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On the Necessity of Pairs and Triplets for the Equivalence between Rationality Axioms

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Abstract This paper is concerned with the axiomatic foundation of the revealed preference theory. Many well-known results in the literature rest upon the ability to choose over budget sets that contain only 2 or 3 elements. This paper shows that for any given choice function, many of the famous consistency requirements, such as those proposed by Arrow, Sen, Samuelson etc., are equivalent if every bundle of goods is chosen from some budget sets, and the domain of the choice function satisfies some set of theoretical properties.

JEL D01, D11

Keywords Revealed preference theory; rationality; preference; choice functions

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Introduction

Revealed preference theory started off as an approach to explain consumers' behaviours by the revealed preference through their actions (Samuelson 1938). By defining preference relations on the bundles rather than specific goods, revealed preference theorists have been able to avoid notions such as marginal utilities, and construct a theory based only on a notion of preference. They also have been trying to pin down the necessary and sufficient conditions for the preference relations to be rationalizable. A few theorems that state the equivalence between some rationality conditions were proved by Sen (1971), Arrow (1959) and Bandyopadhyay (1988) etc. In their proofs, it is assumed that the domain of the choice functions includes all budgets containing finite bundles. In addition, Sen (1971) observes that the equivalence continues to hold if the domain includes all pairs and triplets. However, the assumption of including all pairs and triplets in the domain may not be ideal in every contexts. For example, in the study of behaviours of competitive consumers, it seems more natural to confine the domain to the set of "budget polyhedrons", e.g. "budget triangles" if only considering two different goods. Hence, we are interested if there is another set of assumptions on the choice function and its domain so that the equivalence between the rational axioms remain valid.

To answer this question, we note that Sen (1971) includes in the domain all the budgets with exactly 2 elements in order to compare each pair, and all the budgets with exactly 3 elements in order to avoid cycles. This suggests that first, if we are able to compare every two bundles, not necessarily in a pair but in any set, we should be able to achieve completeness. Secondly, if we are also able to avoid cycles by choosing from more than 3 bundles, we should be able to reach a satisfying result.

To compare two bundles a_1 and a_2, it would be intuitive that when a_1 is chosen from the budget A_1, and a_2 is chosen from the budget A_2, we would have implicitly compared a_1 and a_2 by choosing from the budget $A_1 \cup A_2$. Similarly, for a_1, a_2 and a_3, we can try to choose from the set $A_1 \cup A_2 \cup A_3$. Hence we would replace each singleton budget set by a budget set from which that singleton is chosen, and assume the unions of these budgets are in the choice function's domain. Using this assumption, we can show that several rationality axioms proposed by Arrow (1959), Samuelson (1938), Sen (1971) and Houthakker (1950) etc. are equivalent.

With the similar technique, an analogue of the Bandyopadhyay's theorem (1988) is obtained. The proofs in this paper are partly adapted from Arrow (1959), Sen (1971) and Bandyopadhyay's (1988).

The previous assumption allows the revealed preference theory to be interpreted more easily under some contexts. For example, it better fits the conventional interpretation of consumer theory. Given a disposable income I and prices P_i $(1 \leq i \leq n)$ of n number of goods in the market, the consumer is facing a polyhedral shape budget set under the constraint of $\sum_{i=1}^{n} P_i Q_i \leq I$, where Q_i is the quantity of the ith goods. The assumption requires all of these polyhedrons and finite unions of them (instead of all finite budgets) to be included into the domain of the choice function. Here, the unions of budget sets would become some concave polyhedrons, which can be observed in price cut or whole sale situations.

1 Notations and Definitions

In this paper, we will adopt the following notations. Consider $X \neq \emptyset$ as the set of all bundles, the choice function $C(.)$ is defined on a nonempty subset of the powerset of X, $\mathbb{B}$, called the set of budgets. For any budget $B \in \mathbb{B}$, we require that $C(B) \subseteq B$ and $C(B)$ is not empty, so $\emptyset \notin \mathbb{B}$ (i.e. the empty set cannot be a budget set). Throughout the paper, we use the symbol $\neg$ for mathematical negation.

