Economics
The Open-Access, Open-Assessment E-Journal

Vol. 6, 2012-12 | April 5, 2012 | http://dx.doi.org/10.5018/economics-ejournal.ja.2012-12

Existence of an Exact Walrasian Equilibrium in Nonconvex Economies

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Abstract The existence of an exact Walrasian equilibrium in nonconvex economies is still a largely unexplored issue. This paper shows that such an equilibrium exists in nonconvex economies by following the 'nearby economy' approach introduced by Postlewaite and Schmeidler (*Approximate Walrasian Equilibrium and Nearby Economies*, 1981) for convex economies. More precisely, the paper shows that any equilibrium price of the convexified version of a nonconvex economy is an equilibrium price also for a set of 'perturbed' economies with the same number of agents. It shows that in this set there are economies that differ from the original economy only as regards preferences or initial endowments.

JEL C62, D51
Keywords Exact Walrasian equilibrium; nonconvex economies; perturbed economies

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Citation Antonio D'Agata (2012). Existence of an Exact Walrasian Equilibrium in Nonconvex Economies. *Economics: The Open-Access, Open-Assessment E-Journal*, Vol. 6, 2012-12. http://dx.doi.org/10.5018/economics-ejournal.ja.2012-12

1 Introduction

The existence of an *exact* walrasian equilibrium in non-convex economies is still a largely unexplored issue. Mas-Colell (1977) shows that that in the space of differentiable economies there exists an open (in an appropriate topology) and dense set of economies such that if one considers a sequence of finite economies with an increasing number of consumers and with limit in this set then, eventually, an exact walrasian equilibrium exists. Smale (1974) shows the existence of an extended equilibrium in a nonconvex differentiable economy. In addition to the differentiability of the economies, Mas Colell's work is constrained by the use of sequences of purely competitive economies, while Smale's work relies upon the use of a nonconventional concept of equilibrium.

Postlewaite and Schmeidler (1981) introduce a "nearby economy" approach to deal with the existence issue in convex economies. They show that if an allocation of any *convex* economy is "approximately" walrasian at price *p*, then it is possible to construct an economy "near" (in terms of an "average" metric) the original where that allocation is walrasian at the same price *p*. Postlewaite and Schmeidler's result is obtained constructively by perturbing the preferences of agents in the original convex economy in such a way that the indifference surface passing through the bundle of the approximate walrasian equilibrium coincides with the original indifference surfaces outside the budget set while inside the budget set it is flattened onto the budget surface, with continuous extensions also to neighboring surfaces. The motivation of this approach is that "If we don't know the characteristics [of the agents in an economy], but rather, we must estimate them, it is clearly too much to hope that the allocation would be walrasian with respect to the estimated characteristics even if it were walrasian with respect to the true characteristics. …… [Thus,] one could not easily pronounce that the procedure generating the allocation was not walrasian by examining the allocations unless one is certain that there have been no errors in determining the agents' characteristics" (Postlewaite and Schmeidler 1981:105–106). More recent economic applications of the "nearby economy" approach along Postlewaite and Schmeidler's interpretation have been provided by Kubler and Schmedders (2005) and Kubler (2007).[1]

Since large but finite nonconvex economies exhibit approximate equilibria (see, e.g. Hildenbrand et al. 1971, Anderson et al. 1982), one may wonder whether Postlewaite and Schmeidler's approach can be used to prove that close to nonconvex economies there exists a nonconvex economy with an exact walrasian equilibrium. As a matter of fact, their approach could immediately be extended in this direction,[2] however, their perturbation rule has the disturbing feature that it

[1] Anderson (1986) develops the "nearby economy" argument within a very general framework and, relying on nonstandard analysis and an appropriate formal language, provides an abstract theorem showing that objects "almost" satisfying a property are "near" an object exactly satisfying that property. He emphasizes also that this approach can be used to obtain existence results and applies his abstract result to show the existence of exact decentralization of core allocations (Anderson 1986: 231).

[2] Anderson (1986)'s existential result could be used as well although it does not provide any information concerning the economy with an exact equilibrium.

yields convexity of the individual demand set at the equilibrium price in the nearby economy.

In this paper we introduce a rule for perturbing the original nonconvex economy which allows to retain nonconvexity of preferences in the perturbed economy also at the equilibrium price, and we show that for any nonconvex economy there is a *set* of perturbed nonconvex economies with the same number of agents as the original which exhibit an exact walrasian equilibrium. We provide also an upper bound on the size of perturbation. More specifically, we show that (a) the equilibrium price of the convexified version of the original economy is also the equilibrium price of an exact equilibrium of any economy in this set; (b) as the number of agents tends to infinity this set becomes closer and closer (in terms of an appropriate metric) to the original economy; (c) in this set there are economies which differ from the original one in terms of initial endowments or in terms of preferences only. In addition, the walrasian equilibrium of the convexified version of the original nonconvex economy is also an exact walrasian equilibrium of the economy perturbed only in preferences and the economies perturbed only in endowments exhibit also a no-trade equilibrium.

