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News versus Sunspot Shocks in a New Keynesian Model

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Abstract Separately, news and sunspot shocks have been shown empirically to be determinants of changes in expectations. This paper considers both of them together in a simple New Keynesian monetary business cycle model. A full set of rational expectations solutions is derived analytically. The analytical characterization allows an explicit comparison of news about future monetary policy and sunspots. The key distinction between the shocks lies in their relation to the realized policy shock. If monetary policy is "passive", both types of shocks affect model dynamics through forecast errors. The effect of the news on forecast errors is not unique, and the dynamics induced by news and sunspot shocks can be observationally equivalent. If monetary policy is "active", the sunspots are irrelevant, and the model responses to the news shocks are unique. In both cases, news shocks strengthen the endogenous propagation of the model, since anticipation of future changes prolongs agents' reaction. Keywords: news shocks, sunspots, expectations, monetary policy, indeterminacy.

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Keywords News shocks; sunspots; expectations; monetary policy; indeterminacy

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1 Introduction

Changes in expectations, induced by either sunspots or news about future fundamentals, are potentially important in explaining aggregate fluctuations. While there is some empirical support for both sunspots and news shocks, effects of these shocks have been analyzed separately. This paper makes the first step towards understanding the similarities and differences between these two types of shocks.

A New Keynesian monetary business cycle model is used for analysis. The model exhibits indeterminacy and permits sunspot shocks if a central bank, following an interest rate rule, is not aggressive enough on inflation. Lubik and Schorfheide (2004) argue that indeterminacy and possibly the existence of sunspot shocks may be relevant for understanding the dynamics of U.S. output, inflation and interest rates in a pre-Volcker period. In this paper, the sunspots are compared with news shocks about future monetary policy. These news shocks are partly motivated by the results of Cochrane (1998). He finds that decomposition of monetary policy shocks into unanticipated and anticipated components can influence significantly the measured output responses to changes in monetary policy.

To compare the news and sunspot shocks, a full set of rational expectations solutions is derived analytically following a method of Lubik and Schorfheide (2003). The properties of the news shock are characterized analytically. This is a contribution to the existing literature on news shocks that investigates the role of these shocks only by numerical exercises. The dynamic behaviour of the model in the presence of the news shocks is also studied numerically by conducting impulse response analysis.

The paper is organized as follows. Section 2 describes the model. Sections 3 and 4 characterize analytical solutions under determinacy and indeterminacy. The analytical solutions and impulse response functions are used to study the effects of news and sunspot shocks. Section 5 concludes.

2 Model

News shocks about future monetary policy are introduced into a model studied by Lubik and Schorfheide (2003). The model is summarized by the following four equations:

IS equation	$x_t = E_t x_{t+1} - \sigma \left(R_t - E_t \pi_{t+1} \right),$	(1)
Phillips curve	$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t,$	(2)
Monetary policy rule	$R_t = \psi \pi_t + \varepsilon_t,$	(3)
Monetary policy shock	$\varepsilon_t = \upsilon_t + \mu_{t-n}, n \geq 1.$	(4)

Output x_t, inflation π_t and the nominal interest rate R_t are expressed as log-deviations from the unique non-stochastic steady state. The parameter β, $0 < \beta < 1$, is the discount factor, $\sigma > 0$ is the intertemporal elasticity of substitution, $\kappa > 0$ is related to the speed of price adjustment and $\psi \geq 0$ measures the elasticity of the interest rate response to inflation.

An exogenous policy shock ε_t is partly anticipated in advance. Impulses υ_t and μ_t are uncorrelated over time and with each other. They are observed by the agents in the model. Since μ_t affects the policy shock with a delay, it is called a *news shock*. The impulse υ_t represents an *unexpected policy shock*. In addition, the agents observe an exogenous sunspot shock ζ_t, unrelated to υ_t and μ_t and satisfying $E_{t-1}\zeta_t = 0$.

The model is solved with the method of Lubik and Schorfheide (2003). The analytical solution is derived for $n = 1$. A numerical method is used to conduct impulse response analysis for $n = 3$. The derivations are provided in the Technical Appendix (Appendix 1).

