matrix between products. In this step, the preliminary mapping carried out in step 1 will be cleansed and prepared for the purchasing model that will be applied in the next steps. Market basket analysis, also known as association rule mining, is a useful method of discovering customer purchasing patterns by extracting associations or co-occurrences from firms' databases. The outcome of step 2 is a clear matrix representation A, where binary figures are used to describe whether a statistical correlation exists between the demand rates of a given pair of products (indicating that the two products are substitutable).

$$\mathrm{P}_{\mathrm{ij}}^{\mathrm{b}} = \begin{cases} 0, & \text{there is no dependency between product i and product j} \\ 1, & \text{full or partial dependency between product i and product j} \end{cases}$$

Let us assume, for example, that three products were identified in step 1 as potential candidates for substitute products. In step 2, market basket analysis is applied and confirms that the products' demand rates are indeed correlated with one another, producing the following matrix.

A[G]=

	P_1	P_2	P_3
P_1	0	1	1
P_2	1	0	1
P_3	1	1	0

STEP 3: *Compute dependency factor for each pair of products*

In this step, we compute a dependency factor for each pair of products for which the binary dependency value in matrix A was 1. To this end, we first compute CG_{ij}, which represents the consumption gap between the typical consumption of product i and the consumption of product i that occurs when product j is unavailable over the course of T time periods.

$$CG_{ij} = \frac{\sum_{t=1}^{n}\left(Di_t^{N} - D_{i_t}\right)}{T}$$

By finding CG_{ij}, we can now compute the dependency factor for products i and j, η_{ij} as follows:

$$\eta_{ij} = \frac{CG_{ij}}{D_j}$$

Ultimately, we can create a dependency factor matrix, $A[\eta]$.

$$A[\eta] = \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix}$$

STEP 4: *Compute the demand rate for each product in the event of a stockout of a related product*

Using the matrix obtained in step 3, we compute a new demand rate for product *i* in the case of a shortage of product *j* as follows:

$$D_{i_t}^{N} = D_{i_t} + \eta_{ij} * D_{j_t}$$

STEP 5: *Compute the updated reorder point (ROP)*

Based on the value obtained for the dependency factor between product i and product j and the new probabilistic demand, a dynamic purchasing approach is applied to select the reorder point for product *i* in the event of a shortage of product *j*:

$$ROP_i^N = D_{i_t}{}^N * LT_n$$

This equation above provides a solution for inventory management in a stochastic environment, whereas the old ROP in a deterministic environment is calculated by the multiplication of steady demand times the steady LT. In a disruptive market the deterministic solution is much less effective and responsive.

Therefore, the value obtained in our model will determine the timing of the new purchase order (PO) in a much more accurate way as follows:

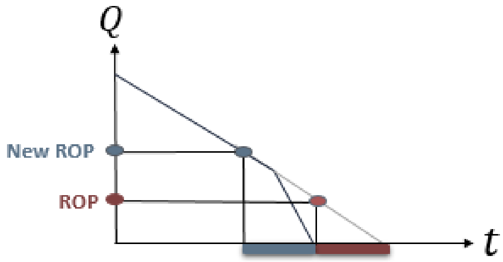
IF: $ROP_i^N < Q_{A_i}$

PO where $Q_{A_i} = ROP_i^N$

ELSE

: **Generate PO immediately**

The result can be illustrated in the below graph, where a comparison between a deterministic and stochastic approaches are applied. Our model detects the demand change rate and automatically determines the new ROP that responses more quickly to market change.



4 Numerical Example

In order to illustrate how the model works, we present a numerical example based on a simulation for a group of two substitute products. Table 1 below (a binary matrix) shows the result of the market basket analysis, which indicates that there is indeed a dependency between the demand rates of P_1 and P_2, and of P_2 and P_1.

Table 1: Binary Dependency Matrix for {P_1, P_2}

	P_1	P_2
P_1	0	1
P_2	1	0

Next, we can analyze the firm's historical data and generate the dependency factor for each pair of products. The dependency factor found between P_1 and P_2 is 0.7, and the dependency factor between P_2 and P_1 is 1, meaning that when P_2 is OOS all of its demand fully passes to P_1 (Table 2).

Each product has a known and constant Lead Time (LT), calculated optimal order quantity (Q*), Revenue (REV) and a known penalty cost for a shortage (PEN) (Table 3).
Table 4 shows the demand rate and the inventory level of P_1 for a 12-week time period. The end of the first week is denoted W1, the end of the second is denoted W2, etc. W0 denotes the initial conditions, before the beginning of week 1 (in Tables 4–9 replenishment is marked in green, stockout (OOS) is marked in red).

P_1 ROP $= LT * D(t) = 60$.

Table 5 shows the demand rate and the inventory level of P_2 for a 12-week time period.

P_2 ROP $= LT * D(t) = 20$.

Table 6 illustrates a scenario of a supply delay for P_2 over the course of a 3-week period; it reaches *OOS* in weeks 6–8.