The intuition behind our results is very simple: consider a n consumer, k good pure exchange economy satisfying all standard assumptions except convexity of preferences. Under our hypotheses, there exists a strictly positive price vector p_n* and an allocation (x_h*) which are a walrasian equilibrium of the convexified version of the original economy (see, e.g., Hildenbrand 1975: 150). It is possible to deformate continuously consumers' indifference curves parallel to the budget surface in such a way that the walrasian equilibrium consumption bundles become optimal with respect to the new preferences. So, p_n* and (x_h*) are an exact walrasian equilibrium of the economy perturbed in preferences only. In addition, Shapley and Folkman Theorem ensure that the number of consumers whose preferences have to be perturbed is independent upon the number of consumers (to be precise, is not greater than k+1.) Therefore, as the number of consumers increases the distance between the original economy and the perturbed economy tends to zero. It is shown that this logic can be extended to perturbations in preferences and/or endowments.

2 Existence of an Exact Walrasian Equilibrium in Nonconvex Economies

Consider the space $\mathcal{E}_n$ of pure exchange economies $\mathcal{E}_n((u_h), (\omega_h))_{h\in N}$ with n consumers and k goods satisfying the assumptions of strict positivity of the initial endowment vector ω_h and of continuity and strict monotonicity of utility function u_h for each consumer $h \in N = \{1,2,\ldots, n\}$. We assume that the consumption set of all consumers is the non-negative orthant of the k-dimensional Euclidean space, and that there exists a compact subset Ω of this space such that $\omega_h \in \Omega$ for every $h \in N$ and every $n \in \infty$, where ∞ is the set of natural numbers. Denote by $A_h(\cdot,\omega_h)$, $A^{(n)}(\cdot,(\omega_h))$ and $\omega^{(n)}$, respectively, the demand correspondence of agent h, the aggregate demand correspondence and the aggregate endowment of economy $\mathcal{E}_n$.

Symbols co, d and d_H indicate, respectively, the convex hull operator, the Euclidean distance and the Hausdorff distance. Symbols K and $(a)_k$ denote, respectively, the index set of goods (i.e. $K = \{1, 2, \ldots, k\}$) and the k-dimensional vector whose elements are all equal to a. Finally, symbol co$\mathcal{E}_n$, denotes the convexified version of economy $\mathcal{E}_n$; i.e. the economy whose demand correspondence of consumer h is co$A_h(\cdot, \omega_h)$. For any utility function u_h, set $P_{u_h} = \{(x, y) \in \Re_+^{2k} \,|\, u_h(x) \geq u_h(y)\}$.

Given a couple of utility functions u_h and $\hat{u}_h$, the distance δ between the preferences underlying these functions is defined as follows (see Debreu 1969):

$\delta(u_h, \hat{u}_h) = d_H(P_{u_h}, P_{\hat{u}_h}) = \inf\{\varepsilon \in (0, \infty) \,|\, P_{u_h} \subseteq N_\varepsilon(P_{\hat{u}_h}) \text{ and } P_{\hat{u}_h} \subseteq N_\varepsilon(P_{u_h})\}$ where $N_\varepsilon(\cdot)$ is the closed ε-ball around a set. We shall use the same metric m used by Postlewaite and Schmeidler (1981): $m(\mathcal{E}_n, \hat{\mathcal{E}}_n) = \frac{1}{n}\sum_{h \in N}\left(\delta(\mathbf{u}_h, \hat{\mathbf{u}}_h) + \frac{\|\omega_h - \hat{\omega}_h\|}{\omega^{(n)} + \hat{\omega}_h^{(n)}}\right)$, where $\mathcal{E}_n\left((u_h),\ (\omega_h)\right)_{h \in N}$ and $\hat{\mathcal{E}}_n\left((\hat{u}_h),\ (\hat{\omega}_h)\right)_{h \in N}$ are economies in $\mathscr{E}_n$. A *walrasian equilibrium* of economy $\mathcal{E}_n$ is a non-negative price vector $p_n{}^*$ and an allocation $(x_{nh}{}^*)_{h \in N}$ such that: $\omega^{(n)} \in A^{(n)}(p_n{}^*, (\omega_h))$ and $x_{nh}{}^* \in A_h(p_n{}^*, \omega_h)$ for every $h \in N$. The set of walrasian equilibria of economy $\mathcal{E}_n$ is indicated by $\mathrm{W}(\mathcal{E}_n)$. A walrasian equilibrium of the convexified economy co$\mathcal{E}_n$ is defined in an obvious way and the set of these equilibria is indicated by $\mathrm{W}(\mathrm{co}\mathcal{E}_n)$. In the following main result one should keep in mind that under our assumptions set $\mathrm{W}(\mathrm{co}\mathcal{E}_n)$ is non-empty for every $n \in \infty$ (see Lemma 2 in Section 3).