Definition 1 a binary relation on X is called an *ordering* if it is transitive and complete (or some authors refer to as connected).

Based on the choice function, the following definitions of preference relations on X have been much discussed in the literature. For the first ones, Sen (1971 p.308) interprets the relation R as "at least as good as", P as "strictly preferred to".

Definition 2 For any $x, y \in X$, xRy if $\exists B \in \mathbb{B}$ such that $x \in C(B)$ and $y \in B$.

Definition 3 For any $x, y \in X$, xPy if xRy and $\neg yRx$.

Another definition of a preference relation was given by Arrow's (1959) "revealed preference". It is denoted as $\widetilde{P}$.

Definition 4 For any $x, y \in X$, $x\widetilde{P}y$ if $\exists B \in \mathbb{B}$ such that $x \in C(B)$ and $y \in B - C(B)$.

Definition 5 For any $x, y \in X$, $x\widetilde{R}y$ if $\neg y\widetilde{P}x$.

Other than the above definitions, there are also some notations about the "wide sense" relations, or, the "transitive closures" of R and $\widetilde{P}$ defined as follows (Ritchter 1966):

Definition 6 For any $x, y \in X$, xWy if there exists a finite sequence $x_0, x_1, \ldots, x_n \in X$ such that $x_0 = x$, $x_n = y$ and $x_{i-1}Rx_i$ for all $i = 1, \ldots n$.

Definition 7 For any $x, y \in X$, $x\widetilde{V}y$ if there exists a finite sequence of $x_0, x_1, \ldots, x_n \in X$ such that $x_0 = x$, $x_n = y$ and $x_{i-1}\widetilde{P}x_i$ for all $i = 1, \ldots n$.

Given that we have the above definitions on relations, we can start stating definitions and axioms on rationality and consistency.

Definition 8 A choice function $C(.)$ is *normal* if $\forall B \in \mathbb{B}, C(B) = \{x \in B | xRy \ \forall y \in B\}$.

If a choice function $C(.)$ is normal and R is an ordering, we say $C(.)$ is *rational* or *rationalizable*.

Definition 9 A choice function is said to satisfy the *Weak Axiom of Revealed Preference* (WARP, Samuelson 1938) if for every $x, y \in X$: $x\widetilde{P}y \Rightarrow \neg yRx$ (or equivalently $yRx \Rightarrow \neg x\widetilde{P}y$).

Definition 10 A choice function is said to satisfy the *Strong Axiom of Revealed Preference* (SARP, Houthakker 1950) if for every $x, y \in X$: $x\widetilde{V}y \Rightarrow \neg yRx$.

Definition 11 A choice function is said to satisfy the *Weak Congruence Ax-*

iom (WCA, Sen 1971) if for every $x, y \in X$: suppose xRy, then for any $B \in \mathbb{B}$, $(x \in B \textit{ and } y \in C(B)\ ) \Rightarrow x \in C(B)$.

Definition 12 A choice function is said to satisfy the *Strong Congruence Axiom* (SCA, Richter 1966) if for every $x, y \in X$: suppose xWy, then for any $B \in \mathbb{B}$, $(x \in B \textit{ and } y \in C(B)\ ) \Rightarrow x \in C(B)$.

Another consistency condition first given by Arrow (1959) is stated as below:

Definition 13 Choice function $C(.)$ is said to satisfy the *Arrow's Condition* (Arrow 1959) if for any $A, B \in \mathbb{B}$, when $A \subseteq B$ and $A \cap C(B) \neq \emptyset$, then $C(A) = C(B) \cap A$.

2 Construction and Main Results

Under the motivation, we make the following assumptions on $C(.)$:

Assumption 1 For a choice function $C(.)$ and its domain $\mathbb{B}$, there exists $\mathbb{B}_1 \subseteq \mathbb{B}$ such that

(i) $\mathbb{B}_3 = \{B_1 \cup B_2 \cup B_3 | B_1, B_2, B_3 \in \mathbb{B}_1\} \subseteq \mathbb{B}$;

(ii) for any bundle $x \in X$, there exists a budget $B_x \in \mathbb{B}_1$ such that $x \in C(B_x)$.

