Economics
The Open-Access, Open-Assessment E-Journal

Vol. 6, 2012-9 | March 30, 2012 | http://dx.doi.org/10.5018/economics-ejournal.ja.2012-9

Measuring Group Disadvantage with Inter-distributional Inequality Indices: A Critical Review and Some Amendments to Existing Indices

Gaston Yalonetzky
University of Leeds

Abstract A long literature on inter-distributional inequality (IDI) has developed statistical tools for measuring the extent of inequality between two groups (e.g. men versus women). The paper reviews some of the most prominent IDI indices proposed in the last four decades. The assessment focuses on how these indices react to inequalities that are disadvantageous to different groups, using two operationalizations of a concept of group-specific disadvantage focus (GDF). Relying on a complementary set of properties, the review also assesses whether these indices are informative about other interesting features related to IDI comparisons, chiefly distributional equality, but also absence of distributional overlap and presence of first-order stochastic dominance. The author proposes amendments to several of these indices in order to render them in fulfillment of GDF properties and more informative on the mentioned distributional features.

Special Issue
The Measurement of Inequality and Well-Being: New Perspectives

JEL D30, J71
Keywords Inter-distributional inequality

Correspondence Gaston Yalonetzky, Maurice Keyworth Building, Leeds University Business School, Leeds LS2 9JT, United Kingdom, e-mail: g.yalonetzky@leeds.ac.uk.

Citation Gaston Yalonetzky (2012). Measuring Group Disadvantage with Inter-distributional Inequality Indices: A Critical Review and Some Amendments to Existing Indices. *Economics: The Open-Access, Open-Assessment E-Journal*, Vol. 6, 2012-9. http://dx.doi.org/10.5018/economics-ejournal.ja.2012-9

1 Introduction

The concern for differences in the distribution of wellbeing characteristics among groups within societies has earned a long-standing interest in the Social Sciences and Political Philosophy. This concern has often emphasized the potential presence of socio-economic discrimination of different natures (e.g. Becker, 1971; Phelps, 1972; Arrow, 1973). In general, it has been associated with concepts of inequality of opportunities.[1] The normative view for between-groups differences, related to ethnicity or gender, states that they are intrinsically unfair (particularly when the groups are defined over characteristics beyond their members' control), and instrumentally detrimental to individuals and societies (e.g. Arneson, 1989; Cohen, 1989; Nussbaum and Glover, 1995; Roemer, 1998; Fleurbaey, 2001; Sen, 2001).

From a quantitative perspective, one way of measuring the extent of differences in wellbeing between groups is to use indices that capture between-group inequalities, and that declare the total absence of between-group inequality when conditional distributions of wellbeing, or some functions of them, are identical across groups.[2] There is also an interest in quantifying between-group inequalities with a focus on capturing inequality if and when it is (more) detrimental to one specific group as opposed to other(s), i.e. a concept of relative economic disadvantage. Even though several authors have focused on inequalities detrimental to one group,[3] only recently formal definitions of the concept have been put forward. Interestingly, these recent definitions emphasize a concern for censoring inequal-

[1] For a good review of the literature on inequality of opportunity see Fleurbaey (2008). Also Roemer (1998).

[2] This condition is consistent with a literalist definition of inequality of opportunity by Roemer (1998, p. 15-6) as well as with Van De Gaer's rule (Ooghe et al., 2007). It is also consistent with Fleurbaey's concept of circumstance neutralization (Fleurbaey, 2008, p. 25). There are alternative ways of measuring between-group inequality. For instance, it could be measured as the residual inequality after within-group inequality has been suppressed (e.g. by replacing individual's wellbeing values with those of their group mean). Such approach has been followed, among others, by Roemer (2006); Elbers et al. (2008); Ferreira and Gignoux (2011); Lanjouw and Rao (2011).

[3] This literature is abundant. Some important examples are Gastwirth (1975), Butler and McDonald (1987), Dagum (1987), Jenkins (1994), van Krem (2009), Gradin et al. (2010) and del Rio et al. (2011).

ities when they are not detrimental to the group of concern. The most recent and neat definition by del Rio et al. (2011), based on the work of Jenkins (1994), applies to comparisons of actual distributions against counterfactuals. This approach effectively deals with distributions of the same population size.

This paper provides a review of inter-distributional inequality (IDI) indices. Considering the renewed interest in measuring between-group inequalities with a focus on those which are detrimental to one specific group, the review explores how we can measure inequalities with metrics that are exclusively sensitive to only one specific group's disadvantage. Since there are several indices of inter-distributional inequality (IDI) already available, I propose some ways of measuring this focused inequality by suggesting some amendments to existing indices, which do not measure IDI with a focus on specific disadvantages in their current forms. Conducting the review with such a concern for exclusive sensitivity to inequality detrimental to *one* group, is also helpful for understanding, and clarifying, how existing IDI indices deal with, and react to, inequalities that are detrimental to *different* groups.

With these two purposes in mind, the paper begins with a discussion of the concept of group-specific disadvantage focus (GDF). Since there are different plausible ways of operationalizing this concept in an axiom, I propose two options for a property of a(n index's) sensitivity to inequality that is detrimental exclusively to one specific group. Both properties are applicable to IDI indices that deal with populations of *different* size. The first property is called quantile group-specific disadvantage focus (QGDF). The second property is called overlap group-specific disadvantage focus (OGDF). An advantage of the first option is that it can be related to indices that measure inequality on quantile space, or on probability space.

This review of existing indices also evaluates whether they are informative, or not, regarding other interesting features related to IDI comparisons. For instance, do they unambiguously pinpoint situations in which two distributions are identical? Remarkably most of them do not. Moreover, the two proposed operationalizations of GDF, i.e. QGDF and OGDF, are related, respectively, with first-order stochastic dominance and the degree of distributional overlap. Hence I also assess whether the IDI indices are informative as to the absence of distributional overlap and/or presence of first-order stochastic dominance. I propose some further amendments that improve the indices' informative content on these features.

There are several indices of inter-distributional differences. Therefore in this paper I focus on IDI indices that are characterized by: i) being useful specifically for two-group comparisons, ii) being more informative than just comparing two means, and iii) being useful when the two distributions have different sample sizes.[4] I first review the PROB index by Gastwirth (1975), followed by the closely related family of indices of relative distributions, discussed by Handcock (1999) and by Le Breton et al. (2008). These indices, which map from probability functions, do not fulfill QGDF because in some cases they compensate inequalities detrimental to one group with inequalities detrimental to the other group, while in other cases they just add up the two forms of inequalities together. In relation to that, most of these indices do not distinguish situations of equal distributions from other situations wherein there is inequality. Finally, while several of these indices are helpful to pinpoint situations of lack of distributional overlap, they are not informative as to the presence of first-order stochastic dominance. I propose some simple amendments to these indices that render them more informative about the aforementioned features; chiefly, the extent of group-specific disadvantage.

I then review the family of quantile-based indices of Ebert (1984) and Vinod (1985). As in the previous case, these indices do not fulfill QGDF, either because they compensate group-specific detrimental inequalities or because they add them up indiscriminately. As for other features, while Ebert's index does differentiate, unambiguously, between distributional equality and other situations, Vinod's does not. Neither index is helpful to detect absolute lack of distributional overlap. I propose simple amendments to these indices that render them both in fulfillment of QGDF, more informative in terms of presence of first-order stochastic dominance, and in compliance with other properties, like scale invariance, which are desirable for certain IDI comparisons.

---

[4] When sample sizes are identical the literature on counterfactual comparisons, e.g. del Rio et al. (2011), provides the relevant indices. However, even without the explicit purpose, mobility indices may also be amendable to render them suitable for the analysis of between-group inequalities with GDF and identical populations. Good examples of such indices are provided by Cowell's measures of distributional change (Cowell, 1985), by Fields (1996); Fields and Ok (1999) and by Schluter and van de Gaer (2011).

Then I turn to an assessment of the family of indices proposed by Dagum (1980, 1987). These indices compare each value of the wellbeing variable in one group against all the values present in the other group. The Dagum family does not fulfill QGDF and does not distinguish a situation of distributional equality from other cases of inequality. However the Dagum family hosts the best examples of indices satisfying OGDF. Accordingly they are useful in pinpointing situations of absence of distributional overlap.