The analytical solution is derived by analyzing the three-dimensional system of the forecasts of output $\xi_t^x \equiv E_t \left(x_{t+1} \right)$, inflation $\xi_t^\pi \equiv E_t \left(\pi_{t+1} \right)$ and the policy shock $\xi_t^\varepsilon \equiv E_t \left(\varepsilon_{t+1} \right)$, described by

$$\xi_t = \underbrace{\begin{bmatrix} 1+\frac{\kappa\sigma}{\beta} & \sigma\left(\psi-\frac{1}{\beta}\right) & \sigma \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Gamma_1^*}\xi_{t-1} + \underbrace{\begin{bmatrix} \sigma & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\Psi^*}\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \underbrace{\begin{bmatrix} 1+\frac{\kappa\sigma}{\beta} & \sigma\left(\psi-\frac{1}{\beta}\right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \\ 0 & 0 \end{bmatrix}}_{\Pi^*}\eta_t \tag{5}$$

Here $\xi_t = [\xi_t^x, \xi_t^\pi, \xi_t^\varepsilon]'$ and the vector $\eta_t = [\eta_t^x, \eta_t^\pi]'$ represents the rational expectations forecast errors $\eta_t^x \equiv x_t - \xi_{t-1}^x$ and $\eta_t^\pi \equiv \pi_t - \xi_{t-1}^\pi$. The stability properties of the model are governed by the eigenvalues of the matrix Γ_1^*, defined by $\lambda_0 = 0$ and $\lambda_{1,2} = \frac{1}{2}\left(1+\frac{\kappa\sigma+1}{\beta}\right) \mp \frac{1}{2}\sqrt{\left(\frac{1+\kappa\sigma}{\beta}-1\right)^2 + \frac{4\kappa\sigma}{\beta}(1-\psi)}$. The eigenvalues λ_1 and λ_2 are identical to the ones in the model without news shocks. Thus, the stability properties of the model are not affected by the presence of news. If $\psi > 1$, the stable solution is unique. If $0 \leq \psi \leq 1$, there are multiple stable solutions.

3 Determinacy

The unique stable solution under determinacy for $n = 1$ is

$$\begin{bmatrix} x_t \\ \pi_t \\ R_t \end{bmatrix} = \frac{1}{1+\kappa\sigma\psi}\begin{bmatrix} -\sigma \\ -\kappa\sigma \\ 1 \end{bmatrix}\underbrace{(\upsilon_t + \mu_{t-1})}_{\varepsilon_t} - \frac{\sigma}{(1+\kappa\sigma\psi)^2}\begin{bmatrix} 1+\kappa\sigma(1-\beta\psi) \\ \kappa(1+\beta+\kappa\sigma) \\ \psi\kappa(1+\beta+\kappa\sigma) \end{bmatrix}\underbrace{\mu_t}_{E_t\varepsilon_{t+1}}. \tag{6}$$

News shocks convey the information about future policy changes. Rational agents will update their beliefs about endogenous variables, based on this information. For $n=1$, the optimal forecasts of output and inflation for one period ahead and their forecast errors are driven by the news shock

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \frac{-\sigma}{1+\kappa\sigma\psi} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} \mu_t, \tag{7}$$

$$\eta_t = -\frac{\sigma}{1+\kappa\sigma\psi} \begin{bmatrix} 1 & \frac{1+\kappa\sigma(1-\beta\psi)}{1+\kappa\sigma\psi} \\ \kappa & \kappa\frac{1+\beta+\kappa\sigma}{1+\kappa\sigma\psi} \end{bmatrix} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix}. \tag{8}$$

The model's implications are best understood by analyzing three experiments: (i) unexpected monetary expansion ($\varepsilon_t = \upsilon_t < 0$, $\mu_{t-n} = 0$), (ii) realized news about future monetary expansion ($\varepsilon_{t+n} = \mu_t < 0$, $\upsilon_{t+n} = 0$) and (iii) unrealized news about future monetary expansion ($\varepsilon_{t+n} = 0$, $\mu_t < 0$, $\upsilon_{t+n} = -\mu_t$).

