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# Optimal Rules for Central Bank Interest Rates Subject to Zero Lower Bound

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**Abstract**
The celebrated Taylor rule provides a simple formula that aims to capture how the central bank interest rate is adjusted as a linear function of inflation and output gap. However, the rule does not take explicitly into account the zero lower bound on the interest rate. Prior studies on interest rate selection subject to the zero lower bound have not produced derivations of explicit formulas. In this work, Taylor-like rules for central bank interest rates bounded below by zero are derived rigorously using a multi-parametric model predictive control framework. This framework is used to derive rules with or without inertia. The proposed approach is illustrated through simulations. Application of the approach to US economy data demonstrates its relevance and provides insight into the objectives underlying central bank interest rate decisions. A number of issues for future study are proposed.

**JEL** E52 C61
**Keywords** Taylor rule; zero lower bound; liquidity trap; model predictive control; multi-parametric programming

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# 1 Introduction and Motivation

The general form of the celebrated Taylor rule suggests that the short-term interest rate $i_t$ applied by the central bank at time $t$ can be set according to the simple formula

$$i_t = \phi_y (y_t - y^*) + \phi_\pi (\pi_t - \pi^*) + r^* + \pi^* \quad (1)$$

While the initial inspiration for the Taylor rule was based on fitting actual historical data, the rule, and a number of related variants, can be derived as a commitment policy through application of optimal control theory on a (quadratic) objective function, subject to the output gap, $y$, and inflation, $\pi$, responding to the interest rate $i$ according to a simple dynamic model of the economy (Ball, 1999; Orphanides and Wieland, 2000; Giannoni and Woodford, 2002; Orphanides, 2003). Such derivations have mainly focused on how various forms of the objective function in the optimal control problem may result in corresponding Taylor-like rules, and have provided considerable insight into the underlying intents for interest rate decisions. However, such derivations cannot be used when the interest rate is explicitly *constrained* by the *zero lower bound* (ZLB), namely $i_t \geq 0$. The usual approaches to handling the ZLB when deriving interest rate policies based on optimal control formulation can be broadly classified into two categories:

- Explicit rules that truncate to zero the interest rate calculated by an unconstrained Taylor rule, when the latter produces a negative interest rate (Reifschneider and Williams, 2000; Williams, 2006; Nakov, 2008); and
- implicit rules through numerical simulation, i.e. repeated numerical solution of a ZLB-constrained optimization problem, to determine the optimal values of interest rate for inflation and output gap values in a range of interest (Orphanides and Wieland, 2000; Hunt and Laxton, 2003; Jung et al., 2005; Kato and Nishiyama, 2005; Adam and Billi, 2007). Most studies in this category rely on a constrained dynamic programming formulation of the underlying optimization problem, whose explicit analytical solution is hard to get.

However, neither of these two approaches is entirely satisfactory. Indeed, truncation to zero the result of an unconstrained Taylor rule can no longer be guaranteed to be the solution of the corresponding *ZLB-constrained* optimal control problem. In addition, numerically derived optimal interest rates lack the appeal of a simple explicit rule. Consequently, the following question can be raised:

*Is it possible to derive Taylor-like rules when explicitly including the ZLB constraint in the original optimal control problem that ordinarily would produce standard Taylor rules in the absence of ZLB?*

The main contribution of this paper is to demonstrate that this is possible, to formulate a corresponding general framework for development of such rules, and to present case studies where such rules are derived. The proposed approach is based on the use of multi-parametric programming for solution of a stabilizing approximation to the corresponding constrained optimal control problem (Pistikopoulos et al., 2002; Darby and Nikolaou, 2007). The key outcome from the proposed approach is that when the ZLB is present, the optimal interest rate can be determined by consulting a look-up table and using one of a finite number of explicit Taylor-like (piecewise linear) formulas, computed once off-line for repeated use over time. These formulas are guaranteed to be stabilizing. Interestingly, while the resulting formulas are similar to truncated versions of the standard Taylor rule in some cases, they are different in others.

In the rest of the paper we provide some preliminaries on optimal control, model predictive control (MPC) and Taylor rules in Section 2. We present our main results in Section 3, namely how multiparametric MPC can produce Taylor-like rules in the presence of ZLB. We illustrate our approach in Section 4, and provide additional discussion and comparison with real data in Section 5. Conclusions and suggestions for further work are provided in Section 6.

## 2 Preliminaries: Constrained Optimal Control and Taylor Rules

While solution of a standard optimal control problem with quadratic objective and a linear model easily leads to the Taylor rule, the same approach cannot be used when the ZLB is present, since rigorously obtaining an explicit (closed-form) optimal solution in that case is not trivial. Existing alternative approaches, as mentioned above, are based on either truncation the standard Taylor rules or on numerical solution of the corresponding constrained optimal control problems. In pertinent studies, it was observed that resulting policies may be nonlinear (rather than piecewise linear, according to truncated Taylor rules) and more aggressive for interest rates close to ZLB (a behavior characterized as pre-emptiveness). However, *a rigorous derivation of optimal explicit Taylor rules subject to ZLB is, to our knowledge, not currently available.*

To develop a related formulation that does lead to an explicit solution in the presence of the ZLB, we rely on a finite-dimensional approximation of the standard optimal control problem and on its moving-horizon implementation, known in the engineering literature as model-predictive control (MPC) or receding horizon control (Maciejowski, 2000; Rawlings and Mayne, 2009). Inequality-constrained MPC involves an objective (cost function) and a model, both of which are discussed next, along with a brief formulation of MPC.

## 2.1 Economy Model Structure

A standard model around a baseline can describe the evolution of the economy as

$$y_{t+1} = \rho y_t - \xi\left(i_t - \pi_t\right) + e_{t+1}^{y}, \tag{2}$$

$$\pi_{t+1} = \pi_t + \alpha y_t + e_{t+1}^{\pi}. \tag{3}$$

Equation (2) comes from the standard IS equation with forward-looking, namely

$$y_{t+1} = \sigma y_t - \xi\left(i_t - E_t[\pi_{t+1}]\right) + e_{t+1}^{y}, \tag{4}$$

where the future expectation $E_t[\pi_{t+1}]$ is approximated by the noise-free counterpart of Eqn. (3) (the standard AS equation), namely

$$E_t[\pi_{t+1}] = \pi_t + \alpha y_t. \tag{5}$$

(Ball, 1999). Note that the sampling period (time interval from time $t$ to $t+1$) is one year.

The above model is similar in spirit to more complicated models used by many central banks. The model's main purpose is to capture the overall dynamic causal relationship between the manipulated input $i$ and the two controlled outputs, $y$, $\pi$.

Note that Eqns. (2) and (3) capture the aggregate effect of the interest rate on the economy. Effects due to phenomena such as rational expectations are assumed to have been incorporated in the model structure. Other kinds of models can also be converted to the aggregate form of Eqns. (2) and (3) (Kato and Nishiyama, 2005).

For the above model, at steady state (equilibrium point), we have $i_t = i^*$, $y_t = 0$ and $\pi_t = \pi^*$, with $r^* = i^* - \pi^*$. Hence, in the terms of deviation variables from the equilibrium point, eqns. (2) and (3), can be written in vector-matrix form as

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}u_t + \boldsymbol{\varepsilon}_{t+1}, \tag{6}$$

where

$$\mathbf{x} \triangleq \begin{bmatrix} \Delta y \\ \Delta \pi \end{bmatrix} \triangleq \begin{bmatrix} y - y^* \\ \pi - \pi^* \end{bmatrix}, \; u \triangleq \Delta i \triangleq i - i^*, \; \boldsymbol{\varepsilon} \triangleq \begin{bmatrix} e^y \\ e^\pi \end{bmatrix}, \tag{7}$$

$$\mathbf{A} \triangleq \begin{bmatrix} \rho & \xi \\ \alpha & 1 \end{bmatrix}, \tag{8}$$

$$\mathbf{B} \triangleq \begin{bmatrix} -\xi \\ 0 \end{bmatrix}. \tag{9}$$

It should be stressed once more that the idea here is not to fully explain the complex dynamics of the economy with such a simple linear model. Rather, the intended use of the above model is to help understand how optimal monetary policies can be derived and how such policies are affected by various objective functions and by a ZLB on the interest rate. The dimension of the state vector $\mathbf{x}$ is also limited to two, so that resulting policies can be easily derived and understood graphically in 2-D and 3-D plots. The methodology presented below can be easily applied to different low-dimensional models as well.

## 2.2 Economy Model Calibration

The economy model expressed by eqns. (2) and (3) is calibrated using US revised economy data over the time period 1976–2007. The annual revised output gap data is taken from the Congressional Budget Office (CBO, 2011). Inflation is calculated as annual percentage change in the GDP deflator Q4/Q4 basis (Bureau of economic Analysis). The real interest rate, $r$, is calculated as the annual average of the interest rate deflated by the annual inflation rate. Interest rates are taken from the database of the Federal Reserve System. Figure 1 plots these data for the time period 1976–2010. Based on these data, Table 1 presents estimated values of parameters for the economy model, obtained using the standard prediction-error method for least-squares estimation (Söderstrom and Stoica, 1989; Ljung, 1999). Based on the parameter estimates in Table 1, the matrices **A** and **B** in eqns. (8) and (9) turn out to be

$$\mathbf{A} = \begin{bmatrix} 0.63 & 0.19 \\ 0.12 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -0.19 \\ 0 \end{bmatrix}. \tag{10}$$

*Figure 1:* Revised data for US output gap, GDP deflator inflation rate and federal fund rates in annual percentage for year 1976–2010 (CBO, 2011).

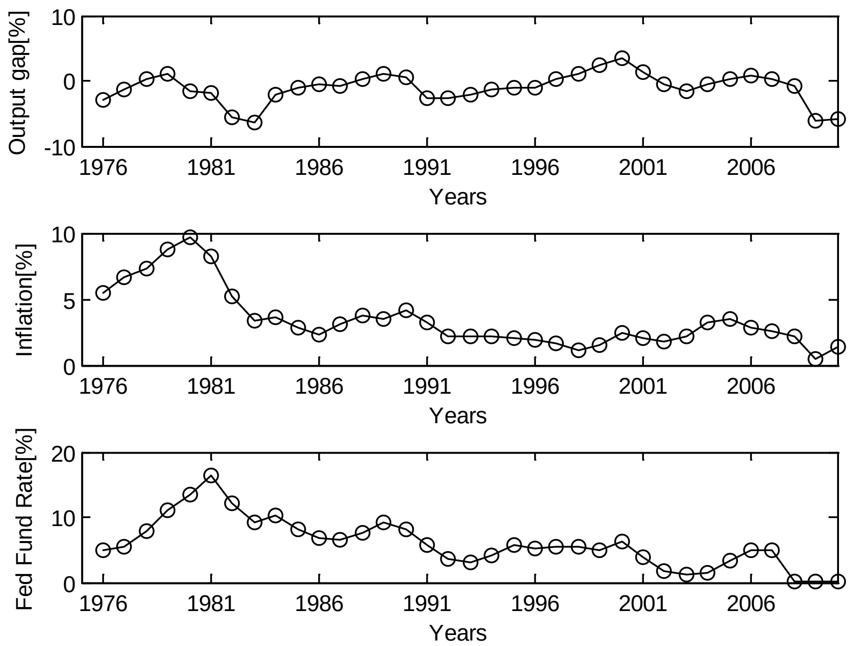


*Table 1:* Parameter Estimates of US Economy Model

| Parameter | Estimate | Standard Error |
|---|---|---|
| $\rho$ | 0.63 | 0.06 |
| $\xi$ | 0.19 | 0.05 |
| $r^*$ | 1.9 | 0.74 |
| $\alpha$ | 0.12 | 0.06 |
| $\sigma_{e^y}$ | 1.4 | |
| $\sigma_{e^\pi}$ | 0.93 | |

The eigenvalues of $\mathbf{A}$ are 0.58 and 1.05, suggesting that the economy model for the US economy is mildly unstable. Whether the economy model is stable or unstable, whatever control policy ones chooses to control the US economy, such a policy must be, at the very least, a stabilizing policy. We develop below such a policy for the above model via MPC.

## 2.3 Formulation of Central Bank's Objective as MPC Optimization

### 2.3.1 Standard Optimal Control Objective

The central bank's generalized cost function projected to infinity at time $t$ is generally assumed to be of the standard optimal control form

$$\sum_{k=0}^{\infty} \beta^k L(\hat{\mathbf{x}}_{t+k|t}, u_{t+k|t}) . \tag{11}$$

where $\beta \in (0,1)$ is the discount factor; $\hat{\mathbf{x}}_{t+k|t}$ is the expected value of $\mathbf{x}$ at time $t+k$ using all information available at time $t$ and a model such as Eqn. (6); $u_{t+k|t}$ refers to the input value at time $t+k$ to be decided on at time $t$; and $L(\hat{\mathbf{x}}_{t+k|t}, u_{t+k|t})$ is usually a quadratic function of $\hat{\mathbf{x}}_{t+k|t}$ and $u_{t+k|t}$ such as

$$L(\hat{\mathbf{x}}_{t+k|t}, u_{t+k|t}) = \hat{\mathbf{x}}_{t+k|t}^T \mathbf{Q} \hat{\mathbf{x}}_{t+k|t} + R^2 u_{t+k|t}^2 . \tag{12}$$

with $R^2 \geq 0$ and $\mathbf{Q} \succeq 0$ (positive semi-definite).

Note that often, but certainly not always (Woodford, 1999; Orphanides and Williams, 2007), the cost function is considered to be of the form $\sum_{k=0}^{\infty} \beta^k L(\hat{\mathbf{x}}_{t+k|t})$ where the term $L(\hat{\mathbf{x}}_{t+k|t})$ is a (typically quadratic) function of $\hat{\mathbf{x}}_{t+k|t}$ alone, without explicitly including a term that contains $u_{t+k|t}$. We claim that explicit inclusion in the cost function of a term that penalizes the input $u_{t+k|t}$, as in Eqn. (12), is important, for the following three reasons:

- The central bank is generally averse to drastic changes in the interest rate, which suggests a relative cost for interest rate changes.
- It is well known in optimal control theory (Kailath, 1980; Anderson and Moore, 2007) that failure to include in the cost function an explicit term containing $u_{t+k|t}$ may result in destabilizing policies (when the inverse of the controlled system is unstable) or excessively large inputs.
- As we discuss below, using a cost function along the lines of Eqn. (12) results in interest rate rules that follow the Federal Reserve Bank's decisions fairly well for over two decades.

Of course, the cost function can easily be made not to depend explicitly on the $u_{t+k|t}$ term by setting $R^2 = 0$ in Eqn. (12).

### 2.3.2 Unconstrained Optimal Control and the Taylor Rule

Solution of the optimal control (minimization) problem comprising a quadratic cost function such as Eqn. (12), a linear model such as Eqn. (6), and no inequality constraints on $u_{t+k|t}$, results in the celebrated linear-quadratic regulator (LQR), namely

$$u_{t|t}^{\text{opt}} = \mathbf{K}\mathbf{x}_t = K_1 \Delta y_t + K_2 \Delta \pi_t \,. \tag{13}$$

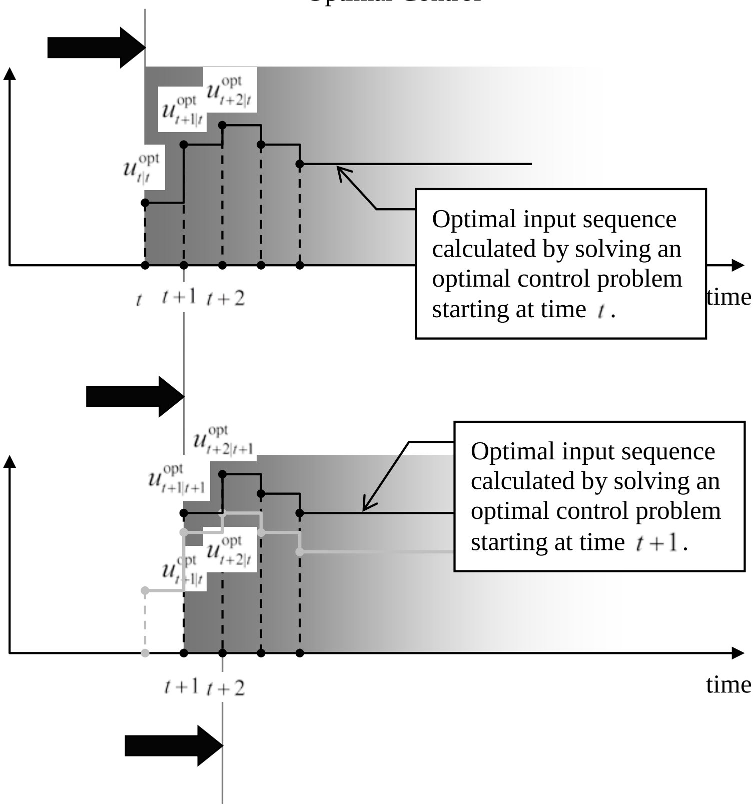
where $u_{t|t}^{\text{opt}}$ is the optimal input to be implemented at time $t$; $\mathbf{x}_t$ is the actual state of the system (assumed to be measured at time $t$); and $\mathbf{K} = [K_1 \quad K_2]^T$ is a *constant* matrix, computed from solution of the Riccati equation (Kailath, 1980; Anderson and Moore, 2007).