The next family under review comprises the indices based on incomplete moments (Butler and McDonald, 1989). These include the indices by Butler and McDonald (1987) and those by Deutsch and Silber (1997). The review shows that these indices do not fulfill QGDF. However, with amendments, some of them can fulfill QGDF, while some others can fulfill OGDF. The review also shows the close relationship between these indices and those of the Dagum family. Indices based on incomplete moments are not informative about distributional equality or first-order stochastic dominance. However, they are also useful for the identification of absence of distributional overlap. Finally, I complete the review with an appraisal of the family of ethical distance indices proposed by Shorrocks (1982), and axiomatically characterized by Chakravarty and Dutta (1987). Ethical distance indices are different from the previous ones in that they compare equally-distributed-equivalent (EDE) standards from the distributions.[5] This requires a first aggregation step in which each distribution is mapped into its respective EDE standard. Then two such standards are compared. Despite this difference, I include these indices in the review because they have been proposed as alternatives to, and contrasted with, some IDI indices (see Shorrocks, 1982). I explain why, notwithstanding their merit and appeal, this family of indices does not fulfill notions of GDF. The indices are also of little help for pinpointing situations of distributional equality, first-order stochastic dominance and/or absolute absence of distributional overlap.

The next section introduces the basic notation and a minimum set of properties that IDI indices are expected to fulfill. The main subsection defines the two properties that operationalize the concept of group-specific disadvantage focus. Then the review and proposal of new amendments is done in subsequent sections:

[5] EDE standards were introduced by Atkinson (1970).

one for the PROB index and indices based on relative distributions; followed by a section on the quantile indices; then followed by a section on the Dagum family, a section on incomplete moment indices, and a section on ethical distance indices. The paper ends with some concluding remarks.

2 Notation and Basic Properties

Consider two population groups, one with distribution X and size M, and the other one with distribution Y and size N. Hence $X(M) := (x_1, x_2, ..., x_M)$ and $Y(N) := (y_1, y_2, ..., y_N)$. Group sizes can be different. The density function of X is $f_X(z)$ and its cumulative distribution function (cdf) is $F_X(z)$, where z is a wellbeing continuous variable. As usual, $\int_{-\infty}^{\infty} f_X(z)dz = 1$ and $F_X(z) = \int_{-\infty}^{z} f_X(s)ds$. The inverse of the cdf yields the quantiles of X. These quantiles are defined as: $x(p) \equiv F_X^{-1}(p)$, where $p \in [0,1]$. Effectively, $p = F_X(x(p))$. An IDI index, $\mathscr{D}(X;Y)$, maps from $\mathbb{R}^M \times \mathbb{R}^N$ to the real line.

Now the first two properties that are reasonable for IDI indices are population invariance (or principle of population) and scale invariance, both traditional axioms from the wellbeing measurement literature. In the case of two group distributions, an IDI index is said to fulfill population invariance if and only if its value is not affected by an identical replication of members *within* each group, although the number of replications can vary *between* groups: $\mathscr{D}(X(M);Y(N)) = \mathscr{D}(X(\lambda_M M);Y(\lambda_N N))$, where λ_M and λ_N are two different scalars. An IDI index is said to fulfill scale invariance if and only if its value is not affected by multiplying all the values of both distributions by the same scalar; i.e. $\mathscr{D}(X(M);Y(N)) = \mathscr{D}(\lambda x_1, ..., \lambda x_M; \lambda y_1, ..., \lambda y_N)$, where λ is a scalar. Fulfillment of these two properties ensures that the IDI comparison is not affected either by changes in relative population sizes per se, or by the unit of measurement used to quantify the wellbeing attribute (e.g. income expressed in different currencies). An alternative, or complement, to scale invariance is translation invariance (e.g. see Ebert, 1984, axiom 2; and Magdalou and Nock, 2011). An IDI index fulfills translation invariance if and only if its value is not affected by adding the same scalar to all the values of both distributions, i.e. if $\mathscr{D}(X(M);Y(N)) = \mathscr{D}(x_1 - \lambda, \ldots, x_M - \lambda; y_1 - \lambda, \ldots, y_N - \lambda)$.

The next desirable property is related to the ability of an IDI index to identify situations of distributional equality (DE). DE holds if and only if $f_X(z) = f_Y(z) \forall z$. DE can also be expressed in terms of cumulative distribution functions (i.e. $F_X(z) = F_Y(z) \forall z$), or in terms of quantiles (i.e. $x(p) = y(p) \forall p \in [0,1]$). Whenever, in the literature, the index is required to be sensitive to the presence of DE, it is designed to take its minimum value under DE, which is usually zero. However this property of sensitivity to DE can take a weak form and a strong form. The weak form of the property is the following:

Axiom 1 *Weak Sensitivity to Distributional Equality (WSDE): An IDI index is weakly sensitive to distributional equality if:* $f_X(z) = f_Y(z) \forall z \rightarrow \mathscr{D}(X;Y) = 0$.

Axiom 1 is basically property (2a) in Shorrocks (1982). WSDE requires the index to take its minimum value (zero) whenever there is distributional equality. However, in principle, an index satisfying WSDE could take that same value in alternative situations of distributional *inequality*. Hence, for an IDI index to be most informative regarding the presence of DE, it should take its minimum value only when DE holds. That is, it should fulfill the following property:

Axiom 2 *Strong Sensitivity to Distributional Equality (SSDE): An IDI index is strongly sensitive to distributional equality if:* $f_X(z) = f_Y(z) \forall z \leftrightarrow \mathscr{D}(X;Y) = 0$.

SSDE is Ebert's reflexivity property (but expressed in terms of densities; Ebert, 1984, p. 268).

A focus on Group-specific Disadvantage

It is much easier to define a situation of DE than to characterize all the different possible forms of IDI, even though the former is rarely observed in practice. Most of the literature on IDI comparisons based on indices for distributions with different population size, has taken one of two conceptual approaches to measure IDI. One approach is to measure IDI as a cumulative departure from DE. In this approach, between-group distributional differences are aggregated without distinguishing whether these differences are favourable, or not, to any specific group. That's the route followed by Ebert (1984) and Chakravarty and Dutta (1987) . A property of *symmetry*, whereby the indices are unaffected by switching the two distributions around, is usually advocated in this first approach; i.e. $\mathscr{D}(X;Y) = \mathscr{D}(Y;X)$.

A second approach acknowledges, more explicitly, that some distributional differences can be said to favour one specific group over another one. But then the measures of these differences, quantifying relative advantage for each group respectively, are pitted against each other, in order to derive an index of *net advantage*. This is the approach followed by Butler and McDonald (1987) , Vinod (1985) and Dagum (1987) , among others. Recently, on the other hand, the literature on discrimination measurement based on counterfactual comparisons is advocating a third approach: indices that are sensitive only to distributional differences that are favourable (or detrimental) to one group in particular (e.g. del Rio et al., 2011). This approach is not new. For instance, the contribution of Dagum (1980) was already in that spirit. However, there has not been an exhaustive discussion of how to operationalize the notions of group-specific disadvantage focus (GDF), i.e. an exclusive sensitivity to inequalities that are detrimental to one specific group, as advocated by the literature on counterfactual comparisons. Without claiming, or aiming, to explore all the possible options, in this section I propose two intuitive and meaningful ways of operationalizing the concept of GDF. These are used in the rest of the review to assess, and better understand, the behaviour of the IDI indices.

In order to introduce the first operationalization, QGDF, it is worth starting by noting that two distributions may be different in many ways. For instance, they may have different means. Or even if they have equal means, they may differ in their average spread, skewness or kurtosis. More importantly, from a wellbeing perspective, these inter-distributional differences may render one distribution more desirable than the other one as a "lottery". The stochastic dominance literature discusses this type of partial-ordering comparisons. But even when stochastic dominance relationships do not hold over the whole admissible range of a wellbeing variable, one may be able to make statements about whether certain parts of a distribution are more advantageous for one group vis-a-vis another one. For instance, consider income distributions A and B. Both are symmetric and have equal means, but people in A are closely clustered around the mean, whereas people in B exhibits significantly higher variance. In that case, one may find that the poorest people in B are poorer than the poorest people in A whereas the richest people in B are richer than the richest people in A. In such situations, one may be interested in measuring only the amount of inequality that is detrimental to, say, A. If that is the purpose then one may want to have an index that is sensitive to the fact that the richest people in A are poorer than the richest people in B, while being *insensitive* to the fact that the poorest people in A are better-off than the poorest people in B.