Theorem. *Let* $\mathcal{E}_n((u_h), (\omega_h))_{h\in N} \in \mathscr{E}_n$ *be a pure exchange economy satisfying the stated assumptions and let* $(p_n^*, (x_{nh}^*)_{h\in N}) \in \mathrm{W}(\mathrm{co}\mathcal{E}_n)$. *Then, for every* $\varepsilon > 0$ *there exists a set* $\mathcal{X}_n(p_n^*) \subset \mathscr{E}_n$ *such that*:

(a) *if* $\hat{\mathcal{E}}_n((\hat{u}_h), (\hat{\omega}_h))_{h\in N}) \in \mathcal{X}_n(p_n^*)$ *then there exists an allocation* $(\hat{x}_{nh})_{h\in H}$ *such that* $(p_n^*, (\hat{x}_{nh})_{h\in H}) \in \mathrm{W}(\hat{\mathcal{E}}_n)$. *In addition,* $m(\mathcal{E}_n, \hat{\mathcal{E}}_n) \to 0$ *as* $n \to \infty$; *in particular,* $m(\mathcal{E}_n, \hat{\mathcal{E}}_n) \le \frac{1}{n}((k+1)K_{n\varepsilon} + 1)$ *with* $K_{n\varepsilon} = \sqrt{2}\max_{h\in N}\max_{i\in K}\left[\frac{p_n^*(\omega_h + (\varepsilon)_k)}{p_{ni}^*}\right]$;

(b) *there exist economies* $\bar{\mathcal{E}}_n(((\bar{u}_h), (\bar{\omega}_h))_{h\in N})$ *and* $\tilde{\mathcal{E}}_n((\tilde{u}_h), (\tilde{\omega}_h))_{h\in N})$ *in* $\mathcal{X}_n(p_n^*)$ *with* $\bar{\omega}_h = \omega_h$ *and* $\tilde{u}_h = u_h$ *for every* $h \in N$. *In addition,* $(p_n^*, (x_{nh}^*)_{h\in N}) \in \mathrm{W}(\bar{\mathcal{E}}_n)$ *and* $(p_n^*, (\tilde{\omega}_h)_{h\in H}) \in \mathrm{W}(\tilde{\mathcal{E}})$.

Remark. It is easy to show that for some economies it is easier to obtain an estimate of the smallest element of the equilibrium vector p_n^*, say $p_n^{\min}$, than the price p_n^* itself. In this case, an upper bound for the distance between the original economy and the perturbed one is $\frac{1}{n}((k+1)\sqrt{2}\max_{h\in N}\frac{\sum_{i\in K}\omega_{hi} + \varepsilon k}{p_n^{\min}} + 1)$.

3 Proofs

The next two results are well-known.

Lemma 1. (see, e.g., Balasko (1988, p. 77)) Let $p \in \Re_+^k$ be a price vector. Then, $A_h(p,\omega_h) = A_h(p,\hat{\omega}_h)$ for every $\hat{\omega}_h \in B_h(p,\omega_h) = \{x \in \Re_+^k \mid p\cdot x = p\cdot\omega_h\}$.

Lemma 2. (see, e.g., Hildenbrand (1974, p. 150)) For every n, $W(\mathrm{co}\mathcal{E}_n) \neq \varnothing$. Moreover, if $(p_n^*, (x_{nh}^*)_{h\in N}) \in W(\mathrm{co}\mathcal{E}_n)$, then $p^* \in \Re^k_{++}$.

From now on p_n^* indicates an equilibrium price vector associated to the convexified economy $\mathrm{co}\mathcal{E}_n((u_h), (\omega_h))_{h\in N}$. It is assumed also that the price vector p_n^* belongs to the (k-1)-dimensional unit simplex. By Lemma 2 the budget surface $B_h(p_n^*,\omega_h)$ of consumer h is compact. By Urysohn's Lemma (see, e.g. Willard 1970: 102), for every real number $\varepsilon > 0$ and every $h \in N$ there exists a continuous function $\gamma_{h\varepsilon}: \Re^k_+ \to [0,1]$ (which depends also on p_n^* and ω_h) such that $\gamma_{h\varepsilon}(x) = 1$ if $x \in B_h(p_n^*,\omega_h)$ and $\gamma_{h\varepsilon}(x) = 0$ if $x \in \Re^k_+ \setminus S_{h\varepsilon}(B_h(p_n^*,\omega_h))$ where $S_{h\varepsilon}(B_h(p_n^*,\omega_h)) = \{y \in \Re^k_+ \mid p_n^* \cdot \omega_h - \varepsilon < p_n^* \cdot y < p_n^* \cdot \omega_h + \varepsilon)\}$ is the open ε-"slice" containing set $B_h(p_n^*,\omega_h)$. The shaded area in Figure 1 illustrates set $S_{h\varepsilon}(B_h(p_n^*,\omega_h))$ while segment B indicates the budget line $B_h(p_n^*,\omega_h)$.