The structure of Eqn. (13) is clearly that of the Taylor rule. The specific values of the coefficients $K_1$, $K_2$ depend on the particular values of the parameters in the linear model and quadratic cost function.

### 2.3.3 Standard Optimal Control and Feedback

It should be noted that the optimal solution sequence $\{u_{t|t}^{\text{opt}}, u_{t+1|t}^{\text{opt}}, \ldots\}$, calculated at time $t$, is *not* what is going to be implemented on the actual system over time beyond $t$, as indicated visually in Figure 2. Rather, the first optimal input value $u_{t|t}^{\text{opt}} = \mathbf{K}\mathbf{x}_t$ of the sequence $\{u_{t|t}^{\text{opt}}, u_{t+1|t}^{\text{opt}}, \ldots\}$ will be implemented at time $t$.

*Figure 2:* Visual Representation of Moving-Horizon Optimization and Associated Optimal Control



Then, the system will be left to run until time $t+1$; a new measurement of the state $\mathbf{x}_{t+1}$ will be obtained at $t+1$; the first optimal input value $u^{\text{opt}}_{t+1|t+1} = \mathbf{K}\mathbf{x}_{k+1}$ of the sequence $\{u^{\text{opt}}_{t+1|t+1}, u^{\text{opt}}_{t+2|t+1}, \ldots\}$, will be implemented at time $t+1$; and so on to infinity. Note that, in general, $u^{\text{opt}}_{t+1|t+1} \neq u^{\text{opt}}_{t+1|t}$ (i.e. the optimal values of the input at time $t+1$ calculated at times $t$ and $t+1$ are not equal) because of various uncertainties (e.g. external disturbances, modeling inaccuracies, etc). Note also that the overall procedure of measuring $\mathbf{x}_t$, deciding on and implementing $u_t$, and repeating these tasks at subsequent times, $t+1$ and onwards, is standard *closed-loop* control.

### 2.3.4 Inequality-Constrained Optimal Control

The preceding discussion in Sections 2.3.2 and 2.3.3 and the resulting optimal rule in Eqn. (13) rely on the assumption that no ZLB is explicitly placed on the input. *What would the optimal rule be if the ZLB were present?*

It is tempting to claim that the optimal rule in the presence of the ZLB would be a truncated version of the standard Taylor rule, namely the interest rate would be computed by Eqn. (13) as either $i_t = \mathbf{K}\mathbf{x}_t + i*$, if $\mathbf{K}\mathbf{x}_t + i* \geq 0$; or $i_t = 0$ otherwise. While this is plausible, it is not necessarily correct. For example, it could well be that, for interest rates near the ZLB, more aggressive action than normal would be required, i.e. the interest rate should be pre-emptively reduced more drastically than recommended by Eqn. (13), in anticipation of (and to counter) near-future situations where the optimal interest rate would be stuck at the ZLB. It is then natural to ask the following:

- *Are there circumstances where the truncated version of the Taylor rule is optimal?*
- *In circumstances where the truncated version of the Taylor rule is not optimal (e.g. in pre-emptive scenarios) would there be an equally simple alternative rule?*
- *Can either kind of circumstances be easily detected and corresponding rules established?*

To answer these questions one can augment the optimal control problem from which The Taylor rule naturally emerges, by explicitly adding the ZLB constraint while retaining the same quadratic cost function and linear model. Unfortunately, the solution of this optimization problem is no longer a simple formula as in Eqn. (13), because of the ZLB constraint. Indeed, one would have to solve a quadratic programming problem with an infinite number of decision variables, namely $\{u_{t+k|t}^{\text{opt}}\}_{k=0,\dots,\infty}$, and with an infinite number of constraints, namely $\{u_{t+k|t}^{\text{opt}} + i^* \geq 0\}_{k=0,\dots,\infty}$. This is not feasible. As we explain below, however, it turns out that this infinite-dimensional constrained optimization problem can be very well approximated by a finite-dimensional counterpart, which, for relatively small linear models, as in Eqn. (6), admits an explicit solution. The benefit from this realization is that *Taylor-like (piecewise linear) rules can be developed when the ZLB is present*. We elaborate on this next.

### 2.3.5 Inequality-Constrained Optimal Control and MPC

It can be rigorously shown (Muske and Rawlings, 1993) that the solution of an *infinite-dimensional* constrained optimal control problem involving summation over infinite time, as in Eqn. (11), can be approximated arbitrarily well by the solution of a corresponding *finite-dimensional* problem, which involves summation of a finite number of terms plus a terminal cost and/or terminal equality constraints. This is also known as finite-horizon approximation of an infinite-time horizon. In fact, it has long been established, through experience from widespread application of MPC (Muske and Rawlings, 1993), that the dimension of the corresponding finite-dimensional problem is practically fairly low. *The preceding statements are crucial for developing Taylor-like rules via multi-parametric programming.*

In the above context, an objective for MPC at time $t$ can be formulated as

$$\min_{\mathbf{u}} \left\{ \sum_{k=0}^{N-1} \beta^k \left( \hat{\mathbf{x}}_{t+k|t}^T \mathbf{Q} \hat{\mathbf{x}}_{t+k|t} + R^2 u_{t+k|t}^2 + S^2 \delta u_{t+k|t}^2 \right) + \hat{\mathbf{x}}_{t+N|t}^T \beta^N \bar{\mathbf{Q}} \hat{\mathbf{x}}_{t+N|t} + \beta^N S^2 \delta u_{t+N|t}^2 \right\} \quad (14)$$

where

$$\mathbf{Q} \hat{=} \begin{bmatrix} 1-\lambda & 0 \\ 0 & \lambda \end{bmatrix} \succ 0, \quad 0 < \lambda < 1 \quad (15)$$

$$\mathbf{u} \hat{=} \begin{bmatrix} \Delta i_{t|t} & \cdots & \Delta i_{t+N-1|t} \end{bmatrix}^T, \quad (16)$$

$$\delta u_{t+k|t} \mathrel{\hat{=}} u_{t+k|t} - u_{t+k-1|t}, \ k = 0,\ldots,N, \tag{17}$$

and $\bar{\mathbf{Q}}$ is defined in Appendix A.

Inclusion of penalties on $\delta u_{t+k|t}$ in the above cost function is justified by the following two arguments.

- The central bank is generally averse to drastic changes in the interest rate, which suggests a relative cost for $\delta u_{t+k|t}$.
- With stronger penalty on $\delta u_{t+k|t}$ rather than on $u_{t+k|t}$, the state $\mathbf{x}$ is guaranteed to asymptotically reach the desired value. (This is known as the *zero-offset property* in engineering literature (Stephanopoulos, 1984)).

Expected values $\hat{\mathbf{x}}_{t+k|t}$, predicted at time $t$, rely on use of the linear model in Eqn. (6) for $k$-step ahead predictions $\hat{\mathbf{x}}_{t+k|t}$ (Söderstrom and Stoica, 1989 ; Ljung, 1999) as

$$\hat{\mathbf{x}}_{t+k|t} = \sum_{\ell=0}^{k-1} \mathbf{A}^{\ell} \mathbf{B} u_{t+k-\ell-1|t} + \mathbf{A}^{k} \mathbf{x}_t, \ k = 1,\ldots,N, \tag{18}$$

with

$$\hat{\mathbf{x}}_{t|t} = \mathbf{x}_t, \tag{19}$$

where $\mathbf{x}_t$ refers to the measured state at time $t$.

The inputs must satisfy the ZLB constraint, namely

$$u_{t+k|t} \geq -i^*, \ k = 0,\ldots,N-1, \tag{20}$$

To complete the MPC formulation, additional equality constraints are included (eqns. (56) and (55) in Appendix A), such that stability of the resulting closed-loop scheme can be guaranteed.

The values of $1-\lambda$ and $\lambda$ in Eqn. (15) determine the relative attention paid by the policy to the output gap and inflation, respectively.

Finally, the values of the weights $R$ and $S$ determine the aggressiveness of the resulting control action, with small values of $R$ and $S$ encouraging more aggressive action and faster closed-loop response, at the cost of decreased closed-loop robustness (Orphanides, 2003; Orphanides and Williams, 2007). In particular, higher values of $S$ are preferred when persistent external disturbances force the interest rate $i$ away from its nominal equilibrium value $i^*$.

In the absence of the ZLB (Eqn. (20)) the standard Taylor rule results for $S = 0$, whereas for $S \neq 0$ the Taylor rule with inertia naturally emerges. The constrained case, namely inclusion of Eqn. (20) in the MPC optimization, is discussed next.

# 3 Taylor Rules from MPC Constrained with ZLB

In this section we show how Taylor-like (piecewise linear) rules can be derived from application of multi-parametric MPC constrained with ZLB. We generate such rules both without and with inertia, in Sections 3.1 and 3.2, respectively. Subsequently, we discuss the effects of various parameters on the resulting rules, before we turn to comparison with real data in the next section.

## 3.1 The General Idea: Taylor Rules in the Presence of ZLB

When the interest rate must satisfy a ZLB constraint, the optimization problem to be solved by MPC at each time $t$ entails

- the objective in Eqn. (14);
- the model-based predictions, Eqn. (18);
- the equality constraints in eqns. (19), (56), and (55); and
- the inequality constraint corresponding to the ZLB, Eqn. (20).

Following (Pistikopoulos et al., 2002), and assuming temporarily that $R > 0$, $S = 0$ in Eqn. (14), it can be shown (see Appendix B) that, after equation reduction by use of all equality constraints, the above optimization problem can be cast in compact form as the inequality-constrained quadratic programming problem

$$\min_{\mathbf{z}} \left\{ \frac{1}{2} \mathbf{z}^T \mathbf{H} \mathbf{z} \right\}, \tag{21}$$

subject to

$$\mathbf{G}\mathbf{z} \leq \mathbf{w} + \mathbf{D}\mathbf{x}_t, \tag{22}$$

where

- the decision variable $\mathbf{z}$ is defined in terms of interest rate as

$$\mathbf{z} \triangleq \mathbf{u}_m + \mathbf{H}^{-1}\mathbf{F}^T\mathbf{x}_t, \tag{23}$$

with

$$\mathbf{u}_m \triangleq \begin{bmatrix} \Delta i_{t|t} \\ \Delta i_{t+1|t} \\ . \\ . \\ \Delta i_{t+m-2|t} \end{bmatrix}. \tag{24}$$

- $\mathbf{H}$, $\mathbf{G}$, $\mathbf{D}$, $\mathbf{F}$ are constant matrices and $\mathbf{w}$ a constant vector, depending on the parameters that appear in the model, objective function, and ZLB, as defined in Appendix B.
- $\mathbf{x}_t$ is the measured state of the system at time $t$, as defined in Eqn. (7).

Eqns. (21) and (22) suggest that the optimization problems solved by MPC at successive points in time *differ only by the right-hand side of Eqn.* (22), which is affine in the state $\mathbf{x}_t$. Note that $\mathbf{x}_t$ is not a decision variable but rather a parameter that is generally different at each time point (hence the term *multi-parametric*). While no single formula exists for the explicit solution of all of these problems, application of multi-parametric programming (Pistikopoulos et al., 2002), implies that the optimal solution can be expressed *explicitly* (in closed form) at each time point $t$ as

$$\mathbf{z}_{t|t}^{\text{opt}} = \mathbf{H}^{-1}\mathbf{G}_A^T\left(\mathbf{G}_A\mathbf{H}^{-1}\mathbf{G}_A^T\right)^{-1}\left(\mathbf{w}_A + \mathbf{D}_A\mathbf{x}_t\right), \tag{25}$$

where $\mathbf{G}_A$, $\mathbf{w}_A$, $\mathbf{D}_A$ correspond to the set of *active* inequality constraints in Eqn. (22), and are finite in number. Which inequality constraints in Eqn. (22) will be active (i.e. satisfied as equalities) at any time point $t$ depends only on $\mathbf{x}_t$, and this can be determined (Pistikopoulos et al., 2002) by checking whether the inequalities

$$\mathbf{G}\mathbf{H}^{-1}\mathbf{G}_A^T\left(\mathbf{G}_A\mathbf{H}^{-1}\mathbf{G}_A^T\right)^{-1}\left(\mathbf{w}_A + \mathbf{D}_A\mathbf{x}_t\right) < \mathbf{w} + \mathbf{D}\mathbf{x}_t \tag{26}$$

and

$$-\left(\mathbf{G}_A\mathbf{H}^{-1}\mathbf{G}_A^T\right)^{-1}\left(\mathbf{w}_A + \mathbf{D}_A\mathbf{x}_t\right) \geq 0, \tag{27}$$

are satisfied for each of the possible choices for $\{\mathbf{G}_A, \mathbf{w}_A, \mathbf{D}_A\}$. While the number of choices for $\{\mathbf{G}_A, \mathbf{w}_A, \mathbf{D}_A\}$ (i.e. the number of combinations of active/inactive inequality constraints) may be generally large, we show in the sequel that this number is fairly small for the problem at hand, resulting in a small set of explicit rules in the form of Eqn. (25), which are shown to be Taylor-like (piecewise linear).

More specifically, for a certain instance of $\{\mathbf{G}_A, \mathbf{w}_A, \mathbf{D}_A\}$, corresponding to a set of active inequality constraints in Eqn. (22), the inequalities in Eqn. (26) and (27) define a linear polytope for $\mathbf{x}_t$, for which the same sets of constraints remain active or inactive, and the same formula, Eqn. (25), can be used to express the optimal solution for any $\mathbf{x}_t$ in that polytope. The collection of all polytopes, whose total number is finite, spans the entire set in which $\mathbf{x}_t$ lies and which is bounded for a stable closed loop. Therefore, determining the active and inactive constraints in Eqn. (22), and consequently the corresponding $\mathbf{G}_A$, $\mathbf{w}_A$, $\mathbf{D}_A$, is a simple matter of using a look-up table, to determine in which polytope $\mathbf{x}_t$ lies, i.e. for which of the finitely many possible instances of $\{\mathbf{G}_A, \mathbf{w}_A, \mathbf{D}_A\}$ in the look-up table eqns. (26) and (27) are satisfied. Eqn. (25) can then be used to determine the optimal interest rate. The corresponding simple rule can then be summarized as follows:

*Piecewise linear interest-rate rule*

*Step 1*: Given the measured value of $\mathbf{x}_t$ at time $t$, select from the finite list of possible instances of $\{\mathbf{G}_A, \mathbf{w}_A, \mathbf{D}_A\}$ the one that satisfies eqns. (26) and (27).

*Step 2*: Depending on $\{\mathbf{G}_A, \mathbf{w}_A, \mathbf{D}_A\}$ determine the optimal interest rate either as

$$\begin{aligned} i_t &= -[1\,\underbrace{0...0}_{m-1}]\mathbf{H}^{-1}\left(\mathbf{G}_A^T\left(\mathbf{G}_A\mathbf{H}^{-1}\mathbf{G}_A^T\right)^{-1}\left(\mathbf{w_A} + \mathbf{D_A}\mathbf{x}_t\right) - \mathbf{F}^T\mathbf{x}_t\right) + r^* + \pi^* \\ &= \phi_y(y_t - y^*) + \phi_\pi(\pi - \pi^*) + r^* + \pi^* \end{aligned} \quad (28)$$

in the familiar Taylor-rule form (with $\phi_y$, $\phi_\pi$ functions of $\{\mathbf{G}_A, \mathbf{w}_A, \mathbf{D}_A\}$) or as

$$i_t = 0 \quad (29)$$

namely at the ZLB value.

*To our knowledge, the above development is the first rigorous derivation of an explicit Taylor-like rule that satisfies the ZLB without resorting either to ad hoc truncation of the interest rate value produced by a Taylor rule* (Reifschneider and Williams, 2000; Williams, 2006; Nakov, 2008) *or to numerical simulation* (Orphanides and Wieland, 2000; Hunt and Laxton, 2003; Jung et al., 2005; Kato and Nishiyama, 2005; Adam and Billi, 2007).

## 3.2 Variant of the General Idea: Taylor Rules with Inertia in the Presence of ZLB

Following the same approach as in the previous Section 3.1 but with $R = 0$, $S > 0$, the MPC optimization problem can be again cast in a similar form as

$$\min_{\tilde{\mathbf{z}}} \frac{1}{2}\tilde{\mathbf{z}}^T \tilde{\mathbf{H}}\tilde{\mathbf{z}}, \tag{30}$$

subject to

$$\mathbf{G}\tilde{\mathbf{z}} \leq \mathbf{w} + \tilde{\mathbf{D}}\tilde{\mathbf{x}}_t, \tag{31}$$

where

- the decision variable $\tilde{\mathbf{z}}$ is defined in terms of interest rate as

  $$\tilde{\mathbf{z}} \triangleq \mathbf{u}_m + \tilde{\mathbf{H}}^{-1}\tilde{\mathbf{F}}^T\tilde{\mathbf{x}}_t, \tag{32}$$

  with $\mathbf{u}_m$ as in Eqn. (24).
- $\tilde{\mathbf{x}}_t$ is defined as

  $$\tilde{\mathbf{x}}_t \triangleq \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta u_{t-1} \end{bmatrix} \triangleq \begin{bmatrix} y_t - y^* \\ \pi_t - \pi^* \\ u_{t-1} - u^* \end{bmatrix}. \tag{33}$$

- $\tilde{\mathbf{H}}$, $\mathbf{G}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{F}}$ are constant matrices and $\mathbf{w}$ a constant vector, depending on the parameters that appear in the model, objective function, and ZLB, as defined in Appendix B.