Such a focused approach could be justified, for instance, by the concept of inequality of opportunity put forward by Roemer (1998). He proposed that in order to measure inequality of opportunity between different groups of people (defined in terms of their specific sets of life circumstances), people in a given percentile within their own group should be compared against people from the same percentile in a different group. The percentile is used as a measure of relative effort within the group, under certain assumptions. An operationalization of GDF, inspired by this inequality-of-opportunity perspective, is the following definition of Quantile Group-specific Disadvantage Focus (QGDF) for an index that is meant to capture only inequalities that are detrimental to a distribution X when compared to a distribution Y:

Definition 1 *An index measuring inter-distributional inequality between Y and X satisfies the property of quantile group-specific disadvantage focus (QGDF) if and only if it is sensitive to the gap $y(p)-x(p)\,\forall p\in[0,1]\mid y(p)\geq x(p)$ and it is insensitive to the gap $y(p)-x(p)\,\forall p\in[0,1]\mid y(p)\leq x(p)$. In particular, the index does not decrease (increase) if the gap $y(p)-x(p)$ increases (decreases) given that initially $y(p)\geq x(p)$ and the index does not react to changes in $y(p)-x(p)$ as long as $y(p)\leq x(p)$ before and after the changes.*

The sensitivity part of Definition 1 is similar to the monotonicity axiom of del Rio et al. (2011) for counterfactual comparisons, while the insensitivity part is similar to their focus axiom. Now Definition 1 can be expressed also in terms of cumulative probabilities. This dual expression is useful for IDI indices that map from cumulative probability space, e.g. the PROB index Gastwirth (1975) and those based on relative distributions (e.g. Le Breton et al. (2008) , and Handcock (1999)). It stems from the fact that, if it is true that $y(p) \geq x(p)$ over the interval $p \in [\underline{p}, \overline{p}]$, and is also the case that $y(\underline{p}) = x(\underline{p})$ and $y(\overline{p}) = x(\overline{p})$, then the following equation holds:

$$\int_{x(\underline{p})}^{x(\overline{p})} [F_X(z) - F_Y(z)]_+ dz = \int_{\underline{p}}^{\overline{p}} [y(p) - x(p)]_+ dp, \tag{1}$$

where $[m]_+ \equiv \max\{m, 0\}$. In words, (1) says that the sum of positive gaps, $y(p) - x(p)$, over the interval $[\underline{p}, \overline{p}]$, is equal to the sum of positive gaps of cdfs, $F_X(z) - F_Y(z)$, in the interval $[x(\underline{p}), x(\overline{p})]$ (or $[y(\underline{p}), y(\overline{p})]$) defined by $[\underline{p}, \overline{p}]$. Hence a dual for definition 1 can be proposed:

Definition 2 *An index measuring inter-distributional inequality between Y and X satisfies the property of quantile group-specific disadvantage focus (QGDF) if and only if it is sensitive to the gap $F_X(z) - F_Y(z) \mid F_X(z) \geq F_Y(z)$ and it is insensitive to the gap $F_X(z) - F_Y(z) \mid F_X(z) \leq F_Y(z)$. In particular, the index does not decrease (increase) if the gap $F_X(z) - F_Y(z)$ increases (decreases) given that initially $F_X(z) \geq F_Y(z)$ and the index does not react to changes in $F_X(z) - F_Y(z)$ as long as $F_X(z) \leq F_Y(z)$ before and after the changes.*

Note also the connection between the two definitions and first-order stochastic dominance. The following three statements are identical:

(i) Distribution Y (weakly) first-order dominates X

(ii) $y(p) \geq x(p) \forall p \in [0, 1]$.

(iii) $F_X(z) \geq F_Y(z) \forall z$.

Hence indices that satisfy QGDF are expected to be informative about the presence of first-order stochastic dominance, especially in its weak form, as is shown below.

Notwithstanding its appeal, QGDF does not exhaust all the possible ways to operationalize GDF. An alternative, to be considered in this review, is implicit in the work of Dagum (1980, 1987) , whose indices compare every value in Y against every value in X and react only to the gaps that favour one specific group. Hence, letting y and x be values from Y and X respectively, an operationalization of GDF, inspired by such notion, is the following definition of Overlap Group-specific Disadvantage Focus (OGDF):

Definition 3 *An index measuring inter-distributional inequality between Y and X satisfies the property of overlap group-specific disadvantage focus (OGDF) if and only if it is sensitive to the gap $y - x$ $\forall x \in X \wedge y \in Y \mid y \geq x$ and it is insensitive to the gap $y - x$ $\forall x \in X \wedge y \in Y \mid y \leq x$. In particular, the index does not decrease (increase) if the gap $y - x$ increases (decreases) given that initially $y \geq x$ and the index does not react to changes in $y - x$ as long as $y \leq x$ before and after the changes.*

The property has the word overlap in its name because indices satisfying it should be informative about the absence of distributional overlap. For instance if all the possible gaps $y - x$ are negative, an index measuring IDI detrimental to X may be insensitive to all possible gaps and take a specific value (e.g. zero) as a result. At the same time whenever that happens, the highest value in Y is clearly below the lowest value in X. Therefore there is no distributional overlap between the two. The family of indices proposed by Dagum provides the best example of this relationship.

The aforementioned provide a minimum set of desirable properties for IDI indices. By contrast, among several properties from the traditional inequality literature, there are some which may not be desirable in the context of IDI comparisons. A prominent one is the principle of transfers, which in the inequality literature states that inequality within a group's distribution should decrease after a progressive transfer from a richer person to a poorer person, that does not affect their relative ranking. Following Bishop et al. (2011) note that when Y is obtained from X using a progressive transfer, Y is going to have some quantiles higher than X's at the bottom of the distribution, while some quantiles lower than X's at the top. An index that satisfies a property based on an operationalization of GDF should not be simultaneously sensitive to the two effects of a progressive transfer. Also considering the contradictory effects of progressive transfers in general IDI comparisons, Magdalou and Nock (2011) state that: "[t]he notion of progressive transfer does not make sense in the general situation where the reference distribution is not egalitarian" (p. 2445).

3 The PROB Measure and Relative Distributions: Review and Amendments

The PROB measure of Gastwirth (1975) is defined as: $PROB_Y \equiv \int_{-\infty}^{\infty} [1 - F_X(z)] f_Y(z)\, dz$. It measures the probability of finding an individual in X having at least as much of z as a random individual in Y (hence Y is the *reference* distribution and X is the *compared* distribution). Since it maps from cumulative probabilities, $PROB_Y$ fulfills the basic properties of population invariance, scale invariance and translation invariance. However, $PROB_Y$ does not fulfill QGDF because it pits inequalities that are detrimental to X against inequalities that are detrimental to Y. To see this notice the following simple decomposition stemming from adding and subtracting $\int_{-\infty}^{\infty} F_Y(z) f_Y(z)\, dz$ and considering that $\int_{0}^{\infty} F_Y(z) f_Y(z)\, dz = 0.5$:

$$PROB_Y = \int_{-\infty}^{\infty} [F_Y(z) - F_X(z)]_+ f_Y(z)\, dz - \int_{-\infty}^{\infty} [F_X(z) - F_Y(z)]_+ f_Y(z)\, dz + 0.5 \tag{2}$$

Hence it is clear from (2) that inequalities detrimental to Y $\left([F_Y(z) - F_X(z)]_+\right)$ are compensated with inequalities detrimental to X $\left([F_X(z) - F_Y(z)]_+\right)$. For this reason $PROB_Y$ cannot distinguish a situation of distributional equality from others of distributional inequality. More precisely, it fulfills WSDE, but not SSDE. Whenever $f_X = f_Y$, $PROB_Y = 0.5$.[6] However the reverse is not true, as is clear from (2). As it stands, $PROB_Y$ does not take any specific value that signals first-order stochastic dominance, which is consistent with its inability to fulfill QDF. By contrast, $PROB_Y$ is useful to pinpoint absences of distributional overlap. For instance: $PROB_Y = 0 \leftrightarrow F_X\left(z^Y_{\min}\right) = 1$, where $z^Y_{\min}$ is the minimum value for which Y has support. When $PROB_Y = 0$ the richest person in X is not better

[6] When $PROB_Y < 0.5$ the distribution of Y has some advantage over X's such that the probability of finding someone in X having at least as much of z as a randomly chosen person from Y is lower than the probability that would ensue from identical distributions. A similar interpretation, favouring X's distribution over Y's, ensues when $PROB_Y > 0.5$.

off than the poorest person in Y (whose value of z is $z^Y_{\min}$). On the other extreme: $PROB_Y = 1 \leftrightarrow F_X\left(z^Y_{\max}\right) = 0$. When $PROB_Y = 1$ the poorest person in X is richer than the richest person in Y.