Table 2: Dependency Factor Matrix Based on Historical Data

	P_1	P_2
P_1	0	0.7
P_2	1	0

Table 3: Parameter Values

	LT	**Q***	**REV**	**PEN**
P_1	3	100	80	40
P_2	2	80	40	20

Table 4: P_1

Period	**W0**	**W1**	**W2**	**W3**	**W4**	**W5**	**W6**	**W7**	**W8**	**W9**	**W10**	**W11**	**W12**
D(t)	--	20	20	20	20	20	20	20	20	20	20	20	20
Inventory level	100	80	60	40	20	100	80	60	40	20	100	80	60

Table 5: P_2

Period	**W0**	**W1**	**W2**	**W3**	**W4**	**W5**	**W6**	**W7**	**W8**	**W9**	**W10**	**W11**	**W12**
D(t)		10	10	10	10	10	10	10	10	10	10	10	10
Inventory level	50	40	30	20	10	80	70	60	50	40	30	20	10

Table 6: Supply Delay for P_2 During Weeks 6–8

Period	W0	W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	W11	W12
D(t)		10	10	10	10	10	10	10	10	10	10	10	10
Inventory level	50	40	30	20	10	0	-10	-20	-30	80	70	60	50

P_1 is a substitute for *P_2*, and the dependency factor is 1, which means that during the time period in which *P_2* is OOS (weeks 6–8), all the demand for *P_2* is transferred to *P_1* (Table 7).

Due to the increased consumption of *P_1*, the inventory of this item will run out more quickly than originally expected. Without adjusting the point of replenishment, *P_1* will go OOS on week 9, leading the decision maker to pay a penalty of $400.

The model provides a solution that enables the inventory manager to avoid these consequences by calculating a new *ROP* for *P_1*, thereby preventing the product from going OOS, and enabling the penalty to be avoided.
The new *ROP* calculation for *P_1*:

$$D_1^N = D_1 + \eta_{12} * D_2 = 20 + 1 * 10 = \mathbf{30}$$

$$ROP_1^N = D_1^N * LT_1 = 30 * 3 = \mathbf{90}$$

The value obtained for the new *ROP* is 90, which means that the reorder point moves to week 6, and replenishment moves from week 10 to week 9. The result is that there is no shortage and no penalty (see Table 8).

Table 9 presents a comparison of the revenues gained and costs incurred by the decision maker in three different scenarios: (i) in the "normal" scenario, in which the supply of *P_2* is not disrupted; (ii) in a scenario in which the supply of *P_2* is delayed but the decision maker does not use the model to determine an updated *ROP* for *P_1*; and (iii) a scenario in which the decision maker does use the model. The comparison shows that, in the presence of a delay, the decision maker can prevent substantial losses by implementing the model.

The research focuses on 2 products so as to simplify the purpose, the model provides support for calculations using multi-products data sets as well. To illustrate further, if product goes out of stock and has let's say 4 substitute items so the consumption probabilities of these 4 will have to be all equal to 1 [1 = P(T) = P(1) +P(2) +P(3)+P(4)]. Based on the dependency factor (correlation strength) we know how to allocate different weights to different items. These differences reflect the consume taste given OOS of specific product.

Table 7: P_1

Period	W0	W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	W11	W12
D(t)	--	20	20	20	20	20	30	30	30	20	20	20	20
Inventory level	100	80	60	40	20	100	70	40	10	-10	100	80	60

Table 8: P_1

Period	W0	W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	W11	W12
D(t)	--	20	20	20	20	20	30	30	30	20	20	20	20
Inventory level	100	80	60	40	20	100	70	40	10	100	80	60	40

Table 9: Comparison Analysis

	Normal demand rate	**P_2 Demand change without using the model**	**P_2 Demand change with new ROP calculation**
REV P_1	19,200	2,0800	**21,600**
REV P_2	4,800	5,200	**5,200**
PEN	0	400	**0**
Total REV	**$24,000**	**$25,600**	**$26,800**

5 Conclusion

The model proposed herein provides an efficient and dynamic solution for inventory managers dealing with multiple substitutable items within an inventory system. The model's main objective is to enable managers to react to market changes in a timely manner by identifying the appropriate points in time at which to reorder their products, given that the demand for some products may increase after other, substitutable products become unavailable. Notably, in contrast to other models of substitutable products, which tend to consider only two items, our model accommodates larger sets of substitutable products, with varying levels of substitutability. An interesting avenue for future research would be to extend our model to other types of product relationships, such as complementarity.

At our model's core is the dependency factor, which indicates the level of substitutability between a given pair of products. The dependency factor is computed on the basis of historical data, taken, for example, from the firm's information systems. We suggest that our model should be programmed into the firm's information systems so that it might continue to gather data and to learn, and thus to improve its accuracy and demand sensing capacities over time. The future research direction can be based on stochastic approach considering product substitutability.

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