Given two vectors $x_h, y_h \in \Re^k_+ \setminus \{0\}$ such that $p_n^* \cdot (x_h - y_h) = 0$, let $t_h(\cdot\,;\varepsilon, p_n^*, x_h, y_h)$ be a mapping defined as follows: $t_h(x;\varepsilon, p_n^*, x_h, y_h) = x + \min_{i\in K}\left(\frac{x_i}{y_{hi}}\right)\gamma_{h\varepsilon}(x)(x_h - y_h)$, where function $\gamma_{h\varepsilon}$ has been previously defined. Intuitively, transformation $t_h(\cdot\,;\varepsilon, p_n^*, x_h, y_h)$ translates any point x in $\Re^l_+$ by the vector $\min_{i\in K}\left(\frac{x_i}{y_{hi}}\right)\gamma_{h\varepsilon}(x)(x_h - y_h)$ perpendicular to p_n^*. In Figure 1 the curved arrows describe the effects of transformation t_h on points on the budget line: for example, point y_h is mapped into point x_h. Moreover, the action of transformation t_h decreases as the axes are approached, and vanishes outside the ε-slice containing the budget line. Given that $p_n^* \cdot (x_h - y_h) = 0$ whenever

$x_h, y_h \in B_h(p_n*, \omega_h)$, mapping t_h is always defined for every $\varepsilon > 0$ and every couple of vectors lying on the budget surface of consumer h at price p_n*.

Lemma 3. *Given* $\varepsilon > 0$ *and* $p_n* \in \Re^k_{++}$, *for every* $x_h, y_h \in B_h(p_n*, \omega_h)$, *map* $t_h(\cdot\,; \varepsilon, p_n*, x_h, y_h)$ *satisfies the following properties*:

(i) $t_h(\cdot; \varepsilon, p_n*, x_h, y_h)$ *maps* $\Re^k_+$ *into itself and is continuous;*

(ii) $p_n* \cdot t_h(x; \varepsilon, p_n*, x_h, y_h) = p_n* \cdot x$ *for every* $x \in \Re^k_+$;

(iii) $t_h(y_h; \varepsilon, p_n*, x_h, y_h) = x_h$;

(iv) $t_h(x; \varepsilon, p_n*, x_h, y_h) = x$ *for every* $x \in \Re^k_+ \backslash S_{h\varepsilon}(B_h(p_n*,\omega_h))$;

(v) *for every* $x \in S_{h\varepsilon}(B_h(p_n*,\omega_h))$ *there exists* $\lambda > 1$ *such that* $\lambda x \in S_{h\varepsilon}(B_h(p_n*,\omega_h))$ *and* $t_h(\lambda x; \varepsilon, p_n*, x_h, y_h) > x$;

Proof. (i) *Continuity is obvious. Take any* $x \in \Re^k_+$, *then, for* $j \in K$,

$$t_{hj}(x; \varepsilon, p_n*, x_h, y_h) = x_j + \min_{i \in K}\left(\frac{x_j}{y_{hi}}\right)\gamma_{h\varepsilon}(x)(x_{hj} - y_{hj}) \geq x_j + \min_{i \in K}\left(\frac{x_j}{y_{hi}}\right)\gamma_{h\varepsilon}(x)x_{hj} - \left(\frac{x_j}{y_{hj}}\right)y_{hj} = \min_{i \in K}\left(\frac{x_j}{y_{hi}}\right)\gamma_{h\varepsilon}(x)x_{hj} \geq 0.$$

Assertions (ii) and (iii) can immediately be verified by substitution. Fact (iv) follows from the properties of function $\gamma_{h\varepsilon}$. As for (v), set: $\lambda* = \sup\{\lambda \in [1, \infty) | \lambda x \in S_{h\varepsilon}(B_h(p_n*, \omega_h))\}$.

Clearly, $\lambda* > 1$. By the fact that $\lambda* x \in \Re^k_+ \backslash \{S_{h\varepsilon}(B_h(p_n*, \omega_h))\}$ and by fact (iv), $t_h(\lambda* x; \varepsilon, p_n*, x_h, y_h) = \lambda* x > x$. By continuity, there exists a real number λ, $1 < \lambda < \lambda*$, such that $t_h(\lambda x; \varepsilon, p_n*, x_h, y_h) > x$. ♦

Transformation $t_h(\cdot;\varepsilon, p_n^*, x_h, y_h)$ is used to "perturbate" preferences by the following rule $\hat{u}_h(\cdot) = u_h(t_h(\cdot;\varepsilon, p_n^*, x_h, y_h))$. Figure 1 illustrates the effects of transformation t_h on the indifference curve $u_h(x) = c$ in case $k = 2$. The dotted curve is the part of the indifference curve of $\hat{u}_h$ in the ε- slice containing the budget line. For any other point outside this set, the indifference curve of $\hat{u}_h$ coincides with the indifference curve of u_h. Intuitively, under transformation $t_h(\cdot;\varepsilon, p_n^*, x_h, y_h)$ vector y_n is optimal with respect to preferences $\hat{u}_h$ at price vector p_n^*, while vector x_h is optimal with respect to preferences u_h at the same price vector.