In summary: $PROB_Y$ does not satisfy QGDF and it does not exclusively identify distributional equality or first-order stochastic dominance, but it does identify lack of distributional overlap. However, some simple measures based on $PROB$ can be used in conjunction with it in order to provide more information on the abovementioned distributional features. I propose the following:

$$PROB^{\alpha}_{Y}(Y-X) \equiv (\alpha+1)\int_{-\infty}^{\infty}[F_Y(z)-F_X(z)]^{\alpha}_{+} f_Y(z)\,dz, \tag{3}$$

$$PROB^{\alpha}_{Y}(X-Y) \equiv (\alpha+1)\int_{-\infty}^{\infty}[F_X(z)-F_Y(z)]^{\alpha}_{+} f_Y(z)\,dz, \tag{4}$$

where α is a parameter and the subindex Y in $PROB^{\alpha}_{Y}(Y-X)$ denotes that the reference distribution is Y.[7] It is straightforward to note that both (3) and (4) fulfill QGDF. It is also the case that: $f_X = f_Y \leftrightarrow \left(PROB^{\alpha}_{Y}(Y-X) = 0 \wedge PROB^{\alpha}_{Y}(X-Y) = 0\right)$. Hence, even though, separately, both indices only fulfill WSDE; used together, they identify distributional equality (for any positive value of α). Two interesting sets of indices are related to the cases when $\alpha = 0$ and $\alpha = 1$. When $\alpha = 0$ the indices help to pinpoint situations of first-order stochastic dominance since: $PROB^{0}_{Y}(Y-X) = 1 \leftrightarrow X \succeq_{FD} Y$, where $\succeq_{FD}$ reads "weakly first-order dominates".[8] When $\alpha = 1$, both $PROB^{1}_{Y}(Y-X)$ and $PROB^{1}_{Y}(X-Y)$ are sensitive to changes in the quantile gaps, and they are helpful to detect absence of distributional overlap because: $PROB^{1}_{Y}(X-Y) = 1 \leftrightarrow F_X\left(z^Y_{\max}\right) = 0$. When $\alpha = 1$ the following relationship holds:

$$2PROB_Y = PROB^{1}_{Y}(Y-X) - PROB^{1}_{Y}(X-Y) + 1 \tag{5}$$

Since these indices map from probability space, it is difficult to ascertain whether they fulfill OGDF.

[7] Analogue indices can be defined using X as the reference distribution.

[8] Likewise: $PROB^{0}_{Y}(X-Y) = 0 \leftrightarrow X \succeq_{FD} Y$.

Relative distributions

The $PROB_Y$ index and this paper's amendments are closely related to indices stemming from discrimination curves based on cumulative relative distributions. A cumulative relative distribution function maps the cumulative distribution of a reference distribution, $F_Y(z)$, into the interval $[0,1]$. Specifically, the cumulative distribution function is: $G_{X/Y}(F_Y) \equiv F_X[y(F_Y)]$ and the discrimination curve is the drawing of $G_{X/Y}(F_Y)$ on an horizontal axis of F_Y .[9] Le Breton et al. (2011) studied dominance conditions for the discrimination curve, and the relationship between $PROB_Y$ and second-order dominance for discrimination curves. In a previous, lenghtier contribution (Le Breton et al., 2008), they proposed some indices based on the area between the discrimination curve and the 45 degree line. Two of their measures are relevant for this paper:

$$AAD = \int_0^1 \left|G_{X/Y}(F_Y) - F_Y\right| dF_Y \tag{6}$$

$$C = \int_0^1 \left[G_{X/Y}(F_Y) - F_Y\right] dF_Y, \tag{7}$$

where AAD is the average absolute deviation between the discrimination curve and the distributional equality line (45 degree).[10] Now note that: $2AAD = PROB_Y^1(Y-X) + PROB_Y^1(X-Y)$ and $2C = PROB_Y^1(X-Y) - PROB_Y^1(Y-X)$.Hence both AAD and C do not fulfill QGDF. In the first case both types of inequalities, i.e those detrimental to X and those detrimental to Y, are added up; while in the second case they compensate each other. As for other distributional features, C fulfills WSDE, but not SSDE. By contrast, $AAD = 0 \leftrightarrow f_X = f_Y$; i.e. AAD fulfills SSDE. Neither AAD nor C are informative about first-order stochastic dominance, but both are informative about the absence of distributional overlap since: $AAD, C = 0.5 \leftrightarrow F_X\left(z^Y_{\min}\right) = 1$.

Given the connection between $PROB_Y$, on one hand, and AAD and C, on the other, the amendments proposed for the former are also relevant for the

[9] Hence when: $f_X = f_Y$ the discrimination curve is a 45 degree line.

[10] Le Breton et al. (2008) use different names for these indices.

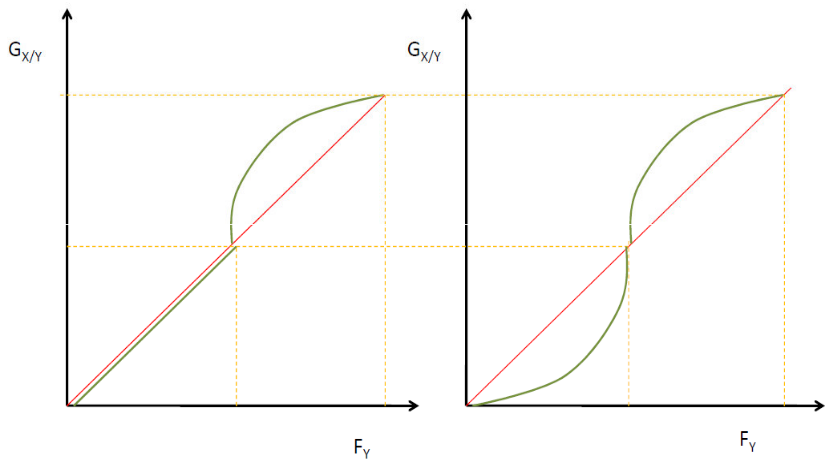
latter. An additional proposal can be made by combining $PROB_Y^1(Y-X)$ and $PROB_Y^1(X-Y)$ with AAD:

$$R_Y(Y-X) = \frac{PROB_Y^1(Y-X)}{2AAD} \tag{8}$$

$$R_Y(X-Y) = \frac{PROB_Y^1(X-Y)}{2AAD} \tag{9}$$

$R_Y(Y-X)$ provides a measure of the proportion of the inter-distributional inequality that is detrimental to Y, when the distribution of Y is taken as reference. An example of its usefulness is provided by the two cases in Figure 1. $PROB_Y^1(X-Y)$ yields the same value for both cases. By contrast, $R_Y(X-Y) = 1$ for the case of the left panel, whereas $R_Y(X-Y) < 1$ for the case of the right panel.

Figure 1: Left panel: the two CDFs overlap in part of their common support. Right panel: the two CDFs cross once



Handcock (1999) make a few further suggestions for indices based on cumulative relative distributions, and inspired by measures of goodness of fit. One such index is the following, based on the stastistic used in the Cramer-von Mises test:

$$CM = \int_0^1 \left| G_{X/Y}(F_Y) - F_Y \right|^2 dF_Y \tag{10}$$

Again, note that $3CM = PROB_Y^1 (Y - X) + PROB_Y^1 (X - Y)$. Hence CM behaves similarly to AAD, i.e it does not fulfill QGDF because it adds the two types of inequalities; but it fulfills SSDE. Likewise, it is not informative about first-order dominance, whereas it is informative about lack of overlap since: $CM = \frac{1}{3} \leftrightarrow F_X\left(z^Y_{\min}\right) = 1$. Handcock (1999) also suggest using several divergence measures from the statistic literature, but mapping from ratios of the density functions (see their Table 5.1, p. 65). For instance, two measures from that list, which they focus on, are:

$$D\chi = \int_0^1 (\frac{f_X}{f_Y} - 1) dF_Y \tag{11}$$

$$DKL = \int_0^1 log(\frac{f_X}{f_Y}) \frac{f_X}{f_Y} dF_Y \tag{12}$$

$D\chi$ is inspired by Pearson's chi-square statistic whereas DKL is based on the Kullback-Leibler divergence measure. Unlike other measures based on relative distributions, it is not easy to render divergence measures like (11) and (12) (and those in Table 5.1 in Handcock, 1999) in fulfillment of any operationalization of GDF due to their mapping from density functions. On the other hand, all these measures satisfy SSDE, i.e. they are good at pinpointing situations of distributional equality. In fact, they have long been used to test the equality of two distributions. Besides distributional equality, these measures are not informative of other distributional features of interest (e.g. stochastic dominance). Moreover, when applied to continuous variables, their computation requires techniques based on Kernel densities (Handcock, 1999). Hence, for the purpose of IDI comparisons based on indices mapping from probabilities, those that rely on cumulative probabilities should be preferred to those depending on densities.