Again, an explicit solution through Taylor-like formulas can be obtained by applying the multi-parametric MPC solution to get direct counterparts of Eqns.

(25) through (27), which eventually lead to the counterparts of eqns. (28) and (29), namely the following rule:

*Piecewise linear interest-rate rule with inertia*

*Step 1*: Given the measured value of $\tilde{\mathbf{x}}_t$ at time $t$, select from the finite list of possible instances of $\{\tilde{\mathbf{G}}_A, \mathbf{w}_A, \tilde{\mathbf{D}}_A\}$ the one that satisfies eqns. (26) and (27).

*Step 2*: Depending on $\{\tilde{\mathbf{G}}_A, \mathbf{w}_A, \tilde{\mathbf{D}}_A\}$ determine the optimal interest rate either as

$$i_t = \phi_y(y_t - y^*) + \phi_\pi(\pi_t - \pi^*) + \phi_i(i_{t-1} - i^*) + r^* + \pi^* . \tag{34}$$

in the familiar form of a Taylor rule with inertia (with $\phi_y$, $\phi_\pi$, $\phi_i$ functions of $\{\tilde{\mathbf{G}}_A, \mathbf{w}_A, \tilde{\mathbf{D}}_A\}$), or

$$i_t = 0 \tag{35}$$

namely at the ZLB value.

## 4 Numerical Simulations

In this section both simulated and actual data are used to illustrate the Taylor-like (piecewise linear) interest rate rules resulting from application of the multi-parametric MPC approach outlined in the previous Section 3, summarized by eqns. (21) and (22). The economic model discussed in Section 2 underlies the multi-parametric MPC problem formulation. The values $N = 80$, $m = 4$, $\beta = 0.99$ are selected throughout this section for the MPC objective, Eqn. (14), as elaborated on in Appendix C. For these values of $N$, $m$, and $\beta$, the resulting Taylor-like rules depend on the values of the weights $\lambda$, $R$, and $S$ in the MPC objective, Eqn. (14). Taylor-like (piecewise linear) rules, resulting from $S = 0$ and a number of combinations of $\lambda$ and $R$, are discussed in Section 4.1. Similar rules with inertia, resulting from $R = 0$ and a number of combinations of $\lambda$ and $S$, are discussed in Section 4.2.

### 4.1 Taylor Rules form MPC with ZLB

For $S=0$, a value for each of $\lambda$ and $R$ results in a Taylor-like (piecewise linear) rule, the specific formula depending on the linear polytope in which the inflation and output gap lie, as presented in Table 2 through Table 7. The corresponding values of $\{\lambda, R\}$ in these tables, selected to illustrate rule patterns, are $\{0.05, 0.07\}$, $\{0.05, 0.55\}$, $\{0.8, 0.07\}$, $\{0.8, 0.55\}$, $\{0.5, 0.07\}$, and $\{0.5, 0.55\}$. For each choice of $\{\lambda, R\}$ in these tables, a collection of linear polytopes is defined through a set of linear inequalities for each polytope. The polytopes corresponding to each $\{\lambda, R\}$ are shown graphically in 2D plots of inflation vs. output gap in Figure 3 through Figure 8. Each polytope is numbered (1, 2, 3,…) for direct reference and comparison.

Inspection of Table 2 through Table 7 and Figure 3 through Figure 8 shows that the following four classes of formulas appear for the optimal interest rate:

a. Polytope 1: Formulas similar in nature to the standard Taylor rule, Eqn. (1). If the economy state, $\mathbf{x}_t$ (output gap and inflation) is in this polytope, the optimal policy is essentially the Taylor rule, and the interest rate does not reach the ZLB.

b. Polytope 2: Formulas that set the interest rate at its ZLB while maintaining closed-loop stability. If the economy state falls in this polytope, the optimal interest rate is zero, and the economy state will eventually return to the desired value. No action in addition to monetary policy is needed.

c. Polytope 3: Formulas that set the interest rate at its ZLB but without maintaining closed-loop stability. If the economy state falls in this polytope, the optimal interest rate is zero, but the economy state will *not* eventually return to the desired value, unless action in addition to monetary policy is taken. This is a case of liquidity trap (Reifschneider and Williams, 2000).

d. Polytope 4 or higher: Formulas that are piecewise linear but different from the corresponding Taylor-like formulas that would result from optimization without anticipation of ZLB activation in the future. This is a case of pre-emptive behavior (Kato and Nishiyama, 2005; Taylor and Williams, 2010, and references therein).

*Table 2*: Multi-Parametric MPC Solution and State-Space Partition for $\lambda = 0.05$ , $R = 0.07$

| No. | Polytope bounds | Interest rate $\Delta i_t$ | Closed-loop Eigenvalues |
|---|---|---|---|
| 1 | $\begin{bmatrix} -0.78 & -0.62 \\ -0.14 & -0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq \begin{bmatrix} 0.98 \\ 1.71 \end{bmatrix}$ | $\begin{bmatrix} 3.12 & 2.49 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$ | 0.07<br>0.96 |
| 2 | $\begin{bmatrix} 0.78 & 0.62 \\ -0.27 & -0.96 \\ 0.76 & 0.65 \\ 0.62 & 0.79 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq \begin{bmatrix} -0.98 \\ 3.70 \\ -1.03 \\ -1.39 \end{bmatrix}$ | $-3.9$ | 0.58<br>1.05 |
| 3 | $\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq -3.70$ | $-3.9$ (Infeasible) | 0.58<br>1.05 |
| 4 | $\begin{bmatrix} -0.76 & -0.65 \\ -0.20 & -0.98 \\ 0.14 & 0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq \begin{bmatrix} 1.03 \\ 2.02 \\ -1.71 \end{bmatrix}$ | $\begin{bmatrix} 3.15 & 2.70 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 0.36$ | 0.07<br>0.96 |
| 5 | $\begin{bmatrix} -0.62 & -0.79 \\ -0.13 & -0.99 \\ 0.20 & 0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq \begin{bmatrix} 1.39 \\ 4.42 \\ -2.02 \end{bmatrix}$ | $\begin{bmatrix} 3.52 & 4.49 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 4.05$ | 0.05<br>0.92 |
| 6 | $\begin{bmatrix} -0.27 & -0.96 \\ 0.13 & 0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq \begin{bmatrix} 3.70 \\ -4.42 \end{bmatrix}$ | $\begin{bmatrix} 5.55 & 19.6 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 71.3$ | 0.00<br>0.58 |

*Table 3*: Multi-Parametric MPC Solution and State-Space Partition for $\lambda = 0.05$, $R = 0.55$

| No. | Polytope bounds | Interest rate $\Delta i_t$ | Closed-loop Eigenvalues |
|---|---|---|---|
| 1 | $\begin{bmatrix} -0.39 & -0.92 \\ -0.28 & -0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 1.47 \\ 1.67 \end{bmatrix}$ | $\begin{bmatrix} 1.03 & 2.44 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$ | 0.50<br>0.93 |
| 2 | $\begin{bmatrix} 0.39 & 0.92 \\ -0.27 & -0.96 \\ 0.37 & 0.93 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} -1.47 \\ 3.70 \\ -1.52 \end{bmatrix}$ | $-3.9$ | 0.58<br>1.05 |
| 3 | $\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le -3.70$ | $-3.9$ (Infeasible) | 0.58<br>1.05 |
| 4 | $\begin{bmatrix} -0.38 & -0.93 \\ -0.32 & -0.95 \\ 0.28 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 1.50 \\ 1.63 \\ -1.67 \end{bmatrix}$ | $\begin{bmatrix} 1.13 & 2.77 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 0.59$ | 0.50<br>0.92 |
| 5 | $\begin{bmatrix} -0.37 & -0.93 \\ 0.32 & 0.95 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 1.52 \\ -1.63 \end{bmatrix}$ | $\begin{bmatrix} 1.34 & 3.39 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 1.65$ | 0.48<br>0.89 |

*Table 4*: Multi-Parametric MPC Solution and State-Space Partition for $\lambda = 0.8$ , $R = 0.07$

| No. | Polytope bounds | Interest rate $\Delta i_t$ | Closed-loop Eigenvalues |
|---|---|---|---|
| 1 | $\begin{bmatrix} -0.35 & -0.94 \\ -0.17 & -0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.40 \\ 0.74 \end{bmatrix}$ | $\begin{bmatrix} 3.39 & 9.09 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$ | 0.22<br>0.76 |
| 2 | $\begin{bmatrix} 0.35 & 0.94 \\ -0.27 & -0.96 \\ 0.28 & 0.96 \\ 0.33 & 0.95 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} -0.98 \\ 3.70 \\ -0.57 \\ -0.45 \end{bmatrix}$ | $-3.9$ | 0.58<br>1.05 |
| 3 | $\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le -3.70$ | $-3.9$ (Infeasible) | 0.58<br>1.05 |
| 4 | $\begin{bmatrix} -0.33 & -0.95 \\ -0.21 & -0.98 \\ 0.17 & 0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.45 \\ 0.76 \\ -0.74 \end{bmatrix}$ | $\begin{bmatrix} 3.65 & 10.6 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 1.13$ | 0.21<br>0.72 |
| 5 | $\begin{bmatrix} -0.28 & -0.96 \\ 0.21 & 0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.57 \\ -0.76 \end{bmatrix}$ | $\begin{bmatrix} 5.17 & 17.5 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 6.48$ | 0.04<br>0.61 |

*Table 5*: Multi-Parametric MPC Solution and State-Space Partition for $\lambda = 0.8$ , $R = 0.55$

| No. | Polytope bounds | Interest rate $\Delta i_t$ | Closed-loop Eigenvalues |
|---|---|---|---|
| 1 | $\begin{bmatrix} -0.28 & -0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le 0.86$ | $\begin{bmatrix} 1.29 & 4.36 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$ | 0.56<br>0.83 |
| 2 | $\begin{bmatrix} 0.28 & 0.96 \\ -0.27 & -0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} -0.86 \\ 3.70 \end{bmatrix}$ | $-3.9$ | 0.58<br>1.05 |
| 3 | $\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le -3.70$ | $-3.9$ (Infeasible) | 0.58<br>1.05 |

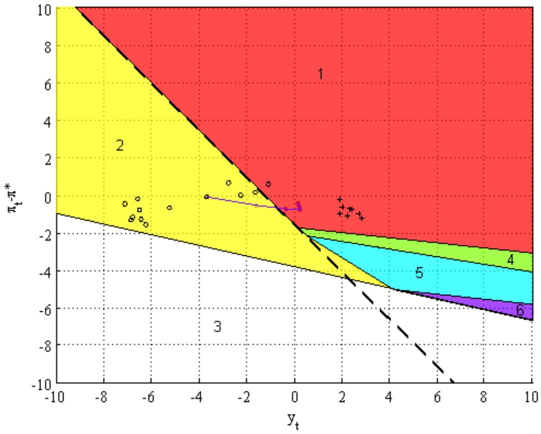
*Table 6*: Multi-Parametric MPC Solution and State-Space Partition for $\lambda = 0.5$, $R = 0.07$

| No. | Polytope bounds | Interest rate $\Delta i_t$ | Closed-loop Eigenvalues |
|---|---|---|---|
| 1 | $\begin{bmatrix} -0.44 & -0.90 \\ 0.14 & -0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.49 \\ 0.72 \end{bmatrix}$ | $\begin{bmatrix} 3.51 & 7.11 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$ | 0.12<br>0.84 |
| 2 | $\begin{bmatrix} 0.44 & 0.90 \\ -0.27 & -0.96 \\ 0.41 & 0.91 \\ 0.32 & 0.95 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} -0.49 \\ 3.70 \\ -0.52 \\ -0.66 \end{bmatrix}$ | $-3.9$ | 0.58<br>1.05 |
| 3 | $\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le -3.70$ | $-3.9$ (Infeasible) | 0.58<br>1.05 |
| 4 | $\begin{bmatrix} -0.44 & -0.91 \\ -0.18 & -0.98 \\ 0.14 & 0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.52 \\ 0.86 \\ -0.72 \end{bmatrix}$ | $\begin{bmatrix} 3.67 & 8.26 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 0.84$ | 0.12<br>0.81 |
| 5 | $\begin{bmatrix} -0.32 & -0.95 \\ 0.18 & -0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.66 \\ -0.86 \end{bmatrix}$ | $\begin{bmatrix} 4.72 & 13.95 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 5.82$ | 0.04<br>0.69 |

*Table 7*: Multi-Parametric MPC Solution and State-Space Partition for $\lambda = 0.5$, $R = 0.55$

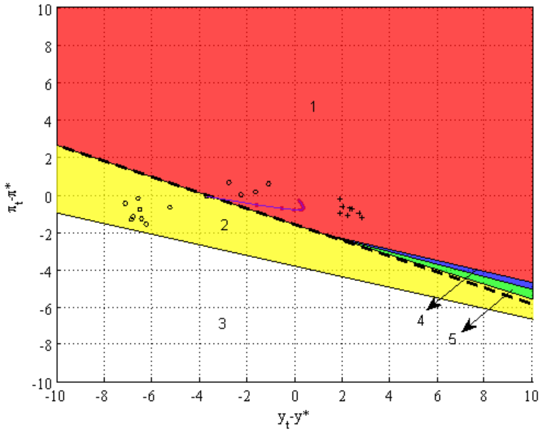
| No. | Polytope bounds | Interest rate $\Delta i_t$ | Closed-loop Eigenvalues |
|---|---|---|---|
| 1 | $\begin{bmatrix} -0.31 & -0.95 \\ -0.29 & -0.96 \\ -0.27 & -0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq \begin{bmatrix} 1.0 \\ 1.09 \\ 1.14 \end{bmatrix}$ | $\begin{bmatrix} 1.21 & 3.71 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$ | 0.53<br>0.87 |
| 2 | $\begin{bmatrix} 0.31 & 0.95 \\ -0.27 & -0.96 \\ 0.30 & 0.95 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq \begin{bmatrix} -1.00 \\ 3.70 \\ -1.04 \end{bmatrix}$ | $-3.9$ | 0.58<br>1.05 |
| 3 | $\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq -3.70$ | $-3.9$ (Infeasible) | 0.58<br>1.05 |
| 4 | $\begin{bmatrix} -0.61 & -0.79 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq \begin{bmatrix} 2.03 \\ 4.51 \\ -2.99 \end{bmatrix}$ | $\begin{bmatrix} 1.4 & 4.38 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 0.8$ | 0.53<br>0.84 |
| 5 | $\begin{bmatrix} -0.61 & -0.79 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \leq \begin{bmatrix} 2.03 \\ 4.51 \\ -2.99 \end{bmatrix}$ | $\begin{bmatrix} 1.77 & 5.63 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 2.23$ | 0.51<br>0.79 |

*Figure 3*: State-Space Partition for $R = 0.07$ and $\lambda = 0.05$



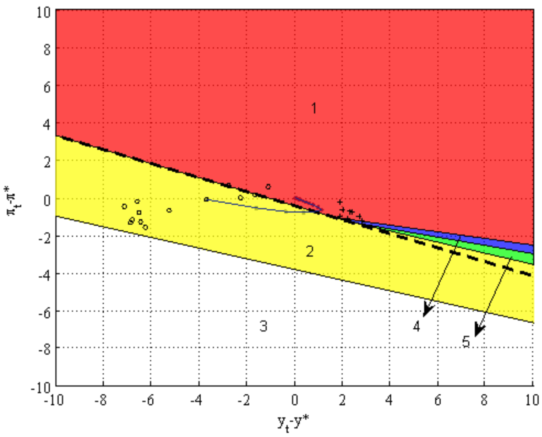

*Note*: Corresponding rules are in Table 2, o represents actual economy data points for period 08Q1–11Q1, + represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (–3.7, 1.9), dashed line represents truncated solution of unconstrained case.

*Figure 4*: State-Space Partition for $R = 0.55$ $\lambda = 0.05$



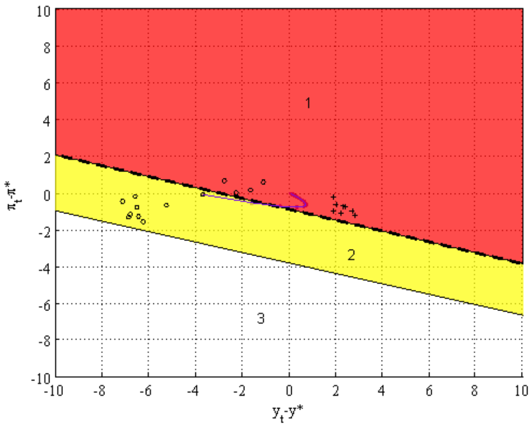

*Note*: Corresponding rules are in Table 3, o represents actual economy data points for period 08Q1–11Q1, + represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (–3.7, 1.9), dashed line represents truncated solution of unconstrained case.

*Figure 5*: State-Space Partition for $R = 0.07$, $\lambda = 0.8$



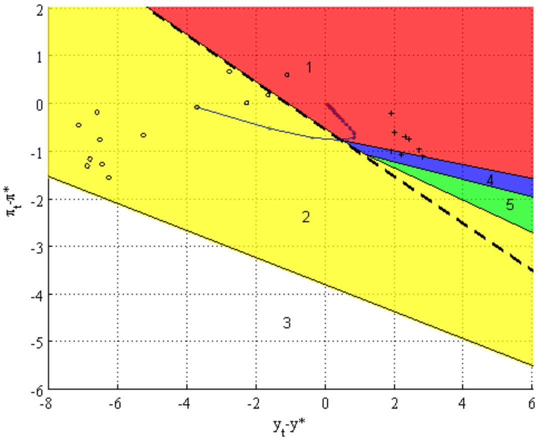

*Note*: Corresponding rules are in Table 4, o represents actual economy data points for period 08Q1–11Q1, + represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (–3.7, 1.9), dashed line represents truncated solution of unconstrained case.