An unexpected monetary expansion leads to a one period increase in output and inflation, but to a fall in the nominal and expected real interest rates. The transmission mechanism for this experiment is well known and explained, for example, in Galí (2003). Figure 1 in Appendix 2 provides a graphical illustration. The solid lines on panels *A* and *B* plot the impulse responses to an unexpected interest rate cut of 25 basis points in period one for two values $\psi = 1.05$ and $\psi = 2.19$. The other parameters are $\beta = 0.99$, $\kappa = 0.5$ and $\sigma = 1$, as in Lubik and Schorfheide (2003). A more active policy, associated with higher ψ, influences the magnitude, but not the direction of the responses.

News shocks generate more interesting dynamics, since beliefs about the future trigger the agents' reactions before the actual policy change. Foreseeing a future expansion, firms increase their prices. Positive inflation raises the nominal interest rate through the policy feedback rule. The responses of real variables are due to nominal rigidities. The expected real interest rate increases immediately by $\frac{-\kappa\sigma(1-\psi(1+\beta))}{(1+\kappa\sigma\psi)^2}$, but is predicted to fall by $\frac{1}{1+\kappa\sigma\psi}$ when the policy change takes place. Output depends on the entire path of the expected real interest rates

$$x_t = -\sigma \sum_{j=0}^{\infty} E_t \left[R_{t+j} - \pi_{t+1+j}\right]. \tag{9}$$

When $n=1$, only two terms in (9) are non-zero. A contemporaneous increase in the expected real rate affects output negatively, while its lower value in the next period has a stimulative effect. If the coefficient ψ in the monetary policy rule is sufficiently high ($\psi > \frac{1+\kappa\sigma}{\beta\kappa\sigma}$), news about a future expansion triggers a temporary contraction.

An expansionary news shock is more likely to decrease output when the number of anticipation periods is larger. As the expected real interest rate remains above its steady state value longer, it becomes easier to overturn the stimulative effect of the future rate decline. Panels A and B of Figure 1 in Appendix 2 plot the impulse responses to news shocks for $n = 3$. The responses correspond to a belief, formed in period one that in period four the interest rate will be cut by 25 basis points. This belief is validated for a realized news shock (R-News), but is followed by no policy change for an unrealized news shock (U-News).

The responses to the realized and unrealized news shocks coincide until period four, when the actual policy shock ε_4 is observed. Along the transition path, the expected real interest rate rises, stimulating output growth due to consumers' preferences for consumption smoothing. In period four, the agents adjust their behavior, depending on the actual policy shock.

Overall, there are noticeable differences in responses of output, inflation and interest rates to unexpected policy and news shocks. News shocks strengthen the endogenous propagation of the model, since anticipation of future changes prolongs agents' reaction. Further, news shocks can generate fluctuations in the endogenous variables without any actual policy changes.

4 Indeterminacy

A full set of stable rational expectations solutions under indeterminacy can be written as

$$\begin{bmatrix} x_t \\ \pi_t \\ R_t \end{bmatrix} = \frac{1}{1+\kappa\sigma\psi} \begin{bmatrix} -\sigma \\ -\kappa\sigma \\ 1 \end{bmatrix} \underbrace{(\upsilon_t + \mu_{t-1})}_{\varepsilon_t}$$
$$-\frac{\sigma}{(1+\kappa\sigma\psi)^2} \begin{bmatrix} 1+\kappa\sigma(1-\beta\psi) \\ \kappa(1+\beta+\kappa\sigma) \\ \psi\kappa(1+\beta+\kappa\sigma) \end{bmatrix} \underbrace{\mu_t}_{E_t\varepsilon_{t+1}} + \frac{1}{d} \begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \\ \psi\kappa\lambda_2 \end{bmatrix} M \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix}$$
$$+\frac{1}{d} \begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \\ \psi\kappa\lambda_2 \end{bmatrix} \zeta_t + \begin{bmatrix} (\beta(\lambda_2-1)-\sigma\kappa)/\kappa \\ 1 \\ \psi \end{bmatrix} \omega_{t-1}, \tag{10}$$

where

$$\begin{aligned} \omega_t &= \lambda_1\omega_{t-1} + \frac{\kappa(1+\kappa\sigma\psi)}{\beta d} M \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \frac{\kappa(1+\kappa\sigma\psi)}{\beta d}\zeta_t, \qquad (11) \\ M &\equiv \begin{bmatrix} m_1 & m_2 \end{bmatrix},\ d \equiv \sqrt{(\kappa\lambda_2)^2 + (\lambda_2 - 1 - \kappa\sigma\psi)^2}. \end{aligned}$$

A particular solution is obtained by assigning specific values to the coefficients m_1 and m_2. The representation $(10)-(11)$ is centered around a solution for which the contemporaneous impact of fundamental shocks is continuous on the boundary of determinacy and indeterminacy region. Thus, the form of the solution coincides with the one under determinacy, given by (6), when $m_1 = 0$ and $m_2 = 0$.