4 The Indices by Ebert and Vinod: Review and Amendments

In a seminal contribution Ebert (1984) axiomatically characterized a family of indices based on Minkowski distances that is useful for IDI measurement. These indices are direct functions of the percentile gaps. The family for two groups with different population sizes is:

$$d^r(X,Y) \equiv \left[\int_0^1 |y(p) - x(p)|^r dp \right]^{\frac{1}{r}} \quad \forall r \geq 1 \tag{13}$$

This proposal is similar to that of Vinod (1985) in that both are direct functions of the percentile gaps. Vinod's measure of "overall economic advantage" is:

$$V(X,Y) \equiv \int_0^1 [y(p) - x(p)] dp = \mu_Y - \mu_X, \tag{14}$$

where μ_Y is the mean of distribution Y.[11] It is easy to check that both d^r and V do not fulfill QGDF. As in the case with AAD, d^r is sensitive to both types of inequalities, which are added up by the index. By contrast, V compensates them. For that reason V fulfills WSDE but not SSDE, i.e. it is not useful to pinpoint distributional equality; whereas, like AAD, d^r satisfies SSDE (property 2a in Ebert, 1984). Here it is worth noting that the lack of fulfillment of QGDF, by d^r, is a logical consequence of the index's fulfillment of a symmetry property which Ebert (1984) adapted from Shorrocks (1982): An IDI index satisfying symmetry should not take a different value when the distributions of X and Y are switched around. Clearly, the property of symmetry rules out any form of GDF.

Neither d^r nor V are informative regarding situations of first-order stochastic dominance or absence of overlap. However simple amendments, which relate to both d^r and V, fulfill QGDF and, combined, provide more information about distributional equality and first-order dominance. The two amended indices are:

[11] Vinod also considered partial measures of economic advantage, e.g. computations of (14) in a restricted quantile range. However, unlike Ebert, Vinod did not characterize his measures axiomatically.

$$d^r_{Y-X} \equiv \int_0^1 |y(p)-x(p)|^r_+ \, dp \;\; \forall r \geq 1 \tag{15}$$

$$d^r_{X-Y} \equiv \int_0^1 |x(p)-y(p)|^r_+ \, dp \;\; \forall r \geq 1 \tag{16}$$

Clearly, both (15) and (16) fulfill QGDF, at the expense of symmetry. Together these indices also pinpoint DE because: $d^r_{Y-X} = d^r_{X-Y} = 0 \leftrightarrow f_X = f_Y$. They also detect first-order dominance since: $\left[d^r_{Y-X} > 0 \wedge d^r_{X-Y} = 0\right] \leftrightarrow X \succeq_{FD} Y$. However neither the amendments nor the original indices take specific values if and only if there is absence of overlap; except, in the odd case for $d^r(X,Y)$, when either $y(p)$ or $x(p)$ are equal to zero for all p. Finally the amendments are related to d^r and V according to the following expressions:

$$[d^r(X,Y)]^r = d^r_{Y-X} + d^r_{X-Y} \;\; \forall r \geq 1 \tag{17}$$

$$V(X,Y) = d^1_{Y-X} - d^1_{X-Y} \tag{18}$$

Unlike the indices in the previous section, d^r and V are not bounded from above and do not fulfill scale invariance. The latter, in the case of d^r, is directly due to Ebert's requirement that his indices fulfill a property of linear homogeneity, whereby multiplying all values of X and Y by a common scalar should translate into a multiplication of d^r by the same scalar (Ebert, 1984, Axiom 1, p. 269). Clearly, V, (15) and (16), also satisfy linear homogeneity. Such property may not always be desirable. For instance, if one does not want an IDI comparison over income to be affected by the choice of currency. However scale invariance can be met, in conjunction with a normalization property that caps the indices from above, by dividing the indices by their maxima:

$$dd^r(X,Y) \equiv \frac{d^r(X,Y)}{\left[\int_0^1 y(p)^r \, dp\right]^{\frac{1}{r}} + \left[\int_0^1 x(p)^r \, dp\right]^{\frac{1}{r}}} \tag{19}$$

$$VV(X,Y) \equiv \frac{V(X,Y)}{\mu_Y + \mu_X} \tag{20}$$

The amended indices, (15) and (16), can be normalized the same way as in (19). Of course, in all cases like (19) and (20), fulfillment of linear homogeneity is relinquished. All indices in this section also fulfill population invariance and translation invariance (Ebert, 1984, Axiom 2, p. 269).

5 The IDI Indices of Dagum

Dagum's rich family of indices deserves special attention for two reasons. Firstly, in the IDI literature working with continuous variables and different distributional sizes, it is the only family whose indices map explicitly from differences in values from the two distributions, e.g. $y-x$. Secondly, some of the indices provide a clear example of fulfillment of some operationalizations of GDF, but not others. In this section I start reviewing Dagum's later contributions first, because, as shown in the section, his earlier contributions are actually better suited to capture notions of GDF. Hence, in a sense, these can work as amendments of the later contributions, if the concern is to measure IDI with some operationalization of GDF.

Dagum (1987) proposed a measure of relative economic affluence (REA) which, in this paper's notation, is defined as: $D_{X/Y} = 1 - \frac{d_Y}{d_X}$, where:

$$d_X = \int_0^\infty dF_Y(y) \int_0^y (y-x)\, dF_X(x), \tag{21}$$

$$d_Y = \int_0^\infty dF_X(x) \int_0^x (x-y)\, dF_Y(y). \tag{22}$$

$D_{X/Y}$ fulfills population invariance, translation invariance and scale invariance, but it does not fulfill either OGDF or QGDF. With respect to OGDF, it is clear that $D_{X/Y}$ depends on $y-x$ when $y>x$ *and* on $x-y$ when $x>y$. As for QDF, first, note that $d_X - d_Y = \mu_Y - \mu_X$.[12] Hence: $D_{X/Y} = \frac{\mu_Y - \mu_X}{d_X} = \frac{V(X,Y)}{d_X}$. This means that D compensates quantile gaps that are detrimental to different groups in the same way that V does. So neither can fulfill QGDF. Like V, D fulfills WSDE but not SSDE, i.e. it is not useful to pinpoint distributional equivalence, since the necessary and sufficient requirement for $D_{X/Y} = 0$ is: $\mu_X = \mu_Y$.

[12] This result stems from equations (4) and (5) of Dagum (1987, p. 6).

$D_{X/Y}$ does not identify situations of first-order dominance because, even though $X \succeq_{FD} Y$ implies $d_Y \geq d_X$, the reverse is not true. By contrast, $D_{X/Y}$ is useful for pinpointing absence of overlap. When the richest person in Y is poorer than the poorest person in X then $d_X = 0$ (and $d_Y = \mu_X - \mu_Y$) and the reverse is also true. When the richest person in X is poorer than the poorest person in Y then $d_Y = 0$ (and $d_X = \mu_Y - \mu_X$) and the reverse is also true. d_X compares every member of Y against all the people in X who have less income and quantifies the respective gaps. That is why the measures are well suited to detect absence of overlaps without resorting to quantiles or probabilities: if everybody in X is richer than everybody in Y then $d_X = 0$. The reverse is true because, unless the two distributions are degenerate and equal to each other, $d_X = 0$ requires that $f_X(y) = 0$ for every value of y on the support of Y (and then $F_X(y) = 0$ over the same support).

The most straightforward alternatives to $D_{X/Y}$ that may fulfill some forms of GDF, are using its basic constituent statistics, i.e. d_X and d_Y. In fact, in an earlier contribution, Dagum (1980) proposed using d_X, or d_Y, as the basic statistics for measures of economic distance normalized by their respective minima and maxima.[13]

Now, clearly, d_X and d_Y fulfill OGDF; and, as other measures fulfilling the notions of GDF considered in this review, they are not symmetric. Like $D_{X/Y}$, d_X and d_Y do not identify situations of first-order stochastic dominance, but they are good for detecting absence of distributional overlap due to the aforementioned reasons.