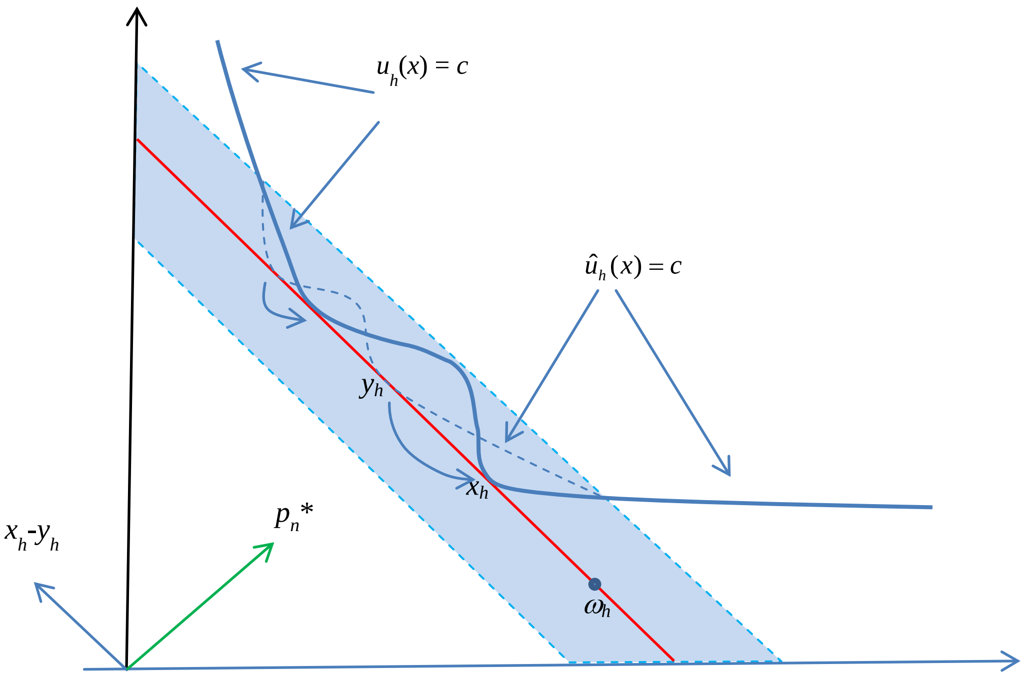

Figure 1. Transformation $t_h(\cdot;\varepsilon, p_n^*, x_h, y_h)$

Lemma 4. *Given* $\varepsilon > 0$ *and for every* $p_n^* \in \Re^k_{++}$ *and every* $x_h, y_h \in B_h(p_n^*, \omega_h)$, *if utility functions* u_h *and* $\hat{u}_h$ *satisfy* $\hat{u}_h(x) = u_h(t_h(x; \varepsilon, p_n^*, x_h, y_h))$, *then* $\delta(u_h, \hat{u}_h) \leq \sqrt{2} \max_{i \in K} \left[\frac{p_n^*(\omega_h + (\varepsilon)_k)}{p_{ni}^*} \right]$.

Proof. By Lemma 3(iv), $\hat{u}_h(x) = u_h(x)$ for $x \in \Re^k_+ \backslash S_{h\varepsilon}(B_h(p_n^*, \omega_h))$, so, preferences of consumer h perturbed by t_h differ only inside set $S_{h\varepsilon}(B_h(p_n^*, \omega_h))$ with respect to the original preferences. Therefore, in order to prove the assertion it is enough to consider couples of vectors only in this set. Take $(x, y) \in P_{u_h}$ with $x, y \in S^2_{h\varepsilon} = S_{nh\varepsilon}(B_h(p_n^*, \omega_h)) \times S_{nh\varepsilon}(B_h(p_n^*, \omega_h))$ and suppose that $(x, y) \notin P_{\hat{u}_h}$ (otherwise there is nothing to prove), i.e. $\hat{u}_h(x) \leq \hat{u}_h(y)$ or $u_h(t_h(x)) \leq u_h(t_h(y))$. Since $x \in S_{h\varepsilon}(B_h(p_n^*, \omega_h))$, then by Lemma 3(ii), $t_h(x) \in S_{h\varepsilon}(B_h(p_n^*, \omega_h))$. By Lemma 3(v) there exists $\lambda > 1$ such that $\lambda t_h(y) \in S_{h\varepsilon}(B_h(p_n^*, \omega_h))$ and $t_h(\lambda t_h(y)) > t_h(y)$. Set $x' = \lambda t_h(y)$ and $y' = y$. By monotonicity, it follows that $u_h(t_h(\lambda t_h(y)) = \hat{u}_h(x') > u_h(t_h(y)) = \hat{u}_h(y')$, that is $(x', y') \in P_{\hat{u}_h}$ with $(x', y') \in S^2_{h\varepsilon}$. Therefore, $d((x, y), (x', y')) \leq \text{diam} S^2_{nh\varepsilon}$ where diam indicates the diameter of a set. Suppose now that $(x, y) \in P_{\hat{u}_h}$ with $(x, y) \in S^2_{nh\varepsilon}$, and, again, $(x, y) \notin P_{u_h}$. Then, $u_h(x) \leq u_h(y)$. Take $x' = \lambda y$ and $y' = y$ where $\lambda > 1$ is such that $\lambda y \in S_{nh\varepsilon}(B_h(p_n^*, \omega_h))$. Hence, $u_h(x') \geq u_h(y')$, that is, there exists a point $(x', y') \in P_{u_h}$ with $(x', y') \in S^2_{h\varepsilon}$. Thus, again $d((x, y), (x', y')) \leq \text{diam} S^2_{nh\varepsilon}$. Therefore, $\delta(u_h, \hat{u}_h) \leq \text{diam} S^2_{nh\varepsilon}$. Notice now that vector $\omega_h + (\varepsilon)_k$ does not belong to $S_{h\varepsilon}(B_h(p_n^*, \omega_h))$ because $p_n^* \cdot (\omega_h + (\varepsilon)_k) - p_n^* \cdot \omega_h = \varepsilon$ where the equality follows from the fact that p_n^* belongs to the (k-1)-dimensional unit simplex. Thus,