Under indeterminacy, both news and sunspot shocks can trigger forecast revisions of the endogenous variables. In particular, the optimal forecasts of output and inflation for one period ahead and their forecast errors are influenced by both the news and sunspot shocks

$$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \frac{-\sigma}{1+\kappa\sigma\psi} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} \mu_t + \begin{bmatrix} (\beta(\lambda_2-1)-\sigma\kappa)/\kappa \\ 1 \end{bmatrix} \omega_t, \tag{12}$$

$$\eta_t = -\frac{\sigma}{1+\kappa\sigma\psi}\begin{bmatrix} 1 & \frac{1+\kappa\beta(1-\beta\psi)}{1+\kappa\sigma\psi} \\ \kappa & \frac{\kappa(1+\beta+\kappa\sigma)}{1+\kappa\sigma\psi} \end{bmatrix}\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \frac{1}{1+\kappa\sigma\psi}\begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{bmatrix}\left(M \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \zeta_t \right) \tag{13}$$

with ω_t given by (11).

In the terminology of Lubik and Schorfheide (2003), the variable ζ_t is a reduced form sunspot shock. The impact of this shock on the endogenous variables is determined uniquely. It can be shown that $\lambda_2 > 1+\kappa\sigma\psi$ and $\lambda_2 > 1+\sigma\kappa/\beta$ for all values of the parameters in the indeterminacy region. Thus, a positive realization of the reduced form sunspot shock increases inflation and output, as well as their future forecasts. To combat inflation, the central bank raises the nominal interest rate. However, the expected real interest rate declines, stimulating current output. Thus, changes in beliefs, induced by the sunspot shock, become self-fulfilling: forecasts of higher output or inflation are validated by the actual increase in inflation and output. The solid lines on Panel *C* of Figure 1 in Appendix 2 plot the impulse responses to a sunspot shock of 0.5% for $\psi = 0.95$.

In contrast to the reduced form sunspot shock, the impacts of news and unexpected policy shocks on the endogenous variables are influenced by arbitrary parameters. Panel *C* of Figure 1 in Appendix 2 shows the impulse responses to a realized news shock for two values of m_2. The news shock corresponds to a belief, formed in period one that in period two the interest rate will be cut by 25 basis points. This belief is confirmed in period two. When $m_2 = 0$, the responses resemble the ones under determinacy. When $m_2 < 0$, a news shock about future monetary expansion increases output, inflation and interest rates. These impact responses coincide qualitatively with the responses to the reduced form sunspot shock. With $m_2 = -2$, the impulses responses to a realized news shock and a positive sunspot are closely matched quantitatively. Ambiguity in the model responses to news shocks points to a potential difficulty in empirical evaluation of news and sunspot shocks as sources of changes in expectations.

News shocks capture the idea that policy changes can be anticipated, for example, from the central bank's announcements. One can imagine that agents receive noisy signals about future monetary policy, update their beliefs, based on these signals, and learn whether their previous beliefs were correct by observing the actual realizations of the policy shock.[1] Under the rational expectations hypothesis, the policy shock is a sum of one-period-ahead expectation revisions $\varepsilon_t = \sum_{j=0}^{\infty} (E_{t-j}\varepsilon_t - E_{t-j-1}\varepsilon_t)$. The decomposition of ε_t in (1) is consistent with the assumption that there is only one signal, n periods before the realization of the policy shock. Thus, beliefs about ε_t are updated only in periods $t-n$ and t. Impulses μ_{t-n} and υ_t represent these updates. Since the correlation between signals and policy shocks need not be perfect, an anticipated policy change may not take place, leading to ex-post mistakes. Modelling these mistakes is one of the motivating factors for studying news shocks, as argued by Beaudry and Portier (2004).