[13] More generally, Dagum also suggested considering the following family of statistics based on generalized means, even though he focused on d_X^1 :

$$d_X^r = \left[\int_0^\infty dF_Y(y) \int_0^y (y-x)^r \, dF_X(x)\right]^{\frac{1}{r}}, \; r \neq 0 \tag{23}$$

$$d_X^0 = e^{\int_0^\infty dF_Y(y) \int_0^y \ln(y-x) dF_X(x)}. \tag{24}$$

However d_X does not fulfill QGDF. For instance, following Shorrocks (1982), d_X can be decomposed in the following way:

$$d_X = \frac{V(X,Y)}{2} + \frac{1}{2}\int_0^1 \int_0^1 \mid y(p_y) - x(p_x) \mid dp_x dp_y, \tag{25}$$

where p_x and p_y are percentiles of X and Y, respectively. Hence d_X compensates *and* adds up quantile gaps that are detrimental to diffferent groups. Likewise, the family of generalized means, i.e. (23) and (24), fulfills OGDF, but not QGDF. The reason for the latter is that the difference $y-x$, in the respective formulas, can be expressed as: $y-x = y(p_y) - y(p_x) + y(p_x) - x(p_x)$, using the notation introduced in (25). Hence, even though $y > x$, in some cases $y(p_x) > x(p_x)$, whereas in others $y(p_x) < x(p_x)$. Therefore quantiles gaps that are detrimental to different groups are compensated, contrary to the requirements of QGDF.

Inability to fulfill QGDF should not be considered a serious drawback for d_X, even if one is interested in IDI indices sensitive to some notion of GDF, because d_X does capture alternative meaningful concepts of GDF, e.g. OGDF. By contrast, a problematic feature of this index is its inability to fulfill even WSDE. As shown by Shorrocks (1982), when there is distributional equality $d_X = d_Y = \mu_X G(X) = \mu_Y G(Y)$, where $G(X)$ is the Gini coefficient of X. However the reverse is not true because two different distributions can have the same mean and Gini coefficient. For instance if Y is obtained from X by performing two transfers of the same amount, but one regressive and one progressive, involving two pairs of individuals in different parts of the distribution, then both distributions, despite being unequal, have the same mean and the same value for the Gini coefficient. [14]

Unlike $D_{X/Y}$, d_X does not fulfill scale invariance. Nor it is normalized. However, the following amendment of (21) fulfills scale invariance and is normalized so that it is equal to 1 if and only if $F_X\left(z^Y_{\min}\right) = 1$, and it is equal to 0 if and only if $F_X\left(z^Y_{\max}\right) = 0$:

$$DN_X = \frac{d_X}{d_X + d_Y}. \tag{26}$$

[14] For instance if $X = (1,2,3,4)$ and $Y = (0.5, 2.5, 3.5, 3.5)$.

A similar amendment is applicable to (24). Interestingly, when $\mu_Y = \mu_X$: $DN_X = DN_Y = 0.5$

6 IDI Indices Based on Incomplete Moments

Some authors have proposed measures based on incomplete moments for IDI comparisons. Incomplete moments take the following form:

$$\phi(x;h) = \frac{\int_0^x y^h dF_Y(y)}{E(y^h)}, \tag{27}$$

where $E\left(y^h\right) \equiv \int_0^\infty y^h dF_Y(y)$. Of the handful of indices proposed, I review the three that use more information from the cumulative distributions of the two compared groups. As this section shows, there is a close connection between some of these indices and the PROB index, and also with the Dagum family. These links make it easier to ascertain which properties are fulfilled by indices based on incomplete moments.

The first index is $P(1,1)$, one from a group of Pietra indices proposed by Butler and McDonald (1987) :

$$P(1,1) = \frac{\int_0^{F_Y(\mu_X)} y(p)\,dp}{\mu_Y} - \frac{\int_0^{F_X(\mu_Y)} x(p)\,dp}{\mu_X} \tag{28}$$

$P(1,1)$ measures the difference between the proportion of total income in Y held by people who have income not higher than the average income in X minus the proportion of total income in X held by people who have income not higher than the average income in Y. Even though the index is not symmetric, it can be shown that $P(1,1)$ does not fulfill QGDF. In order to prove this, imagine that distributions Y and X are both symmetric with equal mean, μ, hence: $F_Y(\mu_X) = F_X(\mu_Y)$. In that situation: $P(1,1) = \frac{\int_0^{0.5} [y(p)-x(p)]dp}{\mu}$. Yet the gaps

$y(p) - x(p)$ can have different signs in the integration interval (e.g. imagine the two distributions differ in their kurtosis). Hence $P(1,1)$ may compensate quantile gaps that are detrimental to different groups.

With a bit more manipulation it is also possible to prove that $P(1,1)$ does not fulfill OGDF either in the case of equal means (μ). The key is to reexpress (28), with equal means, as:

$$\begin{aligned} P(1,1) &= \frac{1}{\mu}[\int_0^\infty \int_0^\mu (y-x)dF_Y(y)dF_X(x) + \int_0^\infty \int_0^\mu (y-x)dF_X(x)dF_Y(y) + \\ & \int_0^\infty xF_Y(\mu)dF_X(x) - \int_0^\infty yF_X(\mu)dF_Y(y)] \end{aligned} \quad (29)$$

Expression (29) is, then, sensitive to gaps $y - x$ of different signs. Hence OGDF is not satisfied.

Now, because $P(1,1)$ may compensate quantile gaps, it does not fulfill SSDE, although it does satisfy WSDE. Likewise it is not difficult to find examples showing that the measure is not helpful in identifying first-order stochastic dominance either. In the absence of overlap $P(1,1) = 1$ if the poorest person in X is richer than the richest person in Y, and $P(1,1) = -1$ if the poorest person in Y is richer than the richest person in X. However the reverse relationships are not true. For instance, it suffices for $P(1,1) = 1$ that the richest person in Y has less than the mean income of X and the poorest person in X has more than the mean income of Y.

Amendments to $P(1,1)$ that may render it in fulfillment of notions of GDF, or more informative about the distributional features under discussion, do not seem to be straightforward.

The second index has been proposed by Deutsch and Silber (1997). For continuous variables, it is:

$$I'_{G2} = \int_0^\infty dF_X(x) \int_0^x dF_Y(y) - \int_0^\infty dF_Y(y) \int_0^y dF_X(x) \quad (30)$$

Interestingly, (30) is the difference between two of the d_0 measures of Dagum (1980). From the definition of $PROB$, it is easy to show that $PROB_Y = 1 -$

$\int_0^\infty dF_X(x)\int_0^x dF_Y(y)$ and $PROB_X = 1-\int_0^\infty dF_Y(y)\int_0^y dF_X(x)$. [15] Hence $I'_{G2} = PROB_Y - PROB_X$. This means that I'_{G2} inherits the inability to fulfill QGDF from the *PROB* measures, as is clear in the following expression:

$$I'_{G2} = 2[\int_0^\infty [F_Y(z)-F_X(z)]_+ f_Y(z)\,dz - \int_0^\infty [F_X(z)-F_Y(z)]_+ f_Y(z)\,dz] \quad (31)$$

Likewise I'_{G2} fulfills WSDE but not SSDE. It also fails to pinpoint situations of first-order stochastic dominance. By contrast, it is helpful for the detection of absence of overlap, since: $PROB_Y = 1 \leftrightarrow PROB_X = 0$, in which case $I'_{G2} = 1$ if and only if the poorest person in X is richer than the richest person in Y. Similarly $I'_{G2} = -1$ if and only if the poorest person in Y is richer than the richest person in X. Amendments to I'_{G2} in order to make it capture notions of GDF (e.g. QGDF) may lead to proposals similar to those in the above section discussing the *PROB* measures.

Finally, the third index of incomplete moments reviewed has also been proposed by Deutsch and Silber (1997). For continuous variables, it is:

$$I'_{G1} = \frac{1}{\mu_X\mu_Y}[\int_0^\infty dF_X(x)x\int_0^x ydF_Y(y) - \int_0^\infty dF_Y(y)y\int_0^y xdF_X(x)] \quad (32)$$

With some manipulation, one can show that (32) is also equal to:

$$\begin{aligned} I'_{G1} \;=\; & \frac{1}{\mu_X\mu_Y}[\int_0^\infty x^2F_Y(x)dF_X(x) - \int_0^\infty y^2F_X(y)dF_Y(y) \\ & -\int_0^\infty dF_X(x)x\int_0^x (x-y)dF_Y(y) \\ & +\int_0^\infty dF_Y(y)y\int_0^y (y-x)dF_X(x)] \end{aligned} \quad (33)$$

Now, with (33), a resemblance to Dagum's $D_{X/Y}$ is apparent. Both are sensitive to gaps $y-x$ with different signs, and compensate for them. Hence I'_{G1} is

[15] As shown also by Dagum (1980).

not expected to fulfill notions of GDF. However, its elements individually, e.g. $d_{XX} \equiv \frac{1}{\mu_X \mu_Y} \int_0^\infty dF_X(x) x \int_0^x y dF_Y(y)$, like d_X, do satisfy OGDF, as is patent in (33). The analogy also follows in terms of sensitivity to distributional equivalence: I'_{G1} fulfills WSDE but not SSDE; whereas its elements do not fulfill WSDE (like d_X). Scale invariance and population invariance are also fulfilled, but not translation invariance. Like, I'_{G2}, I'_{G1} is normalized with extreme values of 1 and -1: $I'_{G1} = 1$ if and only if there is absence of overlap such that the richest person in Y is poorer than the poorest person in X; whereas $I'_{G1} = -1$ if and only if there is absence of overlap such that the richest person in X is poorer than the poorest person in Y.