$S_{nh\varepsilon}(B_h(p_n*,\omega_h)) \subset \left\{x \in \Re_+^k \,\middle|\, p_n * x \le p_n *(\omega_h + (\varepsilon)_k)\right\} \subset \Delta_{nh\varepsilon}$, where $\Delta_{nh\varepsilon}$ is the k-dimensional closed simplex with vertices (0, 0,…, 0), ($W_{nh\varepsilon}$, 0, …, 0), …., (0, 0, …, $W_{nh\varepsilon}$) and $W_{nh\varepsilon} = \max_{i \in K}\left[\dfrac{p_n *(\omega_h + (\varepsilon)_k)}{p_{ni} *}\right]$ is well-defined because of the strict positivity of p_n* (Lemma 2). Hence, $S_{h\varepsilon}^2 \subset \Delta_{nh\varepsilon}^2 = \Delta_{nh\varepsilon} \times \Delta_{nh\varepsilon}$ and, consequently,

$$\text{diam}S_{h\varepsilon}^2 \le \text{diam}\Delta_{nh}^2 = \sqrt{2}\cdot W_{nh\varepsilon} = \sqrt{2}\max_{i\in K}\left[\frac{p_n *(\omega_h + (\varepsilon)_k)}{p_{ni} *}\right]. \ \blacklozenge$$

Proof of Theorem. (a) Set $N(p_n*) = \left\{(x_h)_{h\in N} \in \Re_+^{k\times n} \,\middle|\, \sum_{h\in N} x_h \in \text{co}A^{(n)}(p_n*,(\omega_h)), p_n*\cdot x_h = p_n*\cdot\omega_h, h \in N\right\}$, that is, $N(p_n*)$ denotes the set of allocations which are feasible in terms of vectors in set $\text{co}A^{(n)}(p_n*,(\omega_h))$ and which maintain constant consumers' income with respect to price p_n* and to the initial allocation $(\omega_h)_{h\in N}$. By Lemma 1, $A_h(p_n*,\hat{\omega}_h) = A_h(p_n*,\omega_h)$ for every $h \in H$ and, therefore, $A^{(n)}(p_n*,(\omega_h)) = A^{(n)}(p_n*,(\hat{\omega}_h))$ whenever $(\hat{\omega}_h)_{h\in N} \in N(p_n*)$.

Take any $(\hat{\omega}_h)_{h\in N} \in N(p_n*)$. Hence, $\hat{\omega}^{(n)} \in \text{co}A^{(n)}(p_n*,(\omega_h)) = \text{co}A^{(n)}(p_n*,(\hat{\omega}_h))$. Then $\hat{\omega}^{(n)} = \sum_{i=1}^{\hat{t}} \hat{\alpha}_i^n \hat{x}_i^{(n)} = \sum_{i=1}^{\hat{t}} \hat{\alpha}_i^n \sum_{h\in N} \hat{x}_{nhi}$ with $0 \le \hat{\alpha}_i^n \le 1$, $\sum_{i=1}^{\hat{t}} \hat{\alpha}_i^n = 1$ and $1 \le \hat{t} \le k+1$, $\hat{x}_i^{(n)} = \sum_{h\in N} \hat{x}_{nhi} \in A^{(n)}(p_n*, (\hat{\omega}_h))$, and $\hat{x}_{nhi} \in A_h(p_n*, \hat{\omega}_h)$ for every i = 1,2,…, $\hat{t}$ and every $h \in H$. Therefore, $\hat{\omega}^{(n)} = \sum_{h\in N} \hat{y}_{nh}$ where $\hat{y}_{nh} = \sum_{i=1}^{\hat{t}} \hat{\alpha}_i^n \hat{x}_{nhi} \in \text{co}A_h(p_n*,\omega_h) = \text{co}A_h(p_n*, \hat{\omega}_h)$. By Shapley-Folkman Theorem there exists a subset $\hat{J}_n \subset N$ with $\#\hat{J}_n \le k+1$, such that $\hat{\omega}^{(n)} = \sum_{h'\in N\setminus \hat{J}_n} \hat{y}_{nh'} + \sum_{h''\in \hat{J}_n} \hat{y}_{nh''}$ where $\hat{y}_{nh'} \in A_{h'}(p_n*,\hat{\omega}_{h'})$ and $\hat{y}_{nh''} \in \text{co}A_{h''}(p_n*,\hat{\omega}_{h''})$. Let $\{\hat{x}_{nh}\}_{h\in N}$ be a family of vectors defined as follows:

$\hat{x}_{nh'} = \hat{y}_{nh'}$ for $h' \in N \setminus \hat{J}_n$ and $\hat{x}_{nh''} \in \{x \in A_{h''}(p_n{*}, \hat{\omega}_{h''}) \,|\, d(x, \hat{y}_{nh''}) \leq d(z, \hat{y}_{nh''})$, for every $z \in A_{h''}(p_n{*}, \hat{\omega}_{h''})\}$ for $h'' \in \hat{J}_n$.