5 Conclusions and Implications

This paper compared news and sunspot shocks analytically and numerically in a simple New Keynesian monetary model. Both types of shocks were shown to trigger changes in beliefs. However, there is an importance distinction between the two shocks in their relation to realized fundamental or structural shocks, such as a monetary policy shock analyzed in the paper. A news shock can be interpreted as an anticipated component of a particular fundamental shock. Thus, news shocks are correlated with future values of some fundamental shocks by definition. By contrast, sunspot shocks can be modelled as uncorrelated with any fundamental shocks. This distinction might be potentially exploited in evaluating the relative empirical importance of news and sunspot shocks. The availability of the observed measure of a fundamental shock associated with news shocks would likely be critical for such empirical evaluation.

The New Keynesian example also raises some identification issues in rational expectations models with multiple equilibria. First, non-fundamental expectation revisions caused by sunspots can arise only under indeterminacy. By contrast, news shocks can affect expectation revisions even in a model with a unique solution. Second, the multiplicity of solutions under indeterminacy implies that the propagation of fundamental shocks is influenced by arbitrary parameters. As a

[1] Beaudry and Portier (2004) apply this strategy to study news about future productivity.

result, the dynamic responses of the model variables to news and sunspot shocks can be observationally equivalent, depending on a specific solution. The analysis of the model suggests that it may be difficult (if not impossible) to identify the parameters related to news and sunspot shocks, if both types of shocks are included in the model simultaneously. Further, the dynamic responses of the model variables to sunspot shocks can be observationally equivalent to other fundamental shocks, such as preference or cost shocks, if they introduced into the model.[2] The question then arises whether news and sunspot shocks can be distinguished from other fundamental shocks. Since econometric estimation of dynamic stochastic general equilibrium models is gaining popularity,[3] it would be desirable to establish some conditions under which the parameter identification can be achieved.

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[2] I would like to thank the anonymous referee for pointing this out.

[3] See, for example, Tovar (2009).

Appendix 1: Technical Appendix

This Technical Appendix derives the analytical solution given in the main text of the paper and describes the system of equations used in the numerical simulations for the number of anticipation periods $n = 3$. It also explains why sunspot shocks and belief shocks triggered by these shocks can be treated as uncorrelated with the monetary policy shock.

5.1 Solving the New Keynesian Model Analytically

The initial representation of the model:

IS curve	$x_t = E_t x_{t+1} - \sigma \left(R_t - E_t \pi_{t+1}\right),$
Phillips curve	$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t,$
Monetary policy rule	$R_t = \psi \pi_t + \varepsilon_t,$
Monetary policy shock	$\varepsilon_t = \upsilon_t + \mu_{t-n},\, n \geq 1.$

Substituting the interest rate from the monetary policy rule into the IS equation leads to the following two-dimensional system:

$$\begin{aligned} E_t x_{t+1} + \sigma E_t \pi_{t+1} &= x_t + \sigma \psi \pi_t + \sigma \varepsilon_t, \\ \beta E_t \pi_{t+1} &= \pi_t - \kappa x_t. \end{aligned}$$

Using the definitions of the conditional expectations of output and inflation $\xi_t^x \equiv E_t \left(x_{t+1}\right)$ and $\xi_t^\pi \equiv E_t \left(\pi_{t+1}\right)$, the endogenous forecast errors $\eta_t^x \equiv x_t - \xi_{t-1}^x$ and $\eta_t^\pi \equiv \pi_t - \xi_{t-1}^\pi$, the conditional expectation of future monetary policy shock $\xi_t^\varepsilon \equiv E_t \left(\varepsilon_{t+1}\right)$ and the monetary policy forecast errors, the model can be re-written as a system of six equations for $n = 1$

$$\begin{aligned} x_t &= \xi_{t-1}^x + \eta_t^x, \\ \pi_t &= \xi_{t-1}^\pi + \eta_t^\pi, \\ \varepsilon_t &= \xi_{t-1}^\varepsilon + \upsilon_t, \\ \xi_t^x + \sigma \xi_t^\pi &= \xi_{t-1}^x + \sigma \psi \xi_{t-1}^\pi + \sigma \xi_{t-1}^\varepsilon + \sigma \mu_t^0 + \eta_t^x + \sigma \psi \eta_t^\pi, \\ \beta \xi_t^\pi &= \xi_{t-1}^\pi - \kappa \xi_{t-1}^x + \eta_t^\pi - \kappa \eta_t^x, \\ \xi_t^\varepsilon &= \mu_t. \end{aligned}$$