This is a weaker form of the assumption. It corresponds to the inclusion of all pairs and triplets into the domain of a choice function. There is a stronger version of it that corresponds to the inclusion of all finite budgets:

Assumption 1' For a choice function $C(.)$,

(i) its domain $\mathbb{B}$ is a set closed under finite union;

(ii) there exists $\mathbb{B}_1 \subseteq \mathbb{B}$ such that for any bundle $x \in X$, there exists a budget $B_x \in \mathbb{B}$ such that $x \in C(B_x)$.

Theorem 1. *Suppose $C(.)$ is defined on $\mathbb{B}$ and Assumption 1 is satisfied, then the following are equivalent:*

(i) *$C(.)$ satisfies the Strong Axiom of Revealed Preference;*

(ii) *$C(.)$ satisfies the Weak Axiom of Revealed Preference;*

(iii) *R is an ordering and $C(.)$ is normal;*

(iv) *$C(.)$ satisfies the Strong Congruence Axiom;*

(v) *$C(.)$ satisfies the Weak Congruence Axiom;*

(vi) *$R=\widetilde{R}$.*

Theorem 2. *Suppose $C(.)$ is defined on $\mathbb{B}$ and Assumption 1' is satisfied, then R is an ordering and $C(.)$ is normal if and only if $C(.)$ satisfies Arrow's Condition.*

Uzawa (1956) and Arrow (1959) gave a definition of a relation "generated by comparison over all pairs", by which they mean if a is preferred to b in a pair, then $a \in C(\{a,b\})$. In consistent with the motivation and the previous discussion, we mimic their definition and define a relation "generated by comparison over all pairs of $\mathbb{B}_1$ budgets" to be denoted by $\overline{R}$:

Definition 14 For any $x, y \in X$, $x\overline{R}y$ with respect to a $\mathbb{B}_1$ if there exists $B_x,\ B_y \in \mathbb{B}_1$ such that $x \in C(B_x),\ y \in C(B_y)$ and $x \in C(B_x \cup B_y)$.

In the above definition, if we put $\mathbb{B}$ instead of $\mathbb{B}_1$, then $\overline{R}$ would not be a natural generalization of "preference generated over pairs" (see Uzawa 1956 and Arrow 1959) and $\overline{R}$ with respect to $\mathbb{B}$ would be too weak a relation to establish some desired equivalences. Hence $\overline{R}$ should be dependent upon the chosen $\mathbb{B}_1$ unless $\mathbb{B}_1 = \mathbb{B}$, a case in which the stronger Assumption 1' (i) is satisfied and $\mathbb{B}_1$ would then be replaced by $\mathbb{B}$.

Additionally, in the following generalization, we require some additional assumptions on the choice function $C(.)$ and $\mathbb{B}$ for some of the definitions to be meaningful or applicable.

Assumption 2 *Pre-rationality:* $\forall B_x, B_y \in \mathbb{B}_1,\ C(B_x \cup B_y) \subseteq C(B_x) \cup C(B_y)$.

We can strengthen Assumption 2 in the following way: for any $A, B \in \mathbb{B}, C(A \cup B) \subseteq C(A) \cup C(B)$ whenever $A \cup B \in \mathbb{B}$. However this is not necessary for the proofs in this paper.

Assumption 3 *Finiteness:* each $B \in \mathbb{B}$ has finitely many elements.

It can be seen that all the above assumptions can be implied from assuming that $\mathbb{B}$ consists of all finite subsets of X. In particular, Assumption 3 is required for the following rationality requirement to be meaningful in the context.