7 An Alternative Framework: Indices Based on Welfare Comparisons

Thus far, the indices reviewed are characterized by: i) being useful especifically for two-group comparisons, ii) being more informative than just comparing two means, and iii) being useful when the two distributions have different sample sizes (as mentioned in the introduction). All these indices stem from aggregations of several comparisons of different parts of the two distributions (e.g. in the case of $d^r(X,Y)$ and $V(X,Y)$) pairwise comparisons of quantiles are performed, and then these are aggregated). An alternative to this approach to IDI measurement has been proposed by Shorrocks (1982), and axiomatically characterized by Chakravarty and Dutta (1987). Their proposed indices are characterized by a *different order of aggregation*: first, an equally distributed equivalent (EDE) standard is computed for each distribution separately, and then the two EDE statistics are compared. These indices are also known as "ethical distance functions" and they measure the differences in the welfare provided by two distributions through the metric of the EDE standard introduced by Atkinson (1970).

Let x_{EDE} be the EDE standard of X. Chakravarty and Dutta (1987) show that IDI measures like $d^r(X,Y)$ are not coherent with a welfarist comparison approach. By coherent, they mean that the index should be related monotonically to the absolute value of the difference between the two EDE standards. Instead they propose the following family of EDE-standard-based measures:

$$S(X,Y) = K \mid x_{EDE} - y_{EDE} \mid, K > 0. \tag{34}$$

When $K = 1$, S is the distance index suggested by Shorrocks (1982), and axiomatically characterized by Chakravarty (1990, p. 123). Notwithstanding the merits of these indices in terms of their coherence with an approach based on a social evaluation function, they are unlikely to fulfill general notions of GDF, because the absolute value operator in (34) imposes symmetry. But this could easily be amended by replacing the absolute value operator with the operator $[\ ]_+$ (defined above) and proposing an operationalization of GDF based on EDE standards (as opposed to quantile comparisons, for instance).

As for QGDF and OGDF, note that the ethical indices in (34) do not explicitly document relative advantages at different parts of the distributions, due to the order of aggregation. For instance, consider the following EDE standard: $x_2 = \left[\int_0^1 \sqrt{x(p)}dp\right]^2$.[16] With that standard: $S = K \mid \left(\int_0^1 \left[\sqrt{x(p)} + \sqrt{y(p)}\right] dp\right) \left(\int_0^1 \left[\sqrt{x(p)} - \sqrt{y(p)}\right] dp\right) \mid$. Hence quantile gaps that are detrimental to different groups are compensated. It is not difficult to show the occurrence of this same feature with other choices for the EDE standard. Hence the indices do not fulfill QGDF. Likewise, with that same standard, x_2: $S = K \mid \left(\int_0^\infty \int_0^\infty \left[\sqrt{x} + \sqrt{y}\right] dF_X(x) dF_Y(y)\right) \left(\int_0^\infty \int_0^\infty \left[\sqrt{x} - \sqrt{y}\right] dF_X(x) dF_Y(y)\right) \mid$. Therefore the indices do not fulfill OGDF either, since they are sensitive both to cases when $\sqrt{x} > \sqrt{y}$ (which means $x > y$ for nonnegative values) and cases when $\sqrt{x} < \sqrt{y}$.

As for the other distributional aspects emphasized in this review, measures like $S(X,Y)$ satisfy WSDE but not SSDE, because two different distributions can have the same EDE standard. Likewise, they are not informative as to the presence of first-order stochastic dominance since first-order dominance is sufficient but not necessary in order to have differences between x_{EDE} and y_{EDE}. They are

[16] x_2 is a member of the family of EDE standards based on generalized means, considered by Atkinson (1970).

not helpful either when it comes to detecting absence of overlap. Finally, these measures fulfill population invariance but neither translation invariance nor scale invariance. They could fulfill the latter with appropriate choices for K. One such choice is: $K = (x_{EDE} + y_{EDE})^{-1}$.

8 Concluding Remarks

This paper's review focused on indices that measure IDI involving distributions with different population sizes. This particular literature has not exhausted the discussion around the meaning of the notion of inequality that is detrimental to one specific group. Yet it usually follows, explicitly or implicity, two approaches for the treatment of these inequalities: adding-up and/or compensation. The main question of this review was whether these indices fulfill some notion, or operationalization, of a property of group-specific disadvantage focus (GDF), i.e. a third approach. This property is necessary for indices that quantify inequalities that are exclusively detrimental to one specific group. In that sense, it resembles the focus axiom from the poverty literature. Such concern for an exclusive focus on group-specific disadvantages has been articulated recently in the literature that measures labour market discrimination using counterfactual distribution techniques. The latter compare actual versus counterfactual situations, individual-by-individual. By contrast, traditional IDI indices compare actual distributions of populations with different sizes. Then, it is natural that many of the reviewed IDI indices map from quantiles, or even probabilities. Hence in this paper, I first set out to define GDF, in the context of IDI measurement with different population sizes, by proposing two possible ways of operationalizing it: Quantile group-specific disadvantage focus (QGDF) and Overlap group-specific disadvantage focus (OGDF).

The review also highlighted the ability of the IDI indices to pinpoint the presence of distributional equality. Two properties from the literature have been considered: a weak one, whose fulfillment ensures that the index takes a specific value in the presence of distributional equality; and a strong one, whose fulfillment ensures that the index takes a specific value *if and only if* distributional equality exists. In addition to these features, the review also considered whether the indices were informative about the presence of first-order stochastic dominance and lack of overlap. The former feature has featured prominently in IDI comparisons based on quantiles or probabilities; whereas the latter has been discussed in the context of the Dagum family and in applications of indices based on incomplete moments.

Neither of the indices reviewed satisfies QGDF. Likewise, fulfillment of OGDF is restricted to some members of the Dagum family, and the family of indices based on incomplete moments. Several indices are also limited in the information they provide on the aforementioned distributional features, i.e. distributional equality, presence of first-order stochastic dominance and absence of distributional overlap. However, as the paper shows, in many cases it is straightforward to amend these indices in order to render them in fulfillment of QGDF, or OGDF; and, often, more informative in terms of the additional distributional features mentioned above.

The examination of several indices suggests several patterns of interest regarding the fulfillment of properties. Firstly, the fulfillment of QGDF by IDI indices, for two distributions with different population sizes, may require that the indices map explicitly from either quantiles or probabilities; whereas for OGDF, mapping from the $y-x$ gaps seems to be a natural requirement. Secondly, there is a clear trade-off between fulfillment of symmetry axioms (e.g. as in the cases of indices by Ebert, 1984; and the ethical social indices of Chakravarty and Dutta, 1987) and compliance with any notion, or operationalization, of GDF. Any index fulfilling a notion of GDF, at least as defined in this review, has to be asymmetric. Likewise, there is a tension between fulfillment of GDF and undertaking the compensatory approach of pitting inequalities detrimental to one group against those detrimental to another group (e.g. as in the *PROB* measure, the index by Vinod (1985), indices based on incomplete moments and some of Dagum's indices). Fourthly, the choice between these three approaches has also implications for the sensitivity of the indices to the presence of distributional equality. While symmetric indices that add up inequalities detrimental to different groups (e.g. Ebert's), fulfill SSDE, indices that compensate inequalities only fulfill WSDE.[17] Indices fulfilling QGDF

[17] Ethical social indices are an interesting exception among symmetric indices, in that they only fulfill WSDE. The reason is their order of aggregation, discussed above.

also satisfy WSDE, but not SSDE. However, combinations of such indices fulfill SSDE jointly, as illustrated with some of the amendments proposed in the paper.