This system is block-triangular. Once the values $\xi_t = [\xi_t^x, \xi_t^\pi, \xi_t^\varepsilon]'$, υ_t, μ_t and $\eta_t = [\eta_t^x, \eta_t^\pi]'$ are known, the expressions for output and inflation are easily computed. Thus, the analysis can be focused on the three-dimensional subsystem for the conditional forecasts

$$\underbrace{\begin{bmatrix} 1 & \sigma & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\Gamma_0} \xi_t = \underbrace{\begin{bmatrix} 1 & \sigma\psi & \sigma \\ -\kappa & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Gamma_1} \xi_{t-1} + \underbrace{\begin{bmatrix} \sigma & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\Psi} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & \sigma\psi \\ -\kappa & 1 \\ 0 & 0 \end{bmatrix}}_{\Pi} \eta_t$$

Premultiply the system by

$$\Gamma_0^{-1} = \begin{bmatrix} 1 & -\frac{\sigma}{\beta} & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to obtain

$$\begin{aligned} \xi_t &= \underbrace{\begin{bmatrix} 1 + \frac{\kappa\sigma}{\beta} & \sigma\left(\psi - \frac{1}{\beta}\right) & \sigma \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Gamma_1^*} \xi_{t-1} + \underbrace{\begin{bmatrix} \sigma & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\Psi^*} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} \\ &+ \underbrace{\begin{bmatrix} 1 + \frac{\kappa\sigma}{\beta} & \sigma\left(\psi - \frac{1}{\beta}\right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \\ 0 & 0 \end{bmatrix}}_{\Pi^*} \eta_t \end{aligned}$$

or

$$\xi_t = \Gamma_1^* \xi_{t-1} + \Psi^* \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \Pi^* \eta_t \tag{14}$$

Derivations of analytical solutions of the system (14) closely follows the steps outlined in the appendix to Lubik and Schorfheide (2003). The unstable components of the model can be found by computing the Jordan decomposition of matrix Γ_1^*,

$$\Gamma_1^* = J\Lambda J^{-1},$$

where Λ is the matrix of eigenvalues, and J is the matrix of eigenvectors. The eigenvalues of matrix Γ_1^* solve the equation

$$\det\left[\Gamma_1^* - \lambda I\right] = 0,$$

where I is a 3×3 identity matrix. This condition is equivalent to

$$-\lambda\left\{\lambda^2 - \left[1 + \frac{1}{\beta}(1+\kappa\sigma)\right]\lambda + \frac{1}{\beta}(1+\kappa\sigma)\right\} = 0$$

Thus, one eigenvalue equals to zero and two eigenvalues solve the quadratic equation in curly brackets. The eigenvalues are defined by

$$\begin{aligned}
\lambda_0 &= 0, \\
\lambda_1 &= l_1 - l_2, \\
\lambda_2 &= l_1 + l_2.
\end{aligned}$$

with

$$\begin{aligned}
l_1 &= \frac{1}{2}\left(1 + \frac{\kappa\sigma + 1}{\beta}\right), \\
l_2 &= \frac{1}{2}\sqrt{\left(\frac{1+\kappa\sigma}{\beta} - 1\right)^2 + \frac{4\kappa\sigma}{\beta}(1-\psi)}.
\end{aligned}$$

An eigenvector $v(j)$ corresponding to an eigenvalue λ_j satisfies $\Gamma_1^*\lambda_j = \lambda_j v(j)$, $j = 1,2,3$.