Axiom of Sequential Path Independence was a rationality condition originally proposed by Bandyopadhyay (1988). The idea was that if a choice function is rational, it should be necessary and sufficient that comparing each two bundles in a budget in different orders would give the same final choice. In order to fit in the above settings, we would use the following notations to give a modified version. For all $B \in \mathbb{B}$, let $\Omega(B)$ be the set of all permutations of elements of B, and $|B|$ denote the cardinality of B. Let B_r denote some budget in $\mathbb{B}_1$ such that $r \in C(B_r)$. Suppose Assumption 3 holds, for any choice function $C(.)$ and any $\omega \in \Omega(B)$, define the following sets recursively:

1. $\hat{\omega}(1) = \{\omega(1)\}$
2. For any positive integer $i \leq |B|$, pick arbitrarily $B_a, B_{\omega(i+1)} \in \mathbb{B}_1$ for each $a \in \hat{\omega}(i)$, define $\hat{\omega}(i+1) = B \cap \bigcup_{a \in \hat{\omega}(i)} C(B_a \cup B_{\omega(i+1)})$.

It can be seen that even if we assume rationality of $C(.)$, $\hat{\omega}(i+1)$ is potentially not unique for $1 < i+1 < |B|$. It is dependent upon the B_a's and $B_{\omega(i+1)}$, but the exact notation seems inconvenient to be written down.

Definition 15 If Assumption 3 is satisfied, a choice function $C(.)$ is said to satisfy the *Axiom of Sequential Budget Independence* (ASBI) with respect to a $\mathbb{B}_1$ if for any $B \in \mathbb{B}$, for all $\omega \in \Omega(B)$, and for every choice of $B_a, B_{\omega(i+1)}$ in constructing each $\hat{\omega}(i+1), (1 \leq i < |B|)$, it is always the case that $C(B) = \hat{\omega}(|B|)$.

Theorem 3. *Suppose $C(.)$ is defined on $\mathbb{B}$ and Assumption 1 is satisfied, then with respect to any $\mathbb{B}_1$ satisfying Assumption 1, the following are equivalent:*

(i) *R is an ordering and $C(.)$ is normal;*

(ii) *$\overline{R}$ is an ordering and $C(B) = \{x \in B | x\overline{R}b \ \forall b \in B\}$;*

(iii) *$\overline{R} = \widetilde{R}$ and $C(B) = \{x \in B | x\widetilde{R}b \ \forall b \in B\}$;*

(iv) *Pre-rationality (Assumption 2) is satisfied and $C(.)$ satisfies the Axiom of Sequential Budget Independence.*

Remark 1. *$(ii), (iii)$ are modified from Sen's (1971 p.310) formulations "$\overline{R}$ is an ordering, $C(.)$ is normal" and "$\overline{R} = \widetilde{R}$ and $C(.)$ is normal". But they are respectively equivalent if the domain of choice function consists of all finite subsets of X. To see this, $(ii), (iii)$ imply the corresponding formulations because they imply (i). And if $\mathbb{B}_1$ is all the singletons, then normality would give $\overline{R} = R$ when considering choosing over the pairs. So they are respectively equivalent statements. In addition, Pre-rationality is automatically satisfied if $\mathbb{B}_1$ is the set of all the singleton budgets. In that case, it can be seen that ASBI reduces to ASPI.*

Notice that if $\mathbb{B}$ contains all singletons, pairs and triplets, by taking $\mathbb{B}_1$ to be the set of all singleton budget sets, Assumption 1 is satisfied. (T. 3) of Sen's (1971, p.310) follows from Theorem 1 and Theorem 3 without (iv). Moreover, if all finite budgets are included into the domain of the choice function, Assumption 1' will be satisfied.

Sen's Corollary : Suppose $\mathbb{B}$ consists of all finite budgets, then all the rationality conditions in Theorems 1, 2 and 3 are equivalent.

Example 1 : Let $X = \{a,b,c,d\}$ and $\mathbb{B} = \{\{a,b,d\},\{b,c,d\}, \{a,c,d\},\{d\}, X\}$. Suppose the choice function is

$$C(\{d\}) = \{d\}, \quad C(\{a,b,d\}) = \{a\}, \quad C(\{b,c,d\}) = \{b,c\},$$
$$C(\{a,c,d\}) = \{a\}, \quad C(\{a,b,c,d\}) = \{a\}.$$

Since every budget is finite and $\mathbb{B}$ is closed under finite union, Assumption 3 is satisfied. Assumption 1 holds by letting $\mathbb{B}_1 = \{\{a,b,d\},\{b,c,d\},\{a,c,d\},\{d\}\}$. One can also check that Assumption 2 holds by straight computations. It is also true that Assumption 1' is satisfied. Since $C(.)$ satisfies WARP, therefore, the rationality axioms in Theorem 1, 2 and 3 are satisfied.