*Figure 6*: State-Space Partition for $R = 0.55$, $\lambda = 0.8$



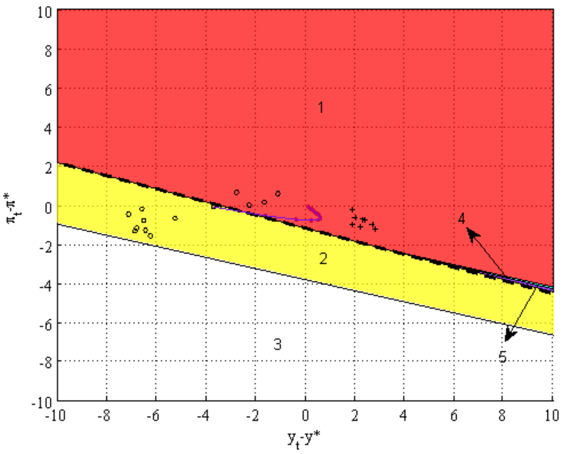

*Note*: Corresponding rules are in Table 5, o represents actual economy data points for period 08Q1–11Q1, + represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (–3.7, 1.9), dashed line represents truncated solution of unconstrained case.

*Figure 7*: State-Space Partition for $R = 0.07 \quad \lambda = 0.5$




*Note*: Corresponding rules are in Table 6, o represents actual economy data points for period 08Q1–11Q1, + represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (–3.7, 1.9), dashed line represents truncated solution of unconstrained case.

*Figure 8*: State-Space Partition for $R = 0.55$, $\lambda = 0.5$




*Note*: Corresponding rules are in Table 7, o represents actual economy data points for period 08Q1–11Q1, + represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (–3.7, 1.9), dashed line represents truncated solution of unconstrained case.

Table 3 and the associated Figure 4 suggest a formula in polytope 1 for $\lambda = 0.05$, $R = 0.55$ that is closest to the standard Taylor rule, in terms of the following two criteria:

- closeness of the values of $\{\phi_y, \phi_\pi\}$ of the formula in polytope 1 vs. the standard Taylor rule (namely $\{1.0, 2.4\}$ vs. $\{0.5, 1.5\}$, respectively); and
- closeness of the eigenvalues of the closed loop resulting from substitution of the interest rate in Eqn. (2) by either the formula in polytope 1 or by the Taylor rule (namely $\{0.50, 0.94\}$ vs. $\{0.56, 0.97\}$, respectively).

Further inspection of the results reveals that the optimal rules follow an *asymmetric* pattern for small values of $R$ (Figure 3, Figure 5, Figure 7), namely the optimal policies are different when the economy state is above or below the desired values. Similar asymmetry has also been observed in literature in a number of numerical studies with $R = 0$ (Orphanides and Wieland, 2000; Kato and Nishiyama, 2005; Williams, 2006; Taylor and Williams, 2010). However, this asymmetry practically disappears (i.e. it would be observable only for unrealistically large output gaps) for large values of $R$ (Figure 6, Figure 8), namely for very sluggish policies.

The asymmetry-related results show that for *negative output gap*, *truncation the standard Taylor rule is the optimal policy*. In addition, for *negative output gap*, when the interest rate is close to zero and future violations of the ZLB are anticipated, *no more aggressive adjustment of interest rate is needed than indicated by the standard Taylor rule.*

On the other hand, for *positive output gap* and *low inflation*, more interesting behavior is observed, namely a small number of piecewise linear formulas result, corresponding to the linear polytopes numbered 4 and above. These formulas become more aggressive as the interest rate approaches the ZLB. This behavior (pre-emptiveness) has also been observed in numerical studies (Kato and Nishiyama, 2005; Taylor and Williams, 2010, and references therein). However, in contrast to these numerical simulation studies, explicit rules are derived here, and the corresponding formulas are (piecewise) linear rather than nonlinear.

## 4.2 Taylor Rules with Inertia form MPC with ZLB

For $S = 0.55$ and $\lambda = 0.5$, the resulting piecewise linear policies and corresponding polytopes are shown in Table 8. The parameter space of multi-parametric MPC, which is now three-dimensional (corresponding to $y_t$, $\pi_t$, and $u_{t-1}$), is partitioned in 6 polytopes shown in Figure 9.

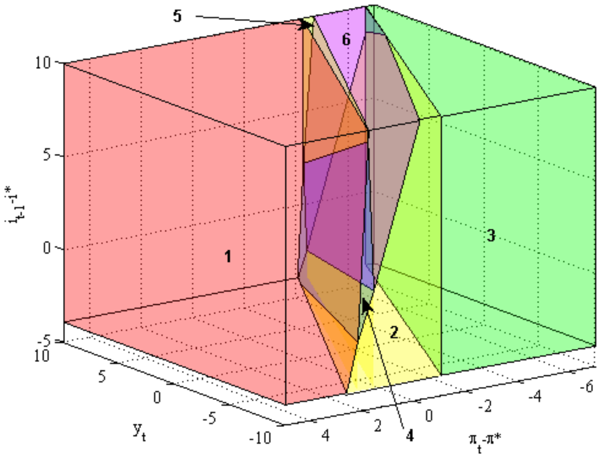
- Polytope 1 corresponds to no constraint being active and hence it produces a rule as in Eqn. (44).
- In polytope 2 the ZLB is active, i.e. the optimal policy is at zero.
- Polytopes 4, 5 and 6 entail rules that are different from the Taylor-like rule of polytope 1, in anticipation of future ZLB activation.
- The infeasibility polytope remains the same.

From Table 8 and Figure 9 it can be concluded that in polytopes of low inflation and negative output gap, if the lagged interest rate $2+2\rho-2\xi\phi_y+\alpha\xi(\phi_\pi-1)>0$ is high (polytopes 4 and 6), the optimal rule becomes less aggressive than the rule in the unconstrained case. However, for low $2+2\rho-2\xi\phi_y+\alpha\xi(\phi_\pi-1)>0$, the optimal rule is just a truncation to zero of the unconstrained case, Eqn. (44). Also, in polytope 5, characterized by low inflation, high output gap, and high $2+2\rho-2\xi\phi_y+\alpha\xi(\phi_\pi-1)>0$, the optimal rule is more aggressive than the rule in the unconstrained case, Eqn. (44). Therefore, an important conclusion is that *for rules with inertia ($S>0$), the optimal policy becomes asymmetrical with respect to both lagged interest rate and output gap for low inflation economic conditions*.

Table 8: Multi-Parametric MPC Solution and State-Space Partition for $S = 0.55$, $\lambda = 0.5$

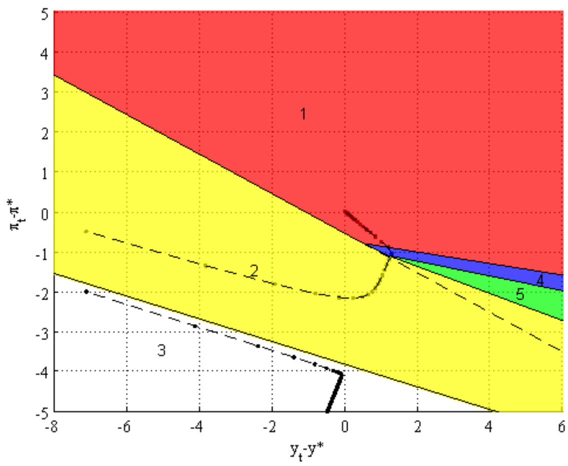
| No. | Polytope bounds | Interest rate $\Delta i_t$ | Closed-loop Eigenvalues |
|---|---|---|---|
| 1 | $\begin{bmatrix} -0.31 & -0.94 & -0.16 \\ -0.33 & -0.95 & -0.04 \\ -0.29 & -0.96 & 0.01 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} 1.27 \\ 0.97 \\ 1.21 \end{bmatrix}$ | $\begin{bmatrix} 0.96 & 2.88 & 0.48 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix}$ | 0.74<br>0.59 + 0.20i<br>0.59 − 0.20i |
| 2 | $\begin{bmatrix} 0.31 & 0.94 & 0.16 \\ -0.27 & -0.96 & 0 \\ 0.31 & 0.93 & 0.19 \\ 0.32 & 0.92 & 0.22 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} -1.27 \\ 3.70 \\ -1.42 \\ -1.43 \end{bmatrix}$ | $-3.9$ | 0.58<br>1.05<br>0 |
| 3 | $\begin{bmatrix} 0.27 & 0.96 & 0 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le -3.7$ | $-3.9$ (Infeasible) | 0.58<br>1.05<br>0 |
| 4 | $\begin{bmatrix} -0.32 & -0.92 & -0.22 \\ -0.28 & -0.96 & 0.03 \\ 0.30 & 0.95 & 0.04 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} 1.43 \\ 1.33 \\ -0.97 \end{bmatrix}$ | $\begin{bmatrix} 0.63 & 1.88 & 0.44 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} - 1.06$ | 0.87<br>0.54 + 0.13i<br>0.54 − 0.13i |
| 5 | $\begin{bmatrix} -0.31 & -0.95 & -0.05 \\ 0.29 & 0.96 & -0.01 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} 0.89 \\ -1.21 \end{bmatrix}$ | $\begin{bmatrix} 1 & 3 & 0.48 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} + 0.16$ | 0.73<br>0.59 + 0.21i<br>0.59 − 0.21i |
| 6 | $\begin{bmatrix} -0.31 & -0.93 & -0.19 \\ -0.27 & -0.96 & 0.02 \\ 0.31 & 0.95 & 0.05 \\ 0.28 & 0.96 & -0.03 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} 1.42 \\ 3.97 \\ -0.89 \\ -1.33 \end{bmatrix}$ | $\begin{bmatrix} 0.71 & 2.1 & 0.43 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} - 0.69$ | 0.85<br>0.54 + 0.13i<br>0.54 − 0.13i |

*Figure 9*: State-Space Partition for $S = 0.55$ and $\lambda = 0.5$




*Note*: Corresponding rules are in Table 8.

*Figure 10*: Closed-Loop Simulation for $(y,\pi)$ = (–7.1, 1.5) 2009Q3 and $(y,\pi)$ = (–7.0, 0) Virtual Point for $R = 0.07$ $\lambda = 0.5$




*Note*: The later state lies in infeasibility polytope and no positive interest rate can stabilize the closed loop.

## 4.3 Remarks on Rules from MPC

The following can be observed in the results of Sections 4.1 and 4.2.

- Polytope 1, where no constraint is active, grows in size with increasing $R$ or $S$. This is intuitively understandable, as increasing $R$ or $S$ makes results in less aggressive policies.
- The policy becomes sluggish and the size of polytopes 2, 4 and higher decreases as $R$ or $S$ increase.
- The state of the economy may reach a point where $\mathbf{x}_t$ or $\tilde{\mathbf{x}}_t$ are such that no feasible interest rate can be found (i.e. the optimal rate would have to be negative). It can be shown (Appendix H) that for an economy model such as described by eqns. (6)-(9) the infeasibility polytope for $\mathbf{x}_t$ is characterized by the inequality

$$\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{x}_t > \frac{\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{B}}{J_u - 1} i^* \tag{36}$$

  It is clear that the state $\mathbf{x}_t$ may satisfy Eqn. (36) fairly easily for economies with low $i^*$, i.e. such economies at corresponding conditions run the risk of falling into the infeasibility polytope where a stabilizing interest rate above the ZLB may not exist. This situation has also been studied in literature numerically (e.g., Williams, 2009).
- For $\mathbf{x}_t$ in a polytope such that a feasible MPC solution exists but not all of the corresponding closed-loop eignevalues are inside the unit disk, the state will definitely escape from that polytope and will enter one where stability is guaranteed. By contrast, for $\mathbf{x}_t$ in a polytope such that no feasible MPC solution exists and not all of the corresponding closed-loop eigenvalues are inside the unit disk, instability will persist. This is illustrated further in Figure 10, discussed below.
- It should be noted that entering into the polytope 2, where the ZLB is active, is an alarming situation, as the infeasibility polytope 3 seats next to this polytope. The longer the economy stays at ZLB, the higher the chance of getting into the infeasibility polytope (a case of liquidity trap) as a result of sudden adverse fluctuations inthe

economy. Similar observations have been made in literature based on numerical simulation (Reifschneider and Williams, 2000).

- In Figure 3 through Figure 8 real-time economy data are plotted for 2008Q1:2011Q1. It is clear from Figure 3, Figure 5, and Figure 7 $(R = 0.07)$ that truncation to zero is the optimal interest rate for nearly all economy points, while in Figure 4, Figure 6, and Figure 8 $(R = 0.55)$ more of the economy data indicate non-zero interest rate due to the policy rule being sluggish.

## 4.4 Illustration of Proposed Approach

The first set of simulations serves to simply illustrate the effects of ZLB on the closed-loop system. Simulations are shown using the rules presented in Table 2 through Table 7, as well as the rules with inertia shown in Table 8 along with five additional rules with similar structure but different MPC weights $R$ and $S$ (not shown in Table 8 for brevity). For this set of simulations the economy is considered to be at $y = -3.7$ and $\pi = 1.9$ in year 1, corresponding to 2009Q1. The results are summarized in Figure 11 and Figure 12. The resulting sums of squared errors (discrepancies between actual and desired values) are summarized in Table 9 and Table 10.

Based on these simulation results, it is clear that for small values of $R$ or $S$, optimal interest rate rules are aggressive and more likely to produce interest rate values at the ZLB when corresponding conditions arise. Conversely, increase in the values of $R$ or $S$ results in sluggish response, as expected.

The second set of simulations illustrates a case of liquidity trap. Figure 10 shows state-space partition for $R = 0.07$ and $\lambda = 0.5$. Two different initial conditions of the economy are considered. For the first case we let the initial point be $y_1 = -7.1$, $\pi_1 = 1.5$ (2009Q3), which lies in polytope 2 in Figure 10 and hence the corresponding optimal interest rate is zero. For the second case we let $y_1 = -7.1$, $\pi_1 = 0$, which lies inside the infeasibility polytope 3, namely no non-negative interest rate can stabilize the economy at that point. A zero interest rate alone results in an unstable closed loop. The only way to stabilize the closed loop would be through additional external stimulus.

*Figure 11*: Closed-Loop Simulation for US Economy (Start Point is 2009Q1) for $\lambda = 0.05$

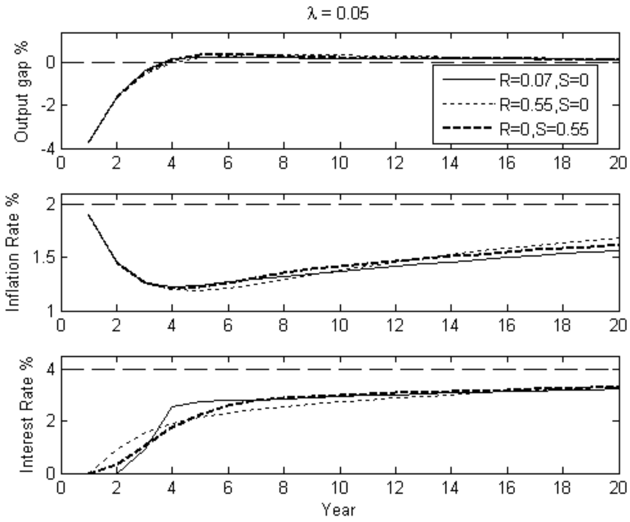



*Figure 12*: Closed-Loop Simulation for US Economy (Start Point is 2009Q1) for $\lambda = 0.8$

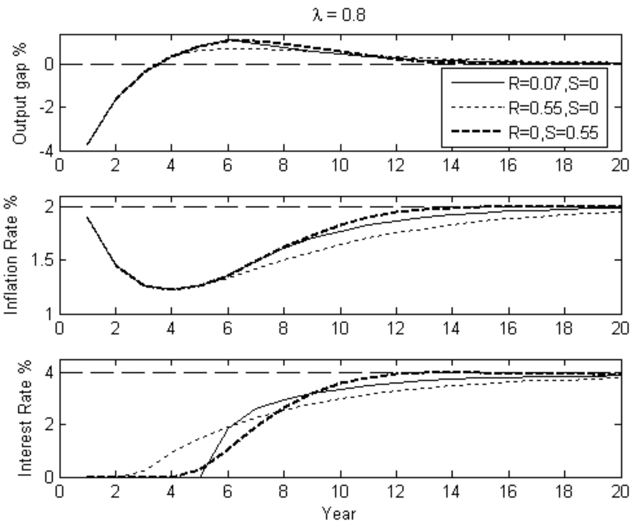


*Table 9*: Sum of Squared Errors for Closed-Loop Simulations with $\lambda = 0.05$

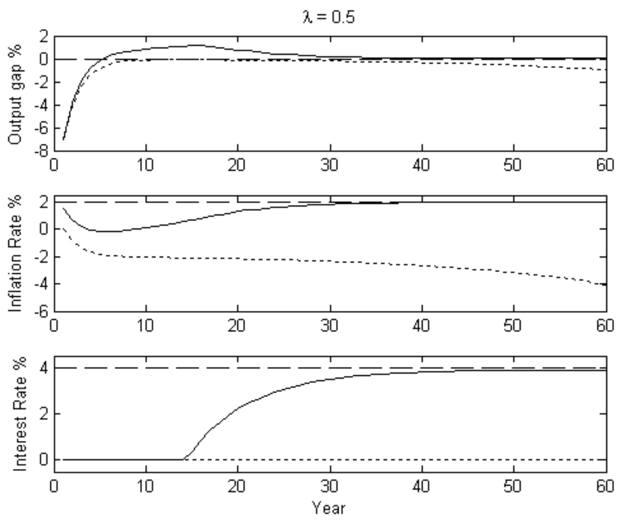
| | $S=0, R=0.07$ | $S=0, R=0.55$ | $S=0.07, R=0$ | $S=0.55, R=0$ |
|---|---|---|---|---|
| $\sum_{t=2}^{20} y_t^2$ | 3.30 | 4.12 | 3.29 | 3.65 |
| $\sum_{t=2}^{20} (\pi_t - 2)^2$ | 7.21 | 6.69 | 7.22 | 6.52 |
| $\sum_{t=1}^{19} (i_t - 3.9)^2$ | 54.5 | 53.2 | 54.4 | 54.3 |
| $\sum_{t=1}^{19} (i_t - i_{t-1})^2$ | 3.63 | 1.52 | 3.32 | 1.54 |

Table 10: Sum of Squared Errors for Closed Loop Simulations with $\lambda = 0.8$

| | $S=0, R=0.07$ | $S=0, R=0.55$ | $S=0.07, R=0$ | $S=0.55, R=0$ |
|---|---|---|---|---|
| $\sum_{t=2}^{20} y_t^2$ | 7.14 | 5.75 | 7.43 | 7.99 |
| $\sum_{t=2}^{20} (\pi_t - 2)^2$ | 3.02 | 3.72 | 2.97 | 2.93 |
| $\sum_{t=1}^{19} (i_t - 3.9)^2$ | 84.0 | 71.2 | 85.8 | 88.4 |
| $\sum_{t=1}^{19} (i_t - i_{t-1})^2$ | 4.27 | 1.43 | 3.59 | 2.47 |

Given the fact that it is practically difficult to exactly quantify the polytope of liquidity trap, the central bank should focus on external stimulus as soon as the ZLB is reached. Closed-loop simulations, the results of which are shown in Figure 13, confirm the preceding assertions for both cases. It is also interesting to note that even though the interest rate in the first case is stabilizing, recovery of the economy is very slow due to the effect of ZLB (inflation stabilization, in particular, takes many years).