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & l_1 - l_2 & 0 \\ 0 & 0 & l_1 + l_2 \end{bmatrix},$$

$$
\begin{aligned}
J &= \frac{1}{\kappa}\begin{bmatrix} 1 & 1-\beta l_1+\beta l_2 & 1-\beta l_1-\beta l_2 \\ \kappa & \kappa & \kappa \\ -\frac{1+\kappa\sigma\psi}{\sigma} & 0 & 0 \end{bmatrix}, \\
&= \frac{1}{\kappa}\begin{bmatrix} 1 & 1-\beta\lambda_1 & 1-\beta\lambda_2 \\ \kappa & \kappa & \kappa \\ -\frac{1+\kappa\sigma\psi}{\sigma} & 0 & 0 \end{bmatrix} \\
&= \frac{1}{\kappa}\begin{bmatrix} 1 & \beta\left(\lambda_2-1\right)-\sigma\kappa & 1-\beta\lambda_2 \\ \kappa & \kappa & \kappa \\ -\frac{1+\kappa\sigma\psi}{\sigma} & 0 & 0 \end{bmatrix},
\end{aligned}
$$

$$
J^{-1} = \frac{1}{2\beta l_2}\begin{bmatrix} 0 & 0 & \frac{-\kappa\sigma\beta}{1+\kappa\sigma\psi}2l_2 \\ \kappa & -1+\beta l_1+\beta l_2 & \frac{\kappa\sigma\beta}{1+\kappa\sigma\psi}\lambda_2 \\ -\kappa & 1-\beta l_1+\beta l_2 & \frac{-\kappa\sigma\beta}{1+\kappa\sigma\psi}\lambda_1 \end{bmatrix}.
$$

Using the transformation $w_t = J^{-1}\xi_t$, the system for the forecasts (14) can be rewritten as follows:

$$
w_t = \Lambda w_{t-1} + J^{-1}\Psi^*\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + J^{-1}\Pi^*\eta_t, \tag{15}
$$

where

$$
\begin{aligned}
J^{-1}\Psi^* &= \frac{1}{2\beta l_2}\begin{bmatrix} 0 & 0 & \frac{-\kappa\sigma\beta}{1+\kappa\sigma\psi}2l_2 \\ \kappa & -1+\beta l_1+\beta l_2 & \frac{\kappa\sigma\beta}{1+\kappa\sigma\psi}\lambda_2 \\ -\kappa & 1-\beta l_1+\beta l_2 & \frac{-\kappa\sigma\beta}{1+\kappa\sigma\psi}\lambda_1 \end{bmatrix}\begin{bmatrix} \sigma & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \frac{\kappa\sigma}{2\beta l_2}\begin{bmatrix} 0 & \frac{-\beta}{1+\kappa\sigma\psi}2l_2 \\ 1 & \frac{\beta}{1+\kappa\sigma\psi}\lambda_2 \\ -1 & \frac{-\beta}{1+\kappa\sigma\psi}\lambda_1 \end{bmatrix},
\end{aligned}
$$

$$J^{-1}\Pi^* = \frac{1}{2\beta l_2}\begin{bmatrix} 0 & 0 & \frac{-\kappa\sigma\beta}{1+\kappa\sigma\psi}2l_2 \\ \kappa & -1+\beta l_1+\beta l_2 & \frac{\kappa\sigma\beta}{1+\kappa\sigma\psi}\lambda_2 \\ -\kappa & 1-\beta l_1+\beta l_2 & \frac{-\kappa\sigma\beta}{1+\kappa\sigma\psi}\lambda_1 \end{bmatrix}\begin{bmatrix} 1+\frac{\kappa\sigma}{\beta} & \sigma\left(\psi-\frac{1}{\beta}\right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \\ 0 & 0 \end{bmatrix}$$
$$\frac{1}{2\beta^2 l_2}\begin{bmatrix} 0 & 0 \\ \kappa(1+\beta+\kappa\sigma-\beta l_1-\beta l_2) & \beta\kappa\sigma\psi-1-\kappa\sigma+\beta l_1+\beta l_2 \\ -\kappa(1+\beta+\kappa\sigma-\beta l_1+\beta l_2) & -(\beta\kappa\sigma\psi-1-\kappa\sigma+\beta l_1-\beta l_2) \end{bmatrix}.$$

The last matrix can be simplified by using the definitions and the properties of the eigenvalues:

$$-1-\kappa\sigma+\beta l_1=\beta(1-l_1),$$

$$J^{-1}\Pi^*=\frac{1}{2\beta l_2}\begin{bmatrix} 0 & 0 \\ \kappa\lambda_1 & 1+\kappa\sigma\psi-\lambda_1 \\ -\kappa\lambda_2 & -(1+\kappa\sigma\psi-\lambda_2) \end{bmatrix}.$$

The general condition determining the forecast errors from Lubik and Schorfheide (2003) is

$$\left[J^{-1}\Psi^*\right]_{2\cdot}\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix}+\left[J^{-1}\Pi^*\right]_{2\cdot}\eta_t=0 \tag{16}$$

The matrices in (16) correspond to the submatrices of $J^{-1}\Psi^*$ and $J^{-1}\Pi^*$. Specifically, the submatrices contain all the columns of $J^{-1}\Psi^*$ and $J^{-1}\Pi^*$. Their number of rows equals to the number of the unstable eigenvalues.