If $C(\{a,c,d\}) = \{a,c\}$ and choices from all other budgets remain unchanged, the $\mathbb{B}_1$ constructed will still satisfy Assumption 1. However, since $\{a,c\} = C(\{a,c,d\})$ but $\{a\} = C(\{a,b,c,d\})$, WARP is not satisfied. Hence, none of the rationality axioms in Theorem 1, 2 and 3 holds.

Example 2 : Fix a positive integer n, let $X = \mathbb{N}^n$ be the n-cartesian product of natural numbers. For each price vector $\mathbf{p}= (p_1, p_2, \dots, p_n)$ such that $p_i > 0$ for every $i = 1,\dots,n$, and each positive income I, let $B^{(I,\mathbf{p})}= \{(q_1,q_2,\dots,q_n) \in X | q_i \in \mathbb{N}, \sum_{i=1}^{n} p_i q_i \leq I\}$. Let $\mathbb{B}_1$ consist of all the budgets of the form $B^{(I,\mathbf{p})}$. That is, $\mathbb{B}_1$ is the set of so called budget triangles. Let $\mathbb{B}$ be the smallest superset of $\mathbb{B}_1$ that is closed under finite union and intersection. We know that each budget in $\mathbb{B}_1$ has finite cardinality, and so is in $\mathbb{B}$. Define the choice function $C(.)$ on $\mathbb{B}$ to be for every $B \in \mathbb{B}$,

$$C(B) = \{(q_1,q_2,\dots,q_n) \in B | \prod_{i=1}^{n} q_i \geq \prod_{i=1}^{n} q_i', \ \forall \left(q_1',q_2',\dots,q_n'\right) \in B\}$$

It can be checked that all Assumptions 1 to 3 and Assumption 1' are satisfied. So Theorems 1, 2 and 3 ensure that the rationality conditions discussed are equivalent.

3 Discussion

Arrow (1959) first suggested that "the demand-function point of view would be greatly simplified if the range over which the choice functions are considered to be determined is broadened to include all finite sets". However with more technical complexity, the equivalence between the discussed rationality axioms can be generalized. In this paper, Assumption 1 is not only an assumption on the budget sets but it is also an assumption about the choice function. The stronger version, Assumption 1', corresponding to the inclusion of "all finite budget sets" is also proposed. One drawback of the assumptions is that in experimental conditions their testability can be difficult because of the *there exists* statement. But this difficulty may be lowered if there is a natural way to interpret $\mathbb{B}_1$ and the interpretation satisfies the assumptions (e.g. in consumer theory, we can take $\mathbb{B}_1$ to be the budget triangles). Hence the assumptions may not be easier applied with data than assuming all pairs and triplets are present. Assumption 2, 3 are technical assumptions, but Assumption 2 may be interesting in its own right and according to Example 1, it is strictly weaker than other rationality axioms discussed.

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Appendix

In the proof of Theorem 1, we make use of the following lemma.

Lemma 1. *Suppose that the WARP holds. Suppose that there exists* $B_x, B_y \in \mathbb{B}$ *such that* $x \in C(B_x)$, $y \in C(B_y)$. *If* $B_x \cup B_y \in \mathbb{B}$, *then* $C(B_x \cup B_y) \cap \{x, y\} \neq \emptyset$

Proof of Lemma 1

Proof. Suppose by contradiction, $x, y \notin C(B_x \cup B_y)$. Then by property of choice function, $\exists z \in C(B_x \cup B_y)$. Without loss of generality, suppose $z \in B_x$. Then by definition, we have $z\widetilde{P}x$. Since we also have xRz, this contradicts WARP. □