*Figure 13*: Closed-Loop Simulation for Figure 10




## 5 Discussion: Parametric Studies and Comparison with Standard Rules and Real Data

In this section we provide additional insight into the proposed MPC approach. In particular, we compare rules generated by MPC to existing Taylor rules, show how rules with inertia arise naturally from MPC, elaborate on closed-loop stability issues, and provide comparison to real data.

### 5.1 Effects of MPC Objective Function Weights on Resulting Taylor Rules

For the choice of $N = 80$, $m = 4$, and $\beta = 0.99$, discussed in the preceding sections, we now proceed to examine the effect of $R$ and $\lambda$ on the resulting Taylor rules, via Eqn. (79). Following the calculations in Appendix D, the matrices $\mathbf{H}$ and $\mathbf{F}$ in Eqn. (77) are calculated as functions of $R$ and $\lambda$, and coefficients of the output gap and inflation in the Taylor rule or Eqn. (1) are expressed analytically in terms of $R$ and $\lambda$, as

$$\phi_y = \frac{q_{y,3}R^6 + q_{y,2}(\lambda)R^4 + q_{y,1}(\lambda)R^2 + q_{y,0}(\lambda)}{p_3R^6 + p_2(\lambda)R^4 + p_1(\lambda)R^2 + p_0(\lambda)}, \qquad (37)$$

$$\phi_\pi = \frac{q_{\pi,3}R^6 + q_{\pi,2}(\lambda)R^4 + q_{\pi,1}(\lambda)R^2 + q_{\pi,0}(\lambda)}{p_3R^6 + p_2(\lambda)R^4 + p_1(\lambda)R^2 + p_0(\lambda)}, \qquad (38)$$

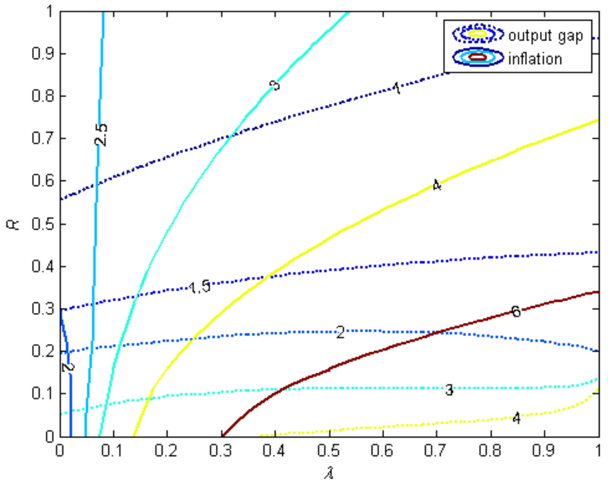
respectively, where the values of the corresponding parameters are shown in Table 11. In general, the numerator and denominator for $\phi_y$ and $\phi_\pi$ are polynomial functions of degree $m-1$ in both $R^2$ and $\lambda$.

Figure 14 employs the preceding eqns. (37) and (38) to calculate the policy coefficients $\phi_y$, $\phi_\pi$ for a range of values of $R$ and $\lambda$. The point corresponding to the original Taylor rule ($\phi_y = 0.5$, $\phi_\pi = 1.5$) is not present in Figure 14.

*Table 11*: Polynomial Coefficients in Equations (37) and (38) as Functions of $\lambda$

| |
|---|
| $q_{y,3} = 1.04$ |
| $q_{y,2} = 0.297 + 0.444\lambda$ |
| $q_{y,1} = -0.04(-1.04+\lambda)(0.420+\lambda)$ |
| $q_{y,0} = 6.11\times10^{-4}(-1.06+\lambda)(-1.01+\lambda)(0.365+\lambda)$ |
| $q_{\pi,3} = 3.67$ |
| $q_{\pi,3} = 0.37 + 2.28\lambda$ |
| $q_{\pi,1} = -0.124(-1.09+\lambda)(0.084+\lambda)$ |
| $q_{\pi,0} = 1.59\times10^{-3}(-1.12+\lambda)(-1.01+\lambda)(0.059+\lambda)$ |
| $p_3 = 1.48$ |
| $p_2 = 0.197 + 0.157\lambda$ |
| $p_1 = -0.0108(-1.02+\lambda)(0.641+\lambda)$ |
| $p_0 = 1.32\times10^{-4}(-1.03+\lambda)(-1.01+\lambda)(0.512+\lambda)$ |

*Figure 14*: Taylor-Like Interest Rate Rule in the Absence of ZLB Constraint on Interest Rate for Various Values of Tuning Parameters $R$ and $\lambda$




*Note*: Solid and dotted lines represent inflation and output gap coefficient respectively based on Eqn. (37)–(38). This solution is also valid when no constraint is active in case of constrained MPC.

However, various values of $R$ and $\lambda$ result in $\phi_y$ in the range of 1 to 3 (Figure 15) and $\phi_\pi$ in the range of 2 to 6 (Figure 16).

The following general observations can be made on Figure 15 and Figure 16:

- When $R$ is small (i.e. control is aggressive) it has a strong effect on $\phi_y$ and $\phi_\pi$.
- The value $R = 0$ results in large values of $\phi_y$ and $\phi_\pi$, i.e. aggressive policy.
- When $R$ is small, the inflation coefficient $\phi_\pi$ is more sensitive to the choice of $\lambda$ than $\phi_y$ is.
- After approximately $R > 1$, further increase in $R$ has very small effect on $\phi_y$ and $\phi_\pi$.

For the economy model under consideration, the nearest point to the original Taylor rule is found at $\phi_y = 1$, $\phi_\pi = 2.4$ for $R = 0.55$ and $\lambda = 0.05$. These values are close to the original Taylor rule and other Taylor-like rules (Rotemberg and Woodford, 1997; Orphanides and Wieland, 2000).

*Figure 15*: Output Gap coefficient $\phi_y$ for Taylor Rule when $\beta = 0.99$

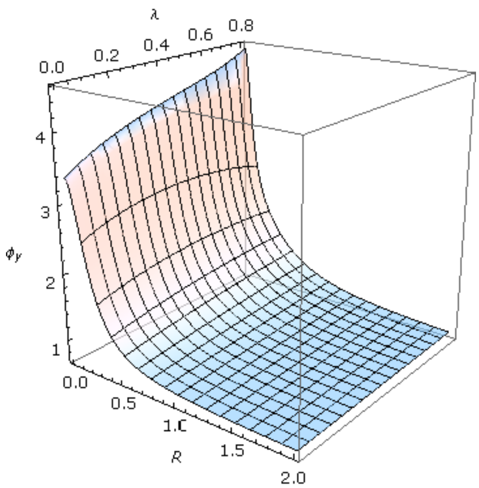
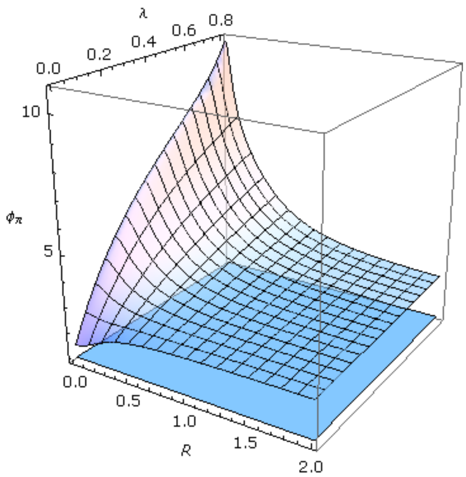



*Figure 16*: Inflation Coefficient $\phi_\pi$ for Taylor Rule when $\beta = 0.99$



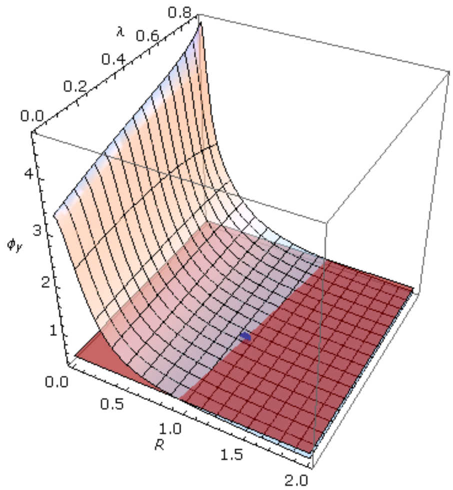

$\phi_\pi = 1$

## 5.2 Original Taylor Rule in MPC Framework

Even though the specific $\phi_y$ and $\phi_\pi$ values of the original Taylor rule were not recovered in the preceding section for the value of $\beta$ used mostly in literature, such values can be obtained if a different value of $\beta$ is considered. It turns out that the original Taylor rule can be recovered for $\beta \leq 0.96$, for which expressions for $\phi_y$ and $\phi_\pi$ similar to eqns. (37) and (38) can be derived in the same way. As shown in Figure 17 and Figure 18, the original Taylor rule values for $\phi_y$ and $\phi_\pi$ can be derived when $\beta = 0.96$ for $R = 1.06$ and $\lambda = 0.36$ in Eqn. (14).

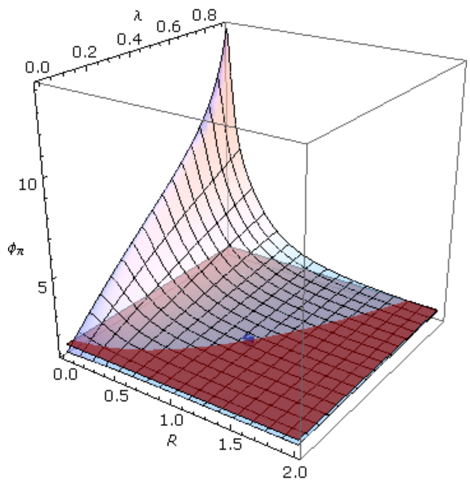
In general, determining values of MPC weights that would correspond to specific values of $\phi_y$ and $\phi_\pi$ is an instance of the inverse linear quadratic regulator problem. An infinite number of solutions generally exist for that problem. Feasibility and characterization of these solutions can be obtained in terms of linear matrix inequality algorithms (Boyd et al., 1994, section 10. 6, p. 147). This issue will be explored elsewhere.

*Figure 17*: Output Gap Coefficient $\phi_y$ for Taylor Rule when $\beta = 0.96$




*Note*: The location of Taylor coefficient $\phi_y = 0.5$ is shown by the circle.

*Figure 18*: Inflation rate coefficient $\phi_\pi$ for Taylor rule when $\beta = 0.96$




*Note*: The location of Taylor coefficient $\phi_\pi = 1.5$ is shown by the circle.

## 5.3 Taylor Rules and Resulting Closed-Loop Stability

For any rule proposed, it is important to determine, at the very least, whether such a rule results in a stable closed loop. Combination of the Taylor rule in Eqn. (1) with the simple economy model, Eqn. (6), yields (Appendix E) the closed loop structure

$$\mathbf{x}_{t+1} = \mathbf{A}_{\text{CL}}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}, \tag{39}$$

where

$$\mathbf{A}_{\text{CL}} \triangleq \mathbf{A} + \mathbf{B}\mathbf{c}^T = \begin{bmatrix} \rho - \xi\phi_y & \xi - \xi\phi_\pi \\ \alpha & 1 \end{bmatrix}. \tag{40}$$

It can be shown (Appendix E) that both eigenvalues of $\mathbf{A}_{\text{CL}}$ are inside the unit disk, i.e. the closed-loop system is stable, if and only if

$$\phi_\pi > 1, \tag{41}$$

$$-2.1 + 0.12\phi_\pi < \phi_y < 8.5 + 0.06\phi_\pi. \tag{42}$$

as illustrated in Figure 19. This is in agreement with the well-established Taylor principle that the central bank should raise its interest rate more than one-for-one with increase in inflation (Woodford, 2001; Davig and Leeper, 2007). Figure 16 shows that this requirement is satisfied for all combinations of the MPC weighting parameters $R$ and $\lambda$. In fact, Figure 20 illustrates that the stability conditions, eqns. (41) and (42), are satisfied for all choices of $R$ and $\lambda$ when $\beta = 0.99$. However, this is not the case for $\beta \leq 0.95$, as illustrated in Figure 21, which shows that as the value of $\beta$ is reduced, the value of $R$ should not be too small, to avoid closed-loop instability.

It is interesting to note that as $R \to \infty$, namely high values of interest rate are heavily penalized, the closed loop remains stable, due to the stabilizing equality constraint, Eqn. (56). For $R \to \infty$, eqns. (37) and (38) suggest that $\phi_y = \frac{q_{y,3}}{p_3} = 0.70$ and $\phi_\pi = \frac{q_{\pi,3}}{p_3} = 2.5$.

Following the preceding observations, it should be noted that the widespread practice of using a discount factor $\beta$ may be more problematic than realized, in the sense that it may not result in robustly stabilizing strategies. This situation, namely the need to shape weights of the terms in the MPC objective in an *increasing* rather than decreasing fashion in order to ensure robustness, has been rigorously analyzed in the past (Genceli and Nikolaou, 1993; Vuthandam et al., 1995) and should be explored further.

## 5.4 Taylor Rules with Inertia from MPC

Taylor-like rules with inertia, Eqn. (34), have been proposed based on empirical arguments and simulation studies, in efforts to reduce large interest rate fluctuations (Goodfriend, 1991; Taylor and Williams, 2010, and references therein). We explain below that such rules result naturally from appropriate tailoring of the MPC objective function to include terms that penalize the rate of change of interest rate.