Consider now the set $\mathscr{X}_n(p_n{*})$ of perturbed economies $\hat{\mathscr{E}}_n((\hat{u}_h), (\hat{\omega}_h))_{h\in N}$ defined as follows: $(\hat{\omega}_h)_{h\in N} \in N(p_n{*})$ and $\hat{u}_h(x) = u_h(t_h(x))$ where $t_h(x) = x + \min_{i\in K}\left(\frac{x_j}{\hat{y}_{nhi}}\right)\gamma_{h\varepsilon}(x)(\hat{x}_{nh} - \hat{y}_{nh})$ for $h \in N$ (in what follows, for the sake of simplicity, we drop parameters ε, $\hat{x}_{nh}$ and $\hat{y}_{nh}$ in t_h.) By construction, $p_n{*} \cdot (\hat{x}_{nh} - \hat{y}_{nh}) = 0$ for $h \in N$ and, moreover, $\hat{u}_{h'} = u_{h'}$ for $h' \in N \setminus \hat{J}_n$.

We show that $(p_n{*}, (\hat{y}_{nh})_{h\in N}) \in W\left(\hat{\mathscr{E}}_n\right)$ for every $\hat{\mathscr{E}}_n((\hat{u}_h), (\hat{\omega}_h))_{h\in N}$ in $\mathscr{X}_n(p_n{*})$. First, by construction $\hat{\omega}^{(n)} = \sum_{h\in N} \hat{\omega}_h = \sum_{h\in N} \hat{y}_{nh}$, so allocation $(\hat{y}_{nh})_{h\in N}$ is feasible. That $\hat{y}_{nh} \in B_h(p_n{*}, \hat{\omega}_h)$ for every $h \in H$, follows again by construction. Vector $\hat{y}_{nh'}$ is optimal for agent $h' \in N \setminus \hat{J}_n$ because $\hat{y}_{nh'} \in A_{h'}(p_n{*}, \hat{\omega}_{h'})$. We now show that $\hat{u}_{h''}(\hat{y}_{nh''}) \geq \hat{u}_{h''}(x)$ for every $x \in B_{h''}(p_n{*}, \hat{\omega}_{h''})$ and every $h'' \in \hat{J}_n$. To this end, notice that, by Lemma 3(ii), transformation t_h maps the budget surface into itself. So, by monotonicity of preferences, we can focus only on vectors on the latter. Thus, suppose that there exists $\tilde{x} \in B_{h''}(p_n{*}, \hat{\omega}_{h''})$ with $\tilde{x} \neq \hat{y}_{nh''}$ such that $\hat{u}_{h''}(\tilde{x}) > \hat{u}_{h''}(\hat{y}_{nh''})$. Hence, by definition, $u_{h''}(t_{h''}(\tilde{x})) > u_{h''}(t_{h''}(\hat{y}_{nh''}))$. By Lemma 3(iii), this implies that $u_{h''}(t_{h''}(\tilde{x})) > u_{h''}(\hat{x}_{nh''})$. Since $t_{h''}(\tilde{x}) \in B_{h''}(p_n{*}, \hat{\omega}_{h''})$, this contradicts the fact that $\hat{x}_{nh''} \in A_{h''}(p_n{*}, \hat{\omega}_{h''})$ for $h'' \in \hat{J}_n$.

We now provide an upper bound for $m(\mathscr{E}_n, \hat{\mathscr{E}}_n)$ and show that $m(\mathscr{E}_n, \hat{\mathscr{E}}_n) \to 0$ as $n \to \infty$. First, as already noticed, in the perturbed economy $\hat{\mathscr{E}}_n$, $\hat{u}_{h'}(x) = u_{h'}(x)$ for $h' \in N \setminus \hat{J}_n$. Hence, $m(\mathscr{E}_n, \hat{\mathscr{E}}_n) = \frac{1}{n}\sum_{h''\in \hat{J}_n} \delta(\mathbf{u}_{h''}, \hat{\mathbf{u}}_{h''}) + \frac{1}{n}\sum_{h\in N}\left(\frac{\|\omega_h - \hat{\omega}_h\|}{\omega^{(n)} + \hat{\omega}^{(n)}}\right)$.