Proof of Theorem 1

Proof. Closely examining the proof of T.3 of Sen's (1971) shows $(v) \Leftrightarrow (ii)$ without any assumption. Theorem 1 will be proven in the following fashion ("$\overset{*}{\Rightarrow}$"

indicates the proof requires Assumption 1):

$$(ii) \overset{*}{\Rightarrow} (iii) \Rightarrow (vi) \Rightarrow (ii)$$
$$(ii) \overset{*}{\Rightarrow} (iii) \Rightarrow (iv) \Rightarrow (v) \overset{Sen(1971)}{\Leftrightarrow} (ii)$$
$$(ii) \overset{*}{\Rightarrow} (iii) \Rightarrow (i) \Rightarrow (ii)$$

$(ii) \overset{*}{\Rightarrow} (iii)$: Suppose WARP holds, we want to show: 1,$C(.)$ is normal; 2, R is a complete binary relation; 3, R is transitive.

To show 1, on one hand, $x \in C(B) \Rightarrow xRy$ for every $y \in B$. Therefore $x \in \{x \in B | \forall y \in B, xRy\}$. On the other hand, $x \in \{x \in B | \forall y \in B, xRy\} \Rightarrow \neg y\widetilde{P}x$ for any $y \in B$ by WARP. If $x \notin C(B)$, then $z\widetilde{P}x$ for some $z \in C(B)$ leads to a contradiction. Hence $C(.)$ is normal.

To show 2, suppose $x, y \in X$, then there exists $B_x, B_y \in \mathbb{B}_1$ such that $x \in C(B_x), y \in C(B_y)$. By Lemma 1, we have x or y belonging to $C(B_x \cup B_y)$. So we have xRy or yRx.

To show 3, suppose xRy and yRz. By construction, we can consider the budgets $B_x, B_y, B_z \in \mathbb{B}_1$ such that

$$r \in C(B_r)\ \forall r \in \{x, y, z\}$$

Let $B = B_x \cup B_y \cup B_z \in \mathbb{B}_3 \subseteq \mathbb{B}$, Lemma 1 implies either x, y or $z \in C(B)$. In the view of WARP, $z \in C(B) \Rightarrow y \in C(B)$ and $y \in C(B) \Rightarrow x \in C(B)$. So we must have xRz.

$(iii) \Rightarrow (vi)$: Suppose R is an ordering and $C(.)$ is normal, we want to show $R = \widetilde{R}$.

Suppose $x\widetilde{R}y$, so $\neg y\widetilde{P}x$. We know R is complete. So we have either yRx or xRy or both. So $\neg y\widetilde{P}x$ implies have xRy.

Suppose xRy, by the normality of $C(.)$, whenever $x, y \in B$ and $y \in C(B)$, we must have $x \in C(B)$ because of the transitivity of R. So it is impossible that $y\widetilde{P}x$. Hence we have $x\widetilde{R}y$.

$(vi) \Rightarrow (ii)$:

$xRy \Rightarrow x\widetilde{R}y \Rightarrow \neg y\widetilde{P}x$ by definition.

$(iii) \Rightarrow (iv)$: Assuming R is an ordering and $C(.)$ is normal, we want to show SCA.

Suppose we have $x_1Rx_2R\ldots Rx_n$, and for some $B \in \mathbb{B}$ we have $x_1, x_n \in B$ and $x_n \in C(B)$. R being an ordering implies x_1Rx_n. $x_n \in C(B)$ and $C(.)$ is normal implies $\forall z \in B$, x_nRz. Therefore x_1Rz by transitivity and $x_1 \in C(B)$ by normality.

$(iv) \Rightarrow (v)$: (trivial).

$(v) \Leftrightarrow (ii)$: See Sen's T.3 (1971 p. 311).

$(iii) \Rightarrow (i)$: Assuming R is an ordering and $C(.)$ is normal, we want to show SARP.

Suppose $x_1\widetilde{P}x_2\ldots\widetilde{P}x_n$, by transitivity of R, we have x_1Rx_n. Now suppose by contradiction, x_nRx_1, then transitivity implies x_iRx_j for any $1 \leq i, j \leq n$. In particular, x_2Rx_1. Now, by $(iii) \Rightarrow (vi)$ we have $R = \widetilde{R}$. Hence $x_2\widetilde{R}x_1$ which contradicts $x_1\widetilde{P}x_2$.