To illustrate this, consider again the MPC optimization problem formulated in Eqn. (14) with $R = 0$ and $S > 0$, namely no penalty on the interest rate itself, but a penalty on its rate of change. It can be shown (Appendix F), that the resulting MPC optimization in this case becomes

*Figure 19*: Closed-Loop Stability Region for the US Economy Model in Terms of Taylor Rule Coefficients $\phi_y$ and $\phi_\pi$ when $\beta = 0.99$

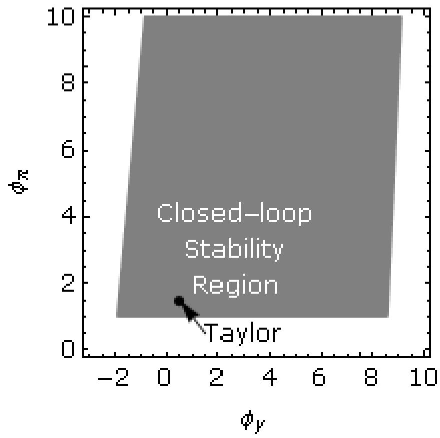



*Figure 20*: Closed-Loop Stability Region in Terms of MPC Tuning Parameters $R$ and $\lambda$ for $\beta = 0.99$

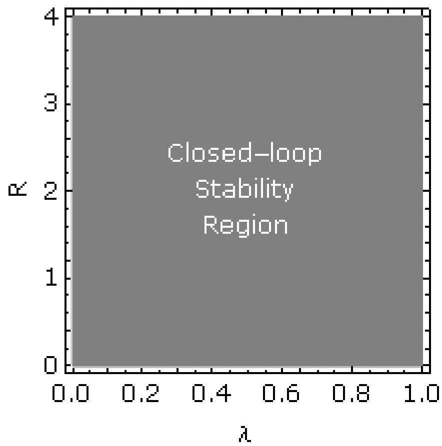
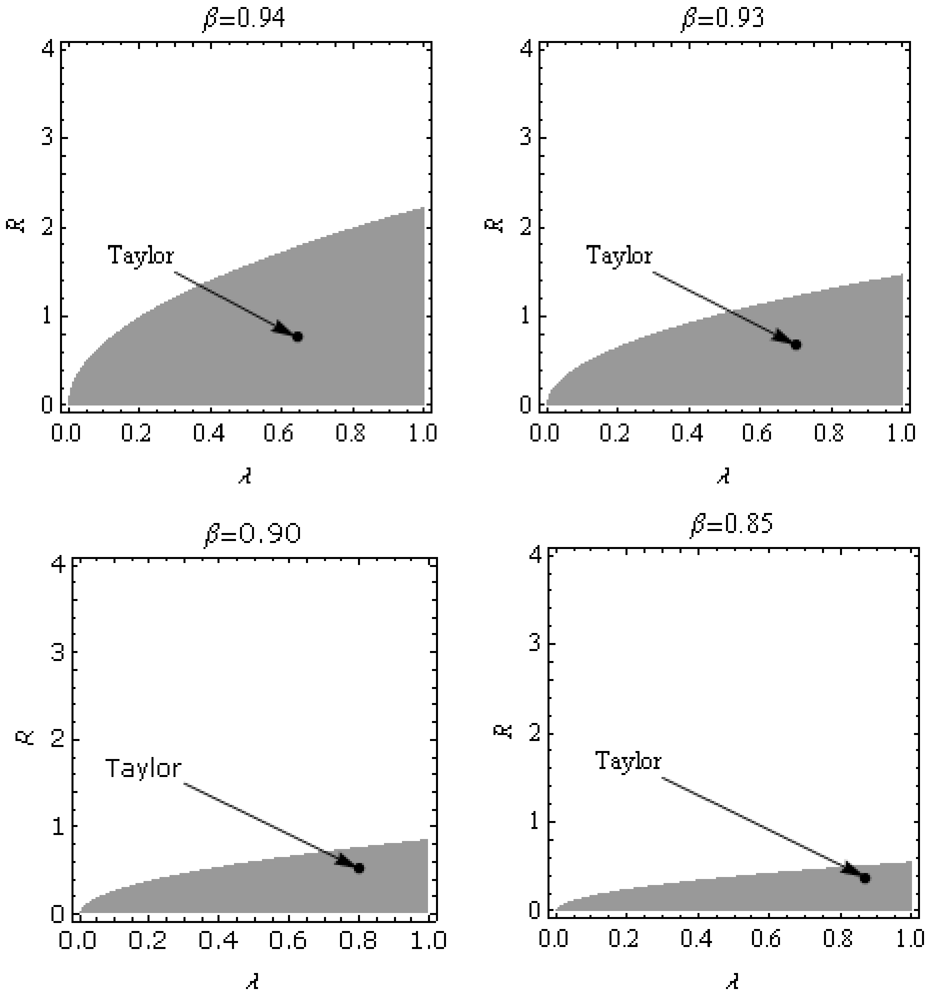


*Figure 21*: Closed-Loop Stability Region (Shaded) in Terms of MPC Weight Parameters $R$ and $\lambda$ for Various Values of $\beta < 0.95$




*Note*: The location of original Taylor rule is shown by circle.

$$\min_{\mathbf{u}_m}\left[\frac{1}{2}\mathbf{u}_m^T\tilde{\mathbf{H}}\mathbf{u}_m+\tilde{\mathbf{x}}_t^T\tilde{\mathbf{F}}\mathbf{u}_m+\frac{1}{2}\tilde{\mathbf{x}}_t^T\tilde{\mathbf{Y}}\tilde{\mathbf{x}}_t\right], \quad (43)$$

where $\tilde{\mathbf{H}}\in\Re^{(m-1)\times(m-1)}$, $\tilde{\mathbf{F}}\in\Re^{3\times(m-1)}$, $\tilde{\mathbf{Y}}\in\Re^{3\times3}$ are functions of $\mathbf{A}$, $\mathbf{B}$, $S$, and $\lambda$; and $\tilde{\mathbf{x}}_t$ is as in Eqn. (33).

In the absence of a ZLB, the minimum in the optimization problem in Eqn. (43) is attained at $\mathbf{u}_m^{\text{opt}}=-\tilde{\mathbf{H}}^{-1}\tilde{\mathbf{F}}^T\tilde{\mathbf{x}}_t$, resulting in the optimal interest rate

$$\begin{aligned} i_t &= -\underbrace{\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}}_{m-1}\tilde{\mathbf{H}}^{-1}\tilde{\mathbf{F}}^T\tilde{\mathbf{x}}_t+r^*+\pi^* \\ &=\phi_y(y_t-y^*)+\phi_\pi(\pi-\pi^*)+\phi_i(i_{t-1}-i^*)+r^*+\pi^* \end{aligned} \quad . \quad (44)$$

which in exactly Eqn. (34).

A parametric analysis similar to that in Section 5.1 can be performed again to assess the effect of the MPC weights $S$ and $\lambda$ on the parameters $\phi_y$, $\phi_\pi$, and $\phi_i$. Similar choices of $N=80$, $m=4$ and $\beta=0.99$ as before yield

$$\phi_y=\frac{\tilde{q}_{y,3}S^6+\tilde{q}_{y,2}(\lambda)S^4+\tilde{q}_{y,1}(\lambda)S^2+\tilde{q}_{y,0}(\lambda)}{\tilde{p}_3S^6+\tilde{p}_2(\lambda)S^4+\tilde{p}_1(\lambda)S^2+\tilde{p}_0(\lambda)}, \quad (45)$$

$$\phi_\pi=\frac{\tilde{q}_{\pi,3}S^6+\tilde{q}_{\pi,2}(\lambda)S^4+\tilde{q}_{\pi,1}(\lambda)S^2+\tilde{q}_{\pi,0}(\lambda)}{\tilde{p}_3S^6+\tilde{p}_2(\lambda)S^4+\tilde{p}_1(\lambda)S^2+\tilde{p}_0(\lambda)}, \quad (46)$$

$$\phi_i=\frac{S^2\left(\tilde{q}_{i,3}S^4+\tilde{q}_{i,2}(\lambda)S^2+\tilde{q}_{i,1}(\lambda)\right)}{\tilde{p}_3S^6+\tilde{p}_2(\lambda)S^4+\tilde{p}_1(\lambda)S^2+\tilde{p}_0(\lambda)}, (47)$$

respectively, where the values of the corresponding parameters are shown in Table 12. From Eqn. (47) it is clear that the inertial term $\phi_i$ is zero for $S=0$. Use of eqns. (45), (46), and (47) yields the patterns shown in Figure 22, Figure 23, and Figure 24 for the coefficients $\phi_y$, $\phi_\pi$, $\phi_i$ as functions of $\lambda$ and $S$. The following trends can be observed:

- The policy coefficients $\phi_y$ and $\phi_\pi$ decrease with increase in $S$.
- When $S$ is small the effect of $\lambda$ on $\phi_\pi$ is dominant compared to the effect on $\phi_y$.

- After approximately $S > 2$ further increase on $S$ does not change the policy coefficients by much.
- The inertial term $\phi_i$ increases with increase in $S$ and eventually converges to 0.7. This result can be explained on the basis of the stabilizing policy criterion. If $\phi_i$ is large compared to $\phi_\pi$ and $\phi_y$, the closed loop will behave like an open loop and due to the unstable nature of the open-loop economy model, related policies will not stabilize the economy. These results are consistent with prior literature observations (Taylor and Williams, 2010, and references therein).

*Table 12*: Polynomial Coefficients in Eqns. (45)–-(47) as Functions of $\lambda$

| |
|---|
| $\tilde{q}_{y,3} = 0.428$ |
| $\tilde{q}_{y,2} = 1.03 + 3.17\lambda$ |
| $\tilde{q}_{y,1} = -0.117(-1.05+\lambda)(0.372+\lambda)$ |
| $\tilde{q}_{y,0} = 6.11\times 10^{-4}(-1.06+\lambda)(-1.01+\lambda)(0.365+\lambda)$ |
| $\tilde{q}_{\pi,3} = 1.51$ |
| $\tilde{q}_{\pi,2} = 0.911 + 14.0\lambda$ |
| $\tilde{q}_{\pi,1} = -0.323(-1.11+\lambda)(0.0653+\lambda)$ |
| $\tilde{q}_{\pi,0} = 1.59\times 10^{-3}(-1.12+\lambda)(-1.01+\lambda)(0.0588+\lambda)$ |
| $\tilde{q}_{i,3} = 3.19$ |
| $\tilde{q}_{i,2} = 0.294 + 0.370\lambda$ |
| $\tilde{q}_{i,1} = -0.00278(-1.02+\lambda)(0.690+\lambda)$ |
| $\tilde{p}_3 = 4.50$ |
| $\tilde{p}_2 = 0.920 + 1.42\lambda$ |
| $\tilde{p}_1 = -0.0326(-1.03+\lambda)(0.553+\lambda)$ |
| $\tilde{p}_0 = 1.32\times 10^{-4}(-1.03+\lambda)(-1.01+\lambda)(0.512+\lambda)$ |

*Figure 22*: Output Gap Coefficient $\phi_y$ for Taylor Rules with Inertia

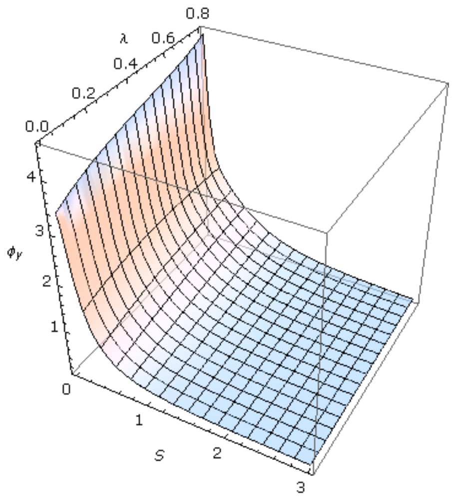



*Figure 23*: Inflation Coefficient $\phi_\pi$ for Taylor Rules with Inertia

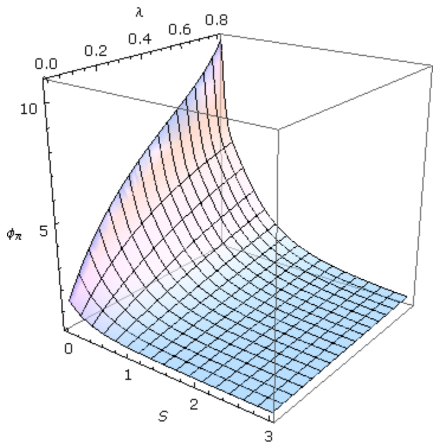
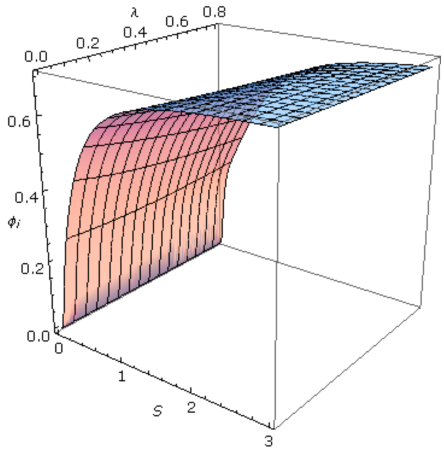


*Figure 24*: Lagged Inertest Rate Coefficient $\phi_i$ for Taylor Rules with Inertia




### 5.4.1 Inertia-Based Rules and Resulting Closed-Loop Stability

For Taylor rules with inertia as in Eqn. (34) the corresponding closed-loop is

$$\begin{bmatrix} \mathbf{x} \\ \psi \end{bmatrix}_{t+1} = \tilde{\mathbf{A}}_{\mathrm{CL}} \begin{bmatrix} \mathbf{x} \\ \psi \end{bmatrix}_t, \tag{48}$$

where

$$\tilde{\mathbf{A}}_{\mathrm{CL}} \mathrel{\hat{=}} \begin{bmatrix} \mathbf{A}+\mathbf{B}\mathbf{c}^T & \mathbf{B} \\ \phi_i \mathbf{c}^T & \phi_i \end{bmatrix} \mathrel{\hat{=}} \left[\begin{array}{cc|c} \rho-\xi\phi_y & \xi-\xi\phi_\pi & -\xi \\ \alpha & 1 & 0 \\ \hline \phi_i\phi_y & \phi_i\phi_\pi & \phi_i \end{array}\right], \tag{49}$$

and $\psi_t = u_t - \mathbf{c}^T \mathbf{x}_t$. It can be shown (Appendix G) that all eigenvalues of $\tilde{\mathbf{A}}_{CL}$ are inside the unit disk if and only if

$$\phi_i + \phi_\pi > 1 \tag{50}$$

$$\phi_\pi > -142 - 142\phi_i + 16.7\phi_y \tag{51}$$

$$176 - 108\phi_i > \phi_\pi \quad (52)$$

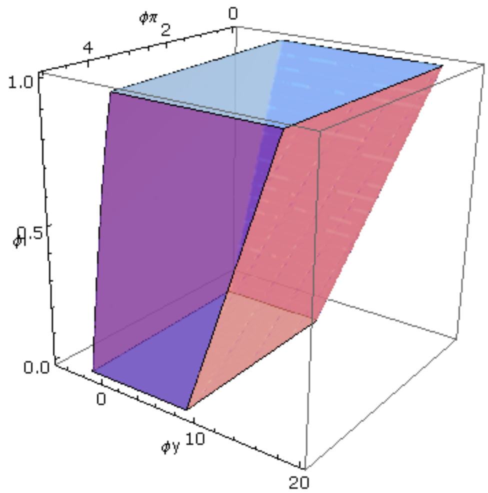
$$33.5 - 35.5\phi_i + 16.7\phi_y > \phi_\pi \tag{53}$$

$$17.2 + 10.5\phi_i^2 + 8.33\phi_y + \phi_i(-28.1 - 5.06\phi_y) > \phi_\pi \qquad (54)$$

as shown in Figure 25. As in Section 5.1, it is also found that all combinations of $S$ and $\lambda$ result in stabilizing monetary policies. Eqn. (50) is the counterpart of Eqn. (41)and has been derived before in a different setting, using a rational expectations approach (Woodford, 2003).

It is again interesting to note that as $S \to \infty$, namely as aggressive changes in the value of interest rate are heavily penalized, the closed loop remains stable, due to the stabilizing equality constraint, Eqn. (56). For $S \to \infty$, eqns. (45)-(47) suggest that $\tilde{\phi}_y = \frac{\tilde{q}_{y,3}}{\tilde{p}_3} = 0.095$, $\tilde{\phi}_\pi = \frac{\tilde{q}_{\pi,3}}{\tilde{p}_3} = 0.34$, and $\tilde{\phi}_i = \frac{\tilde{q}_{i,3}}{\tilde{p}_3} = 0.71$, which satisfy the inequalities in eqns. (51)–(54).

*Figure 25*: Closed-Loop Stability Region for the US Economy Model in Terms of Coefficients $\phi_y$, $\phi_\pi$ and $\phi_i$ for Taylor Rule with Inertia



## 5.5 Comparison with Historical Data

In the original publication (Taylor, 1993) it was assumed that $y^* = 0$, $\pi^* = 2\%$, $r^* = 2\%$, $\phi_\pi = 1.5$, $\phi_y = 0.5$, with quarterly data for output gap, and annual data for inflation rate. Variants of the above basic Taylor rule have been studied in literature, such as rules with an inertia term containing $i_{t-1}$ (Goodfriend, 1991; Woodford, 1999; Orphanides and Williams, 2007), more lagged terms of $i$ (Judd and Rudebusch, 1998; Clarida et al., 2000), and/or with projected future values of $\pi$ and $y$ in the right-hand side of Eqn. (1) (Taylor and Williams, 2010, and references therein).

We use real-time data available to the central bank at the time of making a decision on the interest rate, for the period 1987Q4:2008Q4. For output gap we use Greenbook data over the period 1987Q4:2005Q4; for the remaining period we consider CBO data (Nikolsko-Rzhevskyy and Papell, 2012). The real-time inflation data is also taken from the same publication.

We focus on the interest rate rule with inertia, Eqn. (34), with $r^* = 1.9$ and $\pi^* = 2$. Since the coefficients $\phi_y$, $\phi_\pi$ and $\phi_i$ are functions of the weights $S$ and $\lambda$ as given by eqns. (45)–(47), these weights and corresponding coefficients are estimated using regression to fit the historical data. Estimated values over the entire period of data are shown in Table 13. Figure 26 and Figure 27 compare the interest rate resulting from fitting Eqn. (34) to the interest rate implemented, as well as to the interest rate suggested by the standard Taylor rule (Eqn. (1) with $\phi_y = 0.5$, $\phi_\pi = 1.5$), and by the Taylor rule with values fitted over the entire period of data examined (Eqn. (1) with $\phi_y = 0.77$, $\phi_\pi = 2.0$). It is clear that the inertial rule captures the central bank decisions better, as also demonstrated by the residuals shown in Figure 28.