By Lemma 4, $m(\mathcal{E}_n, \hat{\mathcal{E}}_n) \leq \frac{k+1}{n} K_{n\varepsilon} + \frac{1}{n} \sum_{h \in N} \frac{\|\omega_h\| + \|\hat{\omega}_h\|}{\omega^{(n)} + \hat{\omega}^{(n)}} = \frac{1}{n}((k+1)K_{n\varepsilon} + 1)$ where $K_{n\varepsilon} = \sqrt{2} \max_{h \in N} \max_{i \in K} \left[\frac{p_n * \{\omega_h + (\varepsilon)_k\}}{p_{ni} *} \right]$. Under our assumptions, $p_{ni} * \to p_i > 0$ (see, e,g,, Hildenbrand and Kirman 1988: 93) and all vectors ω_h belong to the compact set Ω, hence $m\left(\hat{\mathcal{E}}_n, \mathcal{E}_n\right) \to 0$ as $n \to \infty$.

(b) In the particular case $(\hat{\omega}_h)_{h \in N} = (\omega_h)_{h \in N} \in N\left(p_n *\right)$, notice that, by definition, the walrasian equilibrium $(p_n *, (x_{nh} *)_{h \in N})$ of $\mathrm{co}\mathcal{E}_n$ satisfies the condition: $\omega^{(n)} = \sum_{h=1}^{n} x_{nh} * \in \mathrm{co}A^{(n)}(p_n *, (\omega_h))$. Therefore, there are $t(1 \leq t \leq k+1)$ real numbers $\alpha_{ni} > 0$ such that $\sum_{i=1}^{t} \alpha_{ni} = 1$ and $\omega^{(n)} = \sum_{i=1}^{t} \alpha_{ni} x_i^{(n)} = \sum_{i=1}^{t} \alpha_{ni} \sum_{h=1}^{n} x_{nhi}$ where $x_i^{(n)} \in A^{(n)}(p_n *, (\omega_h))$ and $x_{nhi} \in A_h(p_n *, \omega_h)$ for every i = 1,2., …, t and every $h \in N$. We have also, $x_{nh} * = \sum_{i=1}^{t} \alpha_{ni} x_{nhi} \in \mathrm{co}A_h(p_n *, \omega_h)$. By Shapley-Folkman Theorem there exists a subset $J_n \subset N$ with $\# J_n \leq k+1$, such that $\omega^{(n)} = \sum_{h' \in N \setminus J_n} x_{nh'} * + \sum_{h'' \in J_n} x_{nh''} *$ where $x_{nh'} * \in A_{h'}(p_n *, \omega_{h'})$ and $x_{nh''} * \in \mathrm{co}A_{h''}(p_n *, \omega_{h''})$. Let $\{\hat{x}_{nh}\}_{h \in N}$ be a family of vectors defined as follows: $\hat{x}_{nh'} = x_{nh'} *$ for $h' \in N \setminus J_n$ and $\hat{x}_{nh''} \in \{x \in A_{h''}(p_n *, \omega_{h''}) \mid d(x, x_{nh''} *) \leq d(z, x_{nh''} *)$, for every $z \in A_{h''}(p_n *, \omega_{h''})\}$ for $h'' \in J_n$. Consider now the perturbed economy $\bar{\mathcal{E}}(((\bar{u}_h), (\omega_h))_{h \in H})$ obtained from the original one by changing only utility functions as follows: $\bar{u}_h(x) = u_h(t_h(x))$ where $t_h(x) = x + \min_{i \in K} \left(\frac{x_i}{x_{nhi} *} \right) \gamma_{h\varepsilon}(x)(\hat{x}_{nh} - x_{nh} *)$ for $h \in N$. By a similar argument used before, it is possible to show that $\bar{u}_h(x_{nh} *) \geq \bar{u}_h(x_h)$ for every $x \in B_h(p_n *, \omega_h)$ and every $h \in N$. Since the allocation $(x_{nh} *)_{h \in N}$ is feasible, one obtains that $(p_n *, (x_{nh} *)_{h \in N}) \in \mathrm{W}\left(\bar{\mathcal{E}}_n\right)$.

Finally, since $\omega^{(n)} \in \mathrm{co}A^{(n)}(p_n*,(\omega_h))$, choose $(\tilde{\omega}_h)_{h\in N} \in N(p_n*)$ such that $\tilde{\omega}^{(n)} \in A^{(n)}(p_n*,(\omega_h))$. Consider the perturbed economy $\tilde{\mathscr{E}}_n(((u_h),(\tilde{\omega}_h))_{h\in N})$. By construction, there exist n vectors such that $\tilde{y}_{nh} \in A_h(p_n*,\tilde{\omega}_h)$ for every $h \in N$ and $\tilde{\omega}^{(n)} = \sum_{h\in N} \tilde{y}_{nh}$. It follows that $(p_n*,(\tilde{y}_{nh})_{h\in N}) \in \mathsf{W}(\tilde{\mathscr{E}}_n)$. It is obviously possible to choose vectors $\tilde{\omega}_h$ in such a way that $\tilde{\omega}_h = \tilde{y}_{nh}$ for every $h \in N$, which implies that $(p_n*,(\tilde{\omega}_{nh})_{h\in N}) \in \mathsf{W}(\tilde{\mathscr{E}}_n)$. ♦

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