$(i) \Rightarrow (ii)$: (trivial).

The above completes the proof.

□

Proof of Theorem 2

Proof. Rationalizability $\Rightarrow$ Arrow's Condition: assuming R is an ordering and $C(.)$ is normal, we want to prove Arrow's Condition.

Suppose $A, B \in \mathbb{B}$, $A \subseteq B$ and $A \cap C(B) \neq \emptyset$. By normality and ordering, it is obvious that $C(A) \supset C(B) \cap A$.

Now suppose $y \in C(A)$ and $a \in A \cap C(B)$. $C(.)$ is normal implies $aRx\ \forall x \in B$ and yRa. Now transitivity implies $yRx\ \forall x \in B$. So normality gives $y \in C(B)$ and $C(A) \subseteq C(B) \cap A$.

Arrow's Condition $\Rightarrow$ rationalizability: Assuming Arrow's Condition, we want to show Weak Congruence Axiom, and by Theorem 1 we have rationalizability. In this direction of the proof, the Assumption 1' is used.

Let xRy and $y \in C(B)$, $x \in B$ for some $B \in \mathbb{B}$, we want to show $x \in C(B)$.

Let $A \in \mathbb{B}$ such that $x \in C(A)$ and $y \in A$. Consider $C(A \cup B)$. If $C(A \cup B) \cap A = \emptyset$, then it is necessary that $C(A \cup B) \cap B \neq \emptyset$ and so $C(B) = C(A \cup B) \cap B$. But $y \in C(B)$, so $y \in C(A \cup B)$ and $y \in A$, which is a contradiction. Hence we have to have $C(A \cup B) \cap A \neq \emptyset$. Then $x \in C(A) = C(A \cup B) \cap A$. Because $x \in B$, so $x \in C(B) = C(A \cup B) \cap B$.

Since Assumption 1' implies Assumption 1, by Theorem 1 we complete the proof. □

Proof of Theorem 3

When it is clear from the context, we will write B_x for the same budget in $\mathbb{B}_1$ where x is chosen. The result will be proven in the following sequence (The "$*$" on the arrow shows when Assumption 1 is used).

$$(i) \overset{*}{\Rightarrow} (iv) \overset{*}{\Rightarrow} (i)$$
$$(i) \overset{*}{\Rightarrow} (iii) \overset{*}{\Rightarrow} (ii) \overset{*}{\Rightarrow} (i)$$

Proof.
$(i) \overset{*}{\Rightarrow} (iv)$: this proof is adapted from the one given by Bandyopadhyay (1988).

It is trivial that (i) implies pre-rationality.

Let R be an ordering and $C(.)$ be normal. Therefore $x \in C(B) \Rightarrow xRb\ \forall b \in B$. Choose an arbitrary $\omega \in \Omega(B)$ and let $\omega(i) = x$ for a fixed $i = 1, \dots |B|$. Then $\hat{\omega}(i) = B \cap \bigcup_{a \in \hat{\omega}(i-1)} C(B_a \cup B_x)$. By R being an ordering and $C(.)$ is normal, we have $x \in \hat{\omega}(i)$. Similarly, we see that $x \in \hat{\omega}(j)$ for every $j \geq i$. So $C(B) \subseteq \hat{\omega}(|B|)$.

Now suppose $x \notin C(B)$, then $\forall y \in C(B)$ transitivity and normality implies yPx. Moreover, normality ensures that $x \notin C(B)$ if and only if yPx. So it follows that $\omega(i) = y \Rightarrow x \notin \hat{\omega}(i)$. Similar reasoning gives $x \notin \hat{\omega}(j)\ \forall j \geq i$. Hence $x \notin \hat{\omega}(|B|)$ and $C(B) \supseteq \hat{\omega}(|B|)$.

$(iv) \overset{*}{\Rightarrow} (i)$: given that ASBI holds, we would first try to show WCA, then by Theorem 1, the claim is proven.

In order to show WCA, suppose xRy, and for some $B \in \mathbb{B}$ there is $x \in B$, $y \in C(B)$, we will try to show $x \in C(B)$.