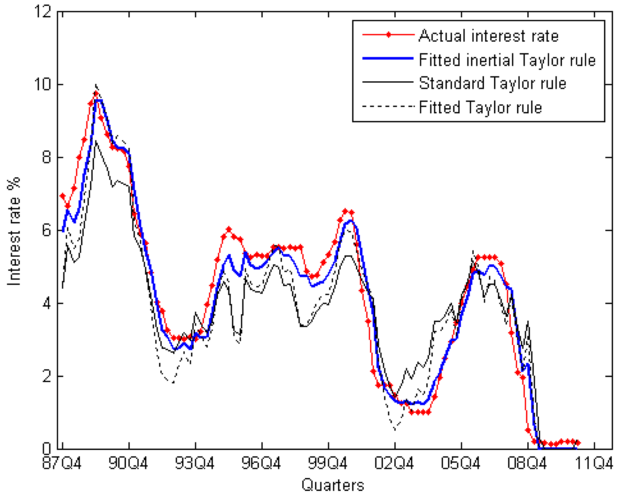
It is also interesting to examine whether additional insight may be gained by fitting data over short periods for which large residuals result from fitting the entire data set. One such period with large residuals is 2000Q1:2004Q4. Table 13 (line 2a) suggests that this period may be problematic, in that the corresponding inertial rule, if applicable, is not stabilizing, i.e. the fitted value of $\phi_\pi + \phi_i$ is greater than 1, thus violating the closed-loop stability condition in Eqn. (50). In fact, it is dubious whether the same objective as on the average was used over that

*Table 13*: Inertial Policy Estimation for US Interest Rate Rule Based on Real-Time Data

| | Period | $S$ | $\lambda$ | $\phi_y$ | $\phi_\pi$ | $\phi_i$ | $\phi_\pi + \phi_i$ |
|---|---|---|---|---|---|---|---|
| | 1987Q4:2008Q4 | 0.83(0.23) | 0.09(0.03) | 0.29 | 0.71 | 0.62 | 1.33 |
| 1 | 1987Q4:1999Q4 | 1.1(0.43) | 0.10(0.06) | 0.24 | 0.67 | 0.64 | 1.31 |
| 2a | 2000Q1:2004Q4 | 0.15(0.08) | –0.07(0.03) | 0.66 | 0.13 | 0.47 | 0.60 |
| 2b | 2000Q1:2004Q4 | 0.3 | 0 | 0.48 | 0.60 | 0.55 | 1.15 |
| 3 | 2005Q1:2008Q4 | 0.44(0.26) | 0.16(0.1) | 0.53 | 1.25 | 0.55 | 1.80 |

*Note*: Standard deviations are reported in parentheses.

*Figure 26*: Federal Funds Rate, Standard Taylor Rule, Fitted Inertial and Fitted Taylor Rules (Fitting Period 1987Q4: 2008Q4) for Period 1987Q4: 2011Q1




*Note*: The interest rate reduction in 2008 suggested by the inertial Taylor rule is more drastic than that suggested by the standard Taylor rule. The actual interest rate over the period 2002–2005 is captured fairly well by the inertial Taylor rule, while the standard Taylor rule produces significantly larger values, as has been studied extensively by Taylor (Taylor, 2009))

*Figure 27*: Magnified View of Figure 26 when Interest Rates are Near Zero

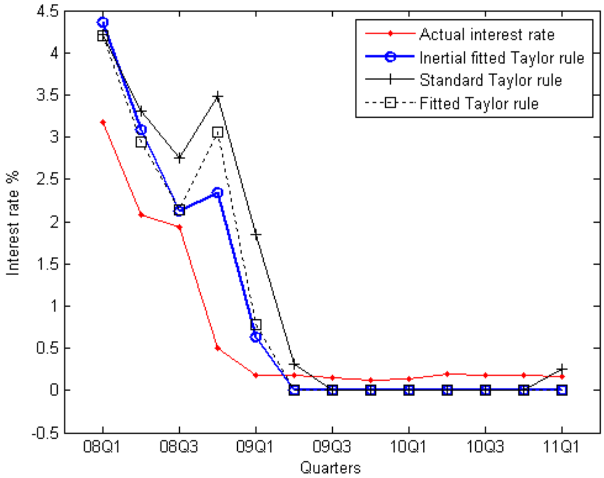
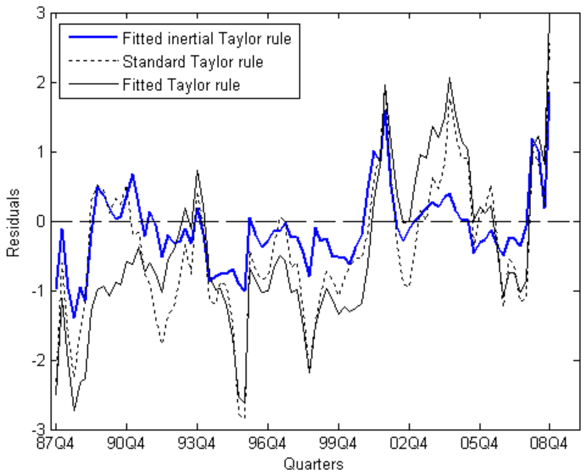



*Figure 28*: Residuals for Policies in Figure 26 for Fitting Period 1987Q4: 2008Q4



period, since the value of $\lambda$ fitted over that period is negative, hence unacceptable. Constrained fitting (i.e. enforcing $0 \le \lambda \le 1$) produces parameter values that do correspond to a stabilizing rule (Table 13, line 2b) but nonetheless places all emphasis on output gap (growth). The actual policy implemented over that period and its role on stimulating over-expansion of the economy has been the subject of intense discussion (Taylor, 2009).

## 6 Conclusions and Future Work

The main issue addressed in this work is the effect of zero lower bound on the optimal interest rate determined by a central bank. We address this issue in a multi-parametric model predictive control framework, which allows the derivation of explicit feedback rules even when inequality constraints are present. Application of this framework to a simple model of the US economy produced a number of Taylor-like rules, depending on the form and parameter values in the objective function employed by MPC. The results suggest that a small number of simple Taylor-like formulas can be applied at each time, depending on the state of the economy. However, it was also shown that simply truncating to zero negative interest rates produced by unconstrained Taylor rules is optimal in situations of negative output gap, as happened recently. Furthermore, it was observed, as has been noted elsewhere, that rules with inertia appear to better capture past decisions by the central bank. Such rules have been systematically derived here by considering penalties on the rate of interest rate change in the MPC objective function.

A number of issues touched in this work warrant further investigation, including the following:

- The inverse problem: Given a suggested Taylor-like rule, what objective function, as in Eqn. (14), is minimized? A promising approach is suggested in Section 5.2.
- Robust stability and performance: There is a vast body of work in the automatic control community addressing the robustness issue, namely how a controller performs when the model assumed in controller design has quantifiable uncertainty.

- Modeling and selection of controlled variables: Should the pair output gap and inflation be the main focus or could variables such unemployment (Orphanides and Williams, 2007) be central in controlling an economy?
- Policy adaptation: The main attractiveness of a fixed rule is its simplicity and predictability (Williams, 2009). However, such a rule may become sub-optimal over time, as the economy or disturbance models change (Orphanides, 2003). Can a fixed rule be replaced by a fixed rule adaptation policy that maintains robustness?

We hope to address the above issues in forthcoming publications.

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***Notation***

| | |
|---|---|
| $y$ | output gap (deviation of real GDP from potential GDP as percent of potential GDP) |
| $\pi$ | inflation rate |
| $t$ | time |
| $i$ | interest rate |
| $r \mathrel{\hat{=}} i - \pi$ | real interest rate |
| $\phi_y$ , $\phi_\pi$ | coefficients associated with the output gap and inflation rate, respectively |
| $\alpha$ | positive constant in economy model |
| $\xi$ | positive constant in economy model |
| $\sigma$ | positive constant in economy model |
| $\rho \in [0,1)$ | constant in economy model |
| $e_{t+1}^{y}$ | zero-mean white noise in output gap equation |
| $e_{t+1}^{\pi}$ | zero-mean white noise in inflation equation |
| * | desired equilibrium value; |

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# Appendices

## Appendix A: MPC Formulation

Following Muske and Rawlings (1993), the following equality constraints are included in the MPC formulation.

- Input move restriction constraints

$$u_{t+k|t} = u_{t+m-1|t}\,,\ k = m, \ldots, N-1, \tag{55}$$

The above equality constraints set all input values, to be decided on at time $t$, equal to each other after time point $t+m-1$ within the moving horizon. This reduces the dimension of the optimization problem without sacrificing performance.

- Unstable mode stabilization constraints

$$\tilde{\mathbf{v}}_{\mathrm{u}}^{T}\left[\mathbf{A}^{N-1}\mathbf{B}, \mathbf{A}^{N-2}\mathbf{B}, \ldots, \mathbf{B}\right]\mathbf{u} = -\tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{A}^{N}\mathbf{x}_{t}\,, \tag{56}$$

with the vectors $\mathbf{v}_{\mathrm{s}}$ (along with $\tilde{\mathbf{v}}_{\mathrm{s}}$) coming from the diagonalization (Jordan form) of the matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1} = \left[\begin{array}{c:c} \mathbf{v}_{\mathrm{u}} & \mathbf{v}_{\mathrm{s}} \end{array}\right]\begin{bmatrix} J_{\mathrm{u}} & 0 \\ 0 & J_{\mathrm{s}} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{v}}_{\mathrm{u}}^{T} \\ \hdashline \tilde{\mathbf{v}}_{\mathrm{s}}^{T} \end{bmatrix}, \tag{57}$$

where $J_{\mathrm{u}}$ and $J_{\mathrm{s}}$ refer to the unstable and stable eigenvalues of the matrix $\mathbf{A}$ with corresponding eigenvectors, $\mathbf{v}_{\mathrm{u}}$ and $\mathbf{v}_{\mathrm{s}}$, respectively.

Finally,

$$\bar{\mathbf{Q}} \mathrel{\hat{=}} \frac{\mathbf{v}_{\mathrm{s}}^{T}\mathbf{Q}\mathbf{v}_{\mathrm{s}}}{1-\beta J_{\mathrm{s}}^{2}}\tilde{\mathbf{v}}_{s}\tilde{\mathbf{v}}_{\mathrm{s}}^{T} \succ 0\,. \tag{58}$$

The main rationale behind the above MPC formulation is that closed-loop stability can be guaranteed by including the terminal penalty term $\hat{\mathbf{x}}_{t+N|t}^{T}\bar{\mathbf{Q}}\hat{\mathbf{x}}_{t+N|t}$ in the objective, Eqn. (14), and by explicitly forcing a terminal constraint, Eqn. (56), to stabilize the unstable mode corresponding to the eigenvalue $J_{\mathrm{u}}$.

## Appendix B: Multi-Parametric MPC Formulation for Taylor Rules with Inertia

The derivation of the formulas will proceed in two steps, namely MPC optimization without and with ZLB, respectively.

### a. MPC Optimization without ZLB

Based on the optimization function in Eqn. (14) and the method discussed in Muske and Rawlings (1993) with discount factor $\beta$, the terminal penalty weight matrix $\bar{\mathbf{Q}}$ is

$$\bar{\mathbf{Q}} = \sum_{i=0}^{\infty} \mathbf{A}^{T^i} \beta^i \mathbf{Q} \mathbf{A}^i \tag{59}$$

Since the unstable mode is constrained to be equal to zero at time $k+N$, it follows that

$$\bar{\mathbf{Q}} = \tilde{\mathbf{v}}_s \Sigma \tilde{\mathbf{v}}_s^T \tag{60}$$

where

$$\Sigma = \frac{\mathbf{v}_s^T Q \mathbf{v}_s}{1-\beta J_s^2}. \tag{61}$$

From Eqns. (60) and (61) it follows that

$$\bar{\mathbf{Q}} = \frac{\mathbf{v}_s^T \mathbf{Q} \mathbf{v}_s}{1-\beta J_s^2} \tilde{\mathbf{v}}_s \tilde{\mathbf{v}}_s^T. \tag{62}$$

Further, Eqn. (56) along with Eqn. (55) results in

$$u_{t+m-1|t} = \mathbf{a}_m^T \mathbf{x}_t + \mathbf{b}_m^T \mathbf{u}_m, \tag{63}$$

where

$$\mathbf{a}_m^T = \frac{-\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{A}^N}{\tilde{\mathbf{v}}_{\mathrm{u}}^T \left(\mathbf{A}^{N-m}\mathbf{B} + \ldots.. + \mathbf{AB} + \mathbf{B}\right)}, \quad \mathbf{b}_m^T = \frac{-\tilde{\mathbf{v}}_{\mathrm{u}}^T \left[\mathbf{A}^{m-2}\mathbf{B} \ldots, \mathbf{AB}, \mathbf{B}\right]}{\tilde{\mathbf{v}}_{\mathrm{u}}^T \left(\mathbf{A}^{N-m}\mathbf{B} + \ldots.. + \mathbf{AB} + \mathbf{B}\right)} \tag{64}$$

and the optimization variable $\mathbf{u}_m$ contains the first $m-1$ elements of $\mathbf{u}$.

Using eqns. (18) and (55) for the case when $k > m$ yields

$$\hat{\mathbf{x}}_{t+k|t} = \mathbf{A}^k x_t + \sum_{\ell=1}^{m-1} \mathbf{A}^{k-\ell} \mathbf{B} u_{t+\ell-1|t} + \left(\sum_{\ell=m}^{k} \mathbf{A}^{k-\ell} \mathbf{B}\right) u_{t+m-1|t}. \tag{65}$$

Using Eqns. (63) and (65) yields

$$\hat{\mathbf{x}}_{t+k|t} = \mathbf{f}_{k,\ell}\mathbf{x}_t + \sum_{\ell=1}^{m-1}\mathbf{h}_{k,\ell}u_{t+\ell-1|t} \,, \tag{66}$$

where

$$\mathbf{h}_{k,\ell} = \begin{cases} \left(\mathbf{A}^{k-\ell} + b_\ell\left(\sum_{\ell=m}^{k}\mathbf{A}^{k-\ell}\right)\right)\mathbf{B} & k \geq m, \ell \leq k \\ \mathbf{A}^{k-\ell}\mathbf{B} & k < m, \ell \leq k \\ 0 & k < m, \ell > k \end{cases}, \tag{67}$$

$$\mathbf{f}_k = \left[\mathbf{f}_{k,1}......\mathbf{f}_{k,m-1}\right] \in \Re^{2\times(m-1)}$$

$$\mathbf{f}_{k,\ell} = \begin{cases} \mathbf{A}^k + \sum_{\ell=m}^{k}\mathbf{A}^{k-l}\mathbf{B}\mathbf{a}^T & \text{for } k \geq m\,, \\ \mathbf{A}^k & \text{for } k < m \end{cases} \tag{68}$$

$$\mathbf{h}_k = \left[\mathbf{h}_{k,1}......\mathbf{h}_{k,m-1}\right] \in \Re^{(m-1)\times(m-1)}\,. \tag{69}$$

Substituting eqns (66)-(69) into Eqn. (14) with $S = 0$ yields Eqn. (77).

The solution to Eqn. (77) is

$$\mathbf{u}_m = -\mathbf{H}^{-1}\mathbf{F}^T\mathbf{x}_t\,. \tag{70}$$

where

$$\mathbf{H} = \sum_{k=1}^{N-1}\mathbf{h}_k^T\beta^k\mathbf{Q}\mathbf{h}_k + \mathbf{h}_N^T\beta^N\bar{\mathbf{Q}}\mathbf{h}_N + R^2\left(\frac{\beta^{m-1}-\beta^N}{1-\beta}\mathbf{b}\mathbf{b}^T + \mathbf{D}_R\right), \tag{71}$$

$$\mathbf{D}_R \hat{=} \operatorname{diag}\begin{bmatrix}1 & \beta & . & . & \beta^{m-2}\end{bmatrix} \tag{72}$$

$$\mathbf{F} = \left(\sum_{k=1}^{N-1}\mathbf{f}_k^T\beta^k\mathbf{Q}\mathbf{h}_k\right) + \mathbf{f}_N^T\beta^N\bar{\mathbf{Q}}\mathbf{h}_N + R^2\frac{\beta^{m-1}-\beta^N}{1-\beta}\mathbf{a}\mathbf{b}^T\,. \tag{73}$$

### *b. MPC Optimization with ZLB*

Using the equality constraints in Appendix D, the ZLB constraint given in Eqn. (20) can be written as,

$$\mathbf{G}\mathbf{u}_m \leq \mathbf{w} + \mathbf{E}\mathbf{x}_t\,, \tag{74}$$

where $\mathbf{G} \hat{=} \begin{bmatrix} -\mathbf{I} \\ -\mathbf{b}^T \end{bmatrix}$; $\mathbf{I}$ is the identity matrix in $\Re^{(m-1)\times(m-1)}$; $\mathbf{w} \hat{=} [i* \quad \cdots \quad i*]^T \in \Re^m$; $\mathbf{E} = \begin{bmatrix} \mathbf{\Theta} \\ \mathbf{a}^T \end{bmatrix}$; $\mathbf{\Theta} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}^T \in \Re^{(m-1)\times 2}$. Therefore, the optimization problem Eqn. (77) subject to the constraint Eqn. (74) can be formulated as

$$\min_{\mathbf{z}} \frac{1}{2}\mathbf{z}^T \mathbf{H}\mathbf{z} \tag{75}$$

$$\mathbf{G}\mathbf{z} \leq \mathbf{w} + \mathbf{D}\mathbf{x}_t \tag{76}$$

where $\mathbf{z} \hat{=} \mathbf{u}_m + \mathbf{H}^{-1}\mathbf{F}^T\mathbf{x}_t$, $\mathbf{D} \hat{=} \mathbf{E} + \mathbf{G}\mathbf{H}^{-1}\mathbf{F}^T$.