Without pretending to provide an exhaustive treatment of the subject, this review has sought to emphasize the importance of deepening the discussion of the relative merits, and drawbacks, of the different approaches used to measure IDI. The aim is also to stimulate an inquiry into the admissible notions of group-specific disadvantage that can be put forward; with an emphasis on the ways in which these could be operationalized.

Acknowledgement: I would like to thank the co-editor Satya Chakravarty, two anonymous referees, Carlos Gradin, and seminar participants at the Human Development Report Seminar Series of the UNDP and the Economics Network Meetings of the Inter-American Development Bank for their very helpful comments and suggestions.

References

Arneson, R. (1989). Equality and equal opportunity for welfare. *Philosophical studies*, 56: 77–93. URL http://philpapers.org/rec/ARNEAE.

Arrow, K. (1973). *Discrimination in the labor market*, chapter The theory of discrimination. Princeton University Press.

Atkinson, A. (1970). On the measurement of economic inequality. *Journal of Economic Theory*, 2(3): 244–263. URL http://darp.lse.ac.uk/papersdb/Atkinson_(JET70).pdf.

Becker, G. (1971). *The economics of discrimination*. The University of Chicago Press.

Bishop, J., Zeager, L., and Zheng, B. (2011). Interdistributional inequality, stochastic dominance, and poverty. URL http://www.ecu.edu/cs-educ/econ/upload/ecu1107-InterdistributionalInequalityAndStochasticDominance-2.pdf. ECU1107 Working Paper, East Carolina University.

Butler, R., and McDonald, J. (1987). Interdistributinal income inequality. *Journal of Business and Economic Statistics*, 5(1): 13–18. URL http://ideas.repec.org/a/bes/jnlbes/v5y1987i1p13-18.html.

Butler, R., and McDonald, J. (1989). Using incomplete moments to measure inequality. *Journal of Econometrics*, 42: 109–119. URL http://ideas.repec.org/a/eee/econom/v42y1989i1p109-119.html.

Chakravarty, S. (1990). *Ethical social index numbers*. Springer-Verlag.

Chakravarty, S., and Dutta, B. (1987). A note on measures of distance between income distributions. *Journal of Economic Theory*, 41: 185–188. URL http://ideas.repec.org/a/eee/jetheo/v41y1987i1p185-188.html.

Cohen, G. (1989). On the currency of egalitarian justice. *Ethics*, 99(4): 906–944. URL http://www.jstor.org/stable/2381239.

Cowell, F. (1985). Measures of distributional change: An axiomatic approach. *Review of Economic Studies*, LII: 135–151. URL http://ideas.repec.org/a/bla/restud/v52y1985i1p135-51.html.

Dagum, C. (1980). Inequality measures between income distributions with applications. *Econometrica*, 48(7): 1791–1803. URL http://ideas.repec.org/a/ecm/emetrp/v48y1980i7p1791-1803.html.

Dagum, C. (1987). Measuring the economic affluence between populations of income receivers. *Journal of Business and Economic Statistics*, 5(1): 5–8. URL http://ideas.repec.org/a/bes/jnlbes/v5y1987i1p5-12.html.

del Rio, C., Gradin, C., and Canto, O. (2011). The measurement of gender wage discrimination: The distributional approach revisited. *Journal of Economic Inequality*, 9(1): 57–86. URL http://ideas.repec.org/a/kap/jecinq/v9y2011i1p57-86.html.

Deutsch, J., and Silber, J. (1997). Gini's "Transvariazione" and the measurement of distance between distributions. *Empirical Economics*, 22: 547–554. URL http://ideas.repec.org/a/spr/empeco/v22y1997i4p547-54.html.

Ebert, U. (1984). Measures of distance between income distributions. *Journal of Economic Theory*, 32: 266–274. URL http://ideas.repec.org/a/eee/jetheo/v32y1984i2p266-274.html.

Elbers, C., Lanjouw, P., Mistiaen, J., and Ozler, B. (2008). Reinterpreting between-group inequality. *Journal of Economic Inequality*, 6: 231–245. URL http://ideas.repec.org/a/kap/jecinq/v6y2008i3p231-245.html.

Ferreira, F., and Gignoux, J. (2011). The measurement of inequality of opportunity: Theory and an application to Latin America. *Review of Income and Wealth*, 57(4): 622–657. URL http://onlinelibrary.wiley.com/doi/10.1111/j.1475-4991.2011.00467.x/abstract.

Fields, E., G.and Ok (1996). The meaning and measurement of income mobility. *Journal of Economic Theory*, 71: 349–377. URL http://ideas.repec.org/a/eee/jetheo/v71y1996i2p349-377.html.

Fields, G., and Ok, E. (1999). Measuring movement of incomes. *Economica*, 66(264): 455–471. URL http://ideas.repec.org/a/bla/econom/v66y1999i264p455-71.html.

Fleurbaey, M. (2001). Egalitarian opportunities. *Law and Philosophy*, 20(5): 499–530. URL http://www.springerlink.com/content/x215616v05j333g2/.

Fleurbaey, M. (2008). *Fairness, responsibility and welfare*. Oxford University Press.

Gastwirth, J. (1975). Statistical measures of earnings differentials. *The American Statistician*, 29(1): 32–35. URL http://www.jstor.org/stable/2683677.

Gradin, C., Del Rio, C., and Canto, O. (2010). Gender wage discrimination and poverty in the EU. *Feminist Economics*, 16(2): 73–110. URL http://ideas.repec.org/a/taf/femeco/v16y2010i2p73-109.html.

Handcock, M., M.and Morris (1999). *Relative distribution methods in the social sciences*. Statistics for Social Science and Public Policy. Springer.

Jenkins, S. (1994). Earnings discrimination measurement: a distributional approach. *Journal of Econometrics*, 61: 81–102. URL http://ideas.repec.org/a/eee/econom/v61y1994i1p81-102.html.

Lanjouw, P., and Rao, V. (2011). Revisiting between-group inequality measurement: An application to the dynamics of caste inequality in two Indian villages. *World Development*, 39(2): 174–187. URL http://ideas.repec.org/a/eee/wdevel/v39y2011i2p174-187.html.

Le Breton, M., Michelangeli, A., and Peluso, E. (2008). Wage discrimination measurement: in defense of a simple but informative statistical tool. URL http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1337986. Econpubblica Working Paper 128.

Le Breton, M., Michelangeli, A., and Peluso, E. (2011). A stochastic dominance approach to the measurement of discrimination. *Journal of Economic Theory*, doi: 10.1016/j/jet.2011.05.003. DOI doi:10.1016/j/jet.2011.05.003. URL http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1337986.

Magdalou, B., and Nock, R. (2011). Income distributions and decomposable divergence measures. *Journal of Economic Theory*, 146: 2440–2454.

Nussbaum, M., and Glover, J. (1995). *Women, culture and development: a study of human capabilities*. WIDER studies in Development Economics. Clarendon Press.

Ooghe, E., Schokkaert, E., and Van de Gaer, D. (2007). Equality of opportunity versus equality of opportunity sets. *Social Choice and Welfare*, 28: 209–230. URL http://ideas.repec.org/a/spr/sochwe/v28y2007i2p209-230.html.

Phelps, E. (1972). The statistical theory of racism and sexism. *The American Economic Review*, 62(4): 659–661. URL http://ideas.repec.org/a/aea/aecrev/v62y1972i4p659-61.html.

Roemer, J. (1998). *Equality of opportunity*. Harvard University Press.

Roemer, J. (2006). Economic development as opportunity equalization. URL http://ideas.repec.org/p/cla/levrem/321307000000000445.html. Cowles Foundation Discussion Paper 1583.

Schluter, C., and van de Gaer, D. (2011). Upward structural mobility, exchange mobility, and subgroup consistent mobility measurement: U.S.-German mobility rankings revisited. *Review of Income and Wealth*, 57(1): 1–22. URL http://ideas.repec.org/a/bla/revinw/v57y2011i1p1-22.html.

Sen, A. (2001). *Development as freedom*. Oxford University Press.

Shorrocks, A. (1982). On the distance between income distributions. *Econometrica*, 50(5): 1337–1339. URL http://www.jstor.org/stable/1911879.

van Krem, P. (2009). Generalized measures of wage differentials. URL http://ideas.repec.org/p/irs/iriswp/2009-08.html. IRISS Working Papers 2009-08.

Vinod, H. (1985). Measurement of economic distance between blacks and whites. *Journal of Business and Economic Statistics*, 3(1): 78–88. URL http://www.jstor.org/stable/1391692.