$xRy \Rightarrow \exists A \in \mathbb{B}$ such that $x \in C(A)$ and $y \in A$. But by ASBI, this means for each $r \in A$, and each $B_r \in \mathbb{B}$, there exists a $B_x \in \mathbb{B}_1$ such that $x \in C(B_r \cup B_x)$. Because otherwise by letting $\omega(|A|) = r$, Pre-rationality gives that $x \notin A \cap \bigcup_{a \in \hat{\omega}(|A|-1)} C(B_a \cup B_r) = \hat{\omega}(|A|)$. In other words, $\forall r \in A, \forall B_r \in \mathbb{B}_1, \exists B_x \in \mathbb{B}_1$ such that $x \in C(B_r \cup B_x)$; in particular, $x \in C(B_y \cup B_x)$ for every $B_y \in \mathbb{B}_1$.

Similarly, because $y \in C(B)$, for every $b \in B$, and $\forall B_b \in \mathbb{B}_1, \exists B_y \in \mathbb{B}_1\ y \in C(B_b \cup B_y)$. So choose $\omega \in \Omega(B)$ such that $y = \omega(|B|-1), x = \omega(|B|)$. Then we have $y \in \hat{\omega}(|B|-1)$. By the previous paragraph,we have $B_y, B_x \in \mathbb{B}_1$ so that $x \in C(B_y \cup B_x)$ for every $B_y \in \mathbb{B}_1$, and hence $x \in \hat{\omega}(|B|) = C(B)$. Therefore, Assumption 2 and ASBI implies WCA.

$(i) \overset{*}{\Rightarrow} (iii)$: Assuming R is an ordering and $C(.)$ is normal, it suffices to show $\overline{R} = \widetilde{R} = R$.

$x\overline{R}y \Rightarrow xRy$. Since R is an ordering and $C(.)$ is normal, Theorem 1 says WARP holds. Therefore $xRy \Rightarrow \neg y\widetilde{P}x$ and hence $x\widetilde{R}y$.

$x\widetilde{R}y \Rightarrow \neg y\widetilde{P}x$. Because R is an ordering and $C(.)$ is normal, we must have $x \; or \; y \in C(B_x \cup B_y)$. So $\neg y\widetilde{P}x \Rightarrow x\overline{R}y$. Therefore $\overline{R} = \widetilde{R}$.

$x\overline{R}y \Rightarrow xRy$ by definition. For the other direction, suppose xRy, normality and transitivity implies $x \in C(B_x \cup B_y)$. Hence $x\overline{R}y$.

$(iii) \overset{*}{\Rightarrow} (ii)$: Assuming (iii) holds, it suffices to show that $\overline{R}$ is an ordering.

To show $\overline{R}$ is complete, suppose by contradiction, $x, y \notin C(B_x \cup B_y)$. Let $z \in C(B_x \cup B_y)$. Since $x \in C(B_x)$, we have $\forall a \in B_x$, $x\widetilde{R}a$ and hence $\neg a\widetilde{P}x$. But since $z\widetilde{P}x$, $z \notin B_x$. Similarly, $z \notin B_y$, which is impossible. So $\overline{R}$ is complete. This argument also shows that for any $B_a, B_b, B_c \in \mathbb{B}_1$, either a, b or $c \in C(B_a \cup B_b \cup B_c)$.

For transitivity, suppose $x\overline{R}y$, $y\overline{R}z$. It follows that $x\widetilde{R}y$, $y\widetilde{R}z$ and $\neg y\widetilde{P}x$, $\neg z\widetilde{P}y$. It follows from the above argument that we must have $x \in C(B_x \cup B_y \cup B_z)$. Therefore $\forall t \in B_x \cup B_y \cup B_z$, $x\overline{R}t$. So we have $x\overline{R}z$ as desired.

$(ii) \overset{*}{\Rightarrow} (i)$: Assuming (ii), it suffices to show that $R = \overline{R}$.

$x\overline{R}y \Rightarrow xRy$ by definition. Suppose xRy, then $\exists B \in \mathbb{B}$ such that $y \in B$, $x \in C(B)$. So $x\overline{R}y$.

This completes the proof.

□

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