## Appendix C: MPC Parameters

In this Appendix we explore choices for the parameters $N$, $m$, $\beta$ that appear in MPC optimization.

### *a. Choice of Prediction Horizon Length, $N$*

For an unstable system such as the one described by eqns. (2) and (3), the horizon length, $N$, should be made long enough to ensure that the MPC optimization problem is feasible and ensure closed-loop stability. Systematic methods can be used for selecting $N$ (Chmielewski and Manousiouthakis, 1996; Scokaert and Rawlings, 1998; Grieder et al., 2004).

In all developments we consider $N = 80$.

### *b. Choice of Control Horizon Length, $m$*

As Eqn. (55) indicates, only a small number of inputs are included as decision variables in the MPC optimization. In addition to convenience (i.e. a small number of decision variables) there are deeper reasons for this choice.

First, increasing the value of $m$ (with $1 \leq m \leq N$) quickly reaches a point of diminishing returns, namely no appreciable change in the closed-loop dynamics. Table 14 substantiates this claim by example, showing that the closed-loop poles remain almost unchanged after increasing the value of $m$ beyond 4.

*Table 14*: Closed-Loop Eigenvalues for Taylor-Like Rules Derived from Unconstrained MPC for $\lambda = 0.05$ and $R = 0.07$

| $m$ | $N$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 20 | | 40 | | 60 | | 80 | |
| | $\mu_1$ | $\mu_2$ | $\mu_1$ | $\mu_2$ | $\mu_1$ | $\mu_2$ | $\mu_1$ | $\mu_2$ |
| 2 | 0.05 | 0.95 | 0.05 | 0.95 | 0.05 | 0.95 | 0.05 | 0.94 |
| 3 | 0.07 | 0.95 | 0.07 | 0.97 | 0.07 | 0.96 | 0.07 | 0.96 |
| 4 | 0.07 | 0.95 | 0.07 | 0.97 | 0.07 | 0.97 | 0.07 | 0.96 |
| 8 | 0.07 | 0.95 | 0.07 | 0.97 | 0.07 | 0.97 | 0.07 | 0.97 |
| 12 | 0.07 | 0.95 | 0.07 | 0.97 | 0.07 | 0.97 | 0.07 | 0.97 |
| 16 | 0.07 | 0.95 | 0.07 | 0.97 | 0.07 | 0.97 | 0.07 | 0.97 |

*Table 15*: Output Gap and Inflation Coefficients in Taylor-Like Rules (Eqn. (1)) Derived from Unconstrained MPC for $\lambda = 0.05$ and $R = 0.07$

| $m$ | $N$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 20 | | 40 | | 60 | | 80 | |
| | $\phi_y$ | $\phi_\pi$ | $\phi_y$ | $\phi_\pi$ | $\phi_y$ | $\phi_\pi$ | $\phi_y$ | $\phi_\pi$ |
| 2 | 3.2 | 2.9 | 3.1 | 2.4 | 3.1 | 2.4 | 3.1 | 2.5 |
| 3 | 3.2 | 2.9 | 3.1 | 2.4 | 3.1 | 2.4 | 3.1 | 2.5 |
| 4 | 3.2 | 2.9 | 3.1 | 2.4 | 3.1 | 2.4 | 3.1 | 2.5 |
| 8 | 3.2 | 2.9 | 3.1 | 2.3 | 3.1 | 2.3 | 3.1 | 2.3 |
| 12 | 3.2 | 2.9 | 3.1 | 2.2 | 3.1 | 2.2 | 3.1 | 2.2 |
| 16 | 3.2 | 2.9 | 3.1 | 2.2 | 3.1 | 2.1 | 3.1 | 2.2 |

The associated Table 15 shows the resulting coefficient for the Taylor-like solution provided by MPC.

Second, it has been rigorously shown that keeping $m$ small improves the robustness of the closed loop, namely it helps maintain closed-loop stability in the presence of discrepancies between the model used by MPC and the actual system under control (Garcia and Morari, 1982; Genceli and Nikolaou, 1993; Vuthandam et al., 1995).

In all developments we consider $m = 4$.

*c. Choice of Discount Factor, $\beta$*

Following the literature (Jung et al., 2005; Adam and Billi, 2007) we use a value of the discount factor $\beta = 0.99$, except in situations where we explicitly specify a different value. We comment in the Discussion section on how different values of $\beta$ affect the resulting Taylor rules, particularly closed-loop stability and performance.

## Appendix D: Multi-Parametric MPC Formulation for Taylor Rules

In the absence of ZLB, Eqn. (20), and without penalty on the change of interest rate $(S = 0)$, the MPC optimization with objective function in Eqn. (14) subject to equality constraints in eqns. (18)-(55) results in the unconstrained quadratic minimization

$$\min_{\mathbf{u}_m} \left[ \frac{1}{2} \mathbf{u}_m^T \mathbf{H} \mathbf{u}_m + \mathbf{x}_t^T \mathbf{F} \mathbf{u}_m + \frac{1}{2} \mathbf{x}_t^T \mathbf{Y} \mathbf{x}_t \right], \tag{77}$$

where $\mathbf{H} \in \Re^{(m-1)\times(m-1)}$, $\mathbf{F} \in \Re^{2\times(m-1)}$, $\mathbf{Y} \in \Re^{2\times 2}$ are function of $\mathbf{A}$, $\mathbf{B}$, $\beta$, $N$, $m$, and the weights $R$ and $\lambda$; and the decision variable is

$$\mathbf{u}_m \mathrel{\hat{=}} \begin{bmatrix} \Delta i_t \\ \Delta i_{t+1|t} \\ . \\ . \\ \Delta i_{t+m-2|t} \end{bmatrix}. \tag{78}$$

The minimum in Eqn. (77) is attained at $\mathbf{u}_m^{\text{opt}} = -\mathbf{H}^{-1}\mathbf{F}^T\mathbf{x}_t$, resulting in the optimal interest rate

$$i_t = -\underbrace{[1\,0...0]}_{m-1}\mathbf{H}^{-1}\mathbf{F}^T\mathbf{x}_t + r^* + \pi^* = \phi_y(y_t - y^*) + \phi_\pi(\pi - \pi^*) + r^* + \pi^* \tag{79}$$

at time $t$, which is clearly a Taylor-like rule, as in Eqn. (1). It is also clear that $\phi_y$, $\phi_\pi$ are functions of the economic model matrices $\mathbf{A}$, $\mathbf{B}$, and of the weights $R$, $\lambda$, given $N$, $m$ and $\beta$.

## Appendix E: Closed-Loop Stability for Taylor Rule

The standard Taylor rule can be written as

$$u_t = \mathbf{c}^T \mathbf{x}_t , \tag{80}$$

where $\mathbf{c}^T \triangleq \begin{bmatrix} \phi_y & \phi_\pi \end{bmatrix}$.

The characteristic equation for the matrix $\mathbf{A}_{\mathrm{CL}}$ in Eqn. (40) is given by

$$f(\mu) \triangleq \mu^2 - \left(1+\rho+\alpha\xi-\xi\phi_y-\alpha\xi\phi_\pi\right)\mu + \left(\rho-\xi\phi_y\right), \tag{81}$$

where $\mu$ is an eigenvalue of the matrix $\mathbf{A}_{\mathrm{CL}}$. For closed-loop stability the eigenvalues of the matrix $\mathbf{A}_{\mathrm{CL}}$ should lie inside the unit disk, which is guaranteed (by the Jury-Routh–Hurwitz stability criterion) if and only if

$$2+2\rho-2\xi\phi_y+\alpha\xi(\phi_\pi-1)>0 , \tag{82}$$

$$1-\rho+\xi\phi_y-\alpha\xi(\phi_\pi-1)>0 , \tag{83}$$

$$\alpha\xi(\phi_\pi-1)>0 . \tag{84}$$

Given that $\alpha\xi>0$, Eqn. (84) is satisfied if and only if $\phi_\pi>1$.

## Appendix F: Multi-Parametric MPC Formulation for Taylor Rules with Inertia

Adopting the same approach as shown in Appendix D, a similar kind of expression for the optimization problem set-up in Eqn. (14) can be derived when $S>0$ as

$$\min_{\mathbf{u}_m}\left[\frac{1}{2}\mathbf{u}_m^T\tilde{\mathbf{H}}\mathbf{u}_m+\tilde{\mathbf{x}}_t^T\tilde{\mathbf{F}}\mathbf{u}_m+\frac{1}{2}\tilde{\mathbf{x}}_t^T\tilde{\mathbf{Y}}\tilde{\mathbf{x}}_t\right], \tag{85}$$

where,

$$\tilde{\mathbf{x}}_t \triangleq \begin{bmatrix}\Delta y_t\\ \Delta\pi_t\\ \Delta u_{t-1}\end{bmatrix}, \tag{86}$$

$$\begin{aligned}\tilde{\mathbf{H}}_{m-1\times m-1} = &\sum_{k=1}^{N-1}\mathbf{h}_k^T\beta^k\mathbf{Q}\mathbf{h}_k + \mathbf{h}_N^T\beta^N\bar{\mathbf{Q}}\mathbf{h}_N + \\ &S^2\left(\beta^{m-1}(\mathbf{b}-\mathbf{b}_0)(\mathbf{b}-\mathbf{b}_0)^T+\beta^{N-1}\mathbf{b}\mathbf{b}^T+\mathbf{S}_0\right)\end{aligned}, \tag{87}$$

where $\mathbf{b}_0 = \begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}^T \in \Re^{m-1}$, $\mathbf{S}_0 \in \Re^{(m-1)\times(m-1)}$ is given by,

$$\mathbf{S}_0 \triangleq \left[ s_{i,j} \right] \begin{cases} s_{i,j} = \beta^{i-1}(1+\beta), i = j, i \neq m-1 \\ s_{i,j} = \beta^{m-2}, i = j, i = m-1 \\ s_{i,j} = -\beta, |i-j| = 1 \\ s_{i,j} = 0, |i-j| > 1 \end{cases}, \quad (88)$$

and

$$\tilde{\mathbf{F}}_{3\times m-1} = \begin{bmatrix} \left( \sum_{k=1}^{N-1} \mathbf{f}_k^T \mathbf{Q} \mathbf{h}_k \right) + \mathbf{f}_N^T \bar{\mathbf{Q}} \mathbf{h}_N + S^2 \left[ \beta^{m-1} \mathbf{a} \left( (\mathbf{b} - \mathbf{b}_0)^T + \beta^{N-1} \mathbf{b}^T \right) \right] \\ -S^2, \underbrace{0.......0}_{m-2} \end{bmatrix}. \quad (89)$$

When there is no inequality constraint, the solution to Eqn. (85) is given by

$$\mathbf{u}_m = -\tilde{\mathbf{H}}^{-1} \tilde{\mathbf{F}}^T \tilde{\mathbf{x}}_t . \quad (90)$$

ZLB constraint given by Eqn. (20) is equivalent to,

$$\mathbf{G}\mathbf{u}_m \leq \mathbf{w} + \tilde{\mathbf{E}} \tilde{\mathbf{x}}_t , \quad (91)$$

where $\tilde{\mathbf{E}} = [\mathbf{E} \quad \mathbf{E}_0]$ and $\mathbf{E}_0 = [0 \quad \cdots \quad 0]^T \in \Re^m$. Eqns. (85) and (91) can be formulated as,

$$\min_{\tilde{\mathbf{z}}} \frac{1}{2} \tilde{\mathbf{z}}^T \tilde{\mathbf{H}} \tilde{\mathbf{z}} \quad (92)$$

$$\mathbf{G}\tilde{\mathbf{z}} \leq \mathbf{w} + \tilde{\mathbf{D}} \tilde{\mathbf{x}}_t \quad (93)$$

where $\tilde{\mathbf{z}} \triangleq \mathbf{u}_m + \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{F}}^T \tilde{\mathbf{x}}_t$, $\tilde{\mathbf{D}} \triangleq \tilde{\mathbf{E}} + \mathbf{G} \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{F}}^T$. Eqn. (92) and inequality constraints Eqn. (93) are used for multi-parametric MPC formulation to derive explicit inertia-based Taylor rules with ZLB constraints.

## Appendix G: Closed-Loop Stability for Inertial Taylor-like Rule

The interest rate rule is

$$u_t = \phi_i u_{t-1} + \mathbf{c}^T \mathbf{x}_t , \quad (94)$$

The characteristic equation for the matrix $\tilde{\mathbf{A}}_{CL}$ is given by

$$\begin{aligned} \tilde{f}(\mu) \triangleq \mu^3 &- \left(1+\rho-\xi\phi_y+\phi_i\right)\mu^2 \\ &+ \left(\rho-\xi\phi_y+(1+\rho)\phi_i-\alpha\xi(1-\phi_\pi)\right)\mu-(\rho-\alpha\xi)\phi_i \end{aligned} \quad (95)$$

Closed-loop stability is guaranteed (by the Jury-Routh–Hurwitz stability criterion) if and only if

$$2+2\phi_i+2\rho(1+\phi_i)-2\xi\phi_y-\alpha\xi(1+\phi_i-\phi_\pi)>0\,, \tag{96}$$

$$4-4\rho\phi_i+\alpha\xi(1+3\phi_i-\phi_\pi)>0\,, \tag{97}$$

$$2-2\phi_i+2\rho(-1+\phi_i)+2\xi\phi_y+\alpha\xi(1-3\phi_i-\phi_\pi)>0\,, \tag{98}$$

$$\alpha\xi(\phi_\pi+\phi_i-1)>0\,, \tag{99}$$

$$\begin{aligned}&-8\Big((\alpha\xi\phi_i)^2+(\rho\phi_i-1)(1-\rho+\rho\phi_i)-\phi_i+\xi\phi_y\Big)\\&+\alpha\xi\Big((1-2\rho)\phi_i^2+\phi_i\left(1+\rho-\xi\phi_y\right)+\phi_\pi-1\Big)>0\end{aligned}\,, \tag{100}$$

## Appendix H: Infeasibility Polytope

The model decomposition of $\mathbf{A}$ is represented by,

$$\mathbf{A}=\mathbf{VJV}^{-1}=\left[\mathbf{v}_\mathrm{u}\;\vdots\;\mathbf{v}_\mathrm{s}\right]\begin{bmatrix}J_\mathrm{u} & 0\\ 0 & J_\mathrm{s}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{v}}_\mathrm{u}^T\\ \hdashline \tilde{\mathbf{v}}_\mathrm{s}^T\end{bmatrix} \tag{101}$$

where

$$J_u=\frac{1+\rho+\sqrt{(1-\rho)^2+4\alpha\xi}}{2}>1 \tag{102}$$

$$J_\mathrm{s}=\frac{1+\rho-\sqrt{(1-\rho)^2+4\alpha\xi}}{2}<1 \tag{103}$$

Equation (101) implies

$$\mathbf{V}^{-1}\hat{\mathbf{x}}_{t+k|t}=\sum_{\ell=0}^{k-1}\mathbf{J}^{\ell}\mathbf{V}^{-1}\mathbf{B}u_{t+k-\ell-1|t}+\mathbf{J}^k\mathbf{V}^{-1}\mathbf{x}_t \quad (104)$$

From Eqn. (104) stable and unstable modes can be treated separately. In terms of the unstable mode

$$\tilde{\mathbf{v}}_\mathrm{u}^T\hat{\mathbf{x}}_{t+k|t}=\sum_{\ell=0}^{k-1}J_u^{\ell}\,\tilde{\mathbf{v}}_\mathrm{u}^T\mathbf{B}u_{t+k-\ell-1|t}+J_u^k\,\tilde{\mathbf{v}}_\mathrm{u}^T\mathbf{x}_t \quad (105)$$

If $\mathbf{x}_t$ lies in the polytope of attraction, then

$$\lim_{k\to\infty}\tilde{\mathbf{v}}_\mathrm{u}^T\hat{\mathbf{x}}_{t+k|t}=0 \tag{106}$$

and

$$\tilde{\mathbf{v}}_\mathrm{u}^T\mathbf{x}_t=-J_u^{-k}\sum_{\ell=0}^{k-1}J_u^{\ell}\,\tilde{\mathbf{v}}_\mathrm{u}^T\mathbf{B}u_{t+k-\ell-1|t} \tag{107}$$

since $-u_{t+k-\ell-1|t} \leq i*$.

The polytope of attraction is given by

$$\tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{x}_{t} \leq \lim_{k\to\infty}\left(\sum_{\ell=0}^{\ell=k-1} J_{u}^{\ell-k}\right)\tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{B}i* \Rightarrow \tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{x}_{t} \leq \frac{\tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{B}}{J_{u}-1}i*. \tag{108}$$

Hence the infeasibility polytope is characterized by,

$$\tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{x}_{t} > \frac{\tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{B}}{J_{u}-1}i*. \tag{109}$$

Similarly, in the case of inertial policy the above exercise can be repeated and the counterpart of Eqn. (108) can be derived.

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