Determinacy

When both λ_1 and λ_2 are greater than one in absolute value, there is only one stable solution. The submatrices of $J^{-1}\Psi^*$ and $J^{-1}\Pi^*$ take the following forms:

$$\left[J^{-1}\Psi^*\right]_{1\cdot} = \frac{\kappa\sigma}{2\beta l_2}\begin{bmatrix} 0 & \frac{-\beta}{1+\kappa\sigma\psi}2l_2 \end{bmatrix}.$$
$$\left[J^{-1}\Psi^*\right]_{2\cdot} = \frac{\kappa\sigma}{2\beta l_2}\begin{bmatrix} 1 & \frac{\beta}{1+\kappa\sigma\psi}\lambda_2 \\ -1 & \frac{-\beta}{1+\kappa\sigma\psi}\lambda_1) \end{bmatrix}$$

and

$$\begin{aligned}
\left[J^{-1}\Pi^*\right]_{1\cdot} &= \begin{bmatrix} 0 & 0 \end{bmatrix}, \\
\left[J^{-1}\Pi^*\right]_{2\cdot} &= \frac{1}{2\beta l_2}\begin{bmatrix} \kappa\lambda_1 & 1+\kappa\sigma\psi-\lambda_1 \\ -\kappa\lambda_2 & -(1+\kappa\sigma\psi-\lambda_2) \end{bmatrix}.
\end{aligned}$$

The non-explosive solution must satisfy

$$\begin{aligned}
w_0 &= 0, && (17) \\
\left[J^{-1}\Pi^*\right]_{2\cdot}\eta_t &= -\left[J^{-1}\Psi^*\right]_{2\cdot}\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix}. && (18)
\end{aligned}$$

The endogenous expectation errors are uniquely determined by the news and unexpected monetary policy shocks

$$\begin{aligned}
\eta_t &= -\left[J^{-1}\Pi^*\right]_{2\cdot}^{-1}\left[J^{-1}\Psi^*\right]_{2\cdot}\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} \\
&= -\frac{\sigma}{1+\kappa\sigma\psi}\begin{bmatrix} 1 & \frac{1+\kappa\sigma(1-\beta\psi)}{1+\kappa\sigma\psi} \\ \kappa & \kappa\frac{1+\beta+\kappa\sigma}{1+\kappa\sigma\psi} \end{bmatrix}\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix}
\end{aligned}$$

From the law of motion (15), the initial restrictions on w_0 and the values for η_t, the solution to the model in terms of the transformed variables w_t is

$$\begin{aligned}
w_{1t} &= -\frac{\kappa\sigma}{1+\kappa\sigma\psi}\mu_t, \\
w_{2t} &= 0, \\
w_{3t} &= 0.
\end{aligned}$$

Then the vector of the conditional forecasts is

$$\xi_t = Jw_t = \frac{1}{\kappa}\begin{bmatrix} 1 \\ \kappa \\ -\frac{1+\kappa\sigma\psi}{\sigma} \end{bmatrix} w_{1,t} = \frac{\sigma}{1+\kappa\sigma\psi}\begin{bmatrix} -1 \\ -\kappa \\ \frac{1+\kappa\sigma\psi}{\sigma} \end{bmatrix}\mu_t.$$

By definition, output and inflation are linked to the conditional forecasts as follows

$$
\begin{aligned}
\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \xi^x_{t-1} \\ \xi^\pi_{t-1} \end{bmatrix} + \begin{bmatrix} \eta^x_t \\ \eta^\pi_t \end{bmatrix} = \\
&= -\frac{\sigma}{1+\kappa\sigma\psi}\begin{bmatrix} 1 \\ \kappa \end{bmatrix}\mu_{t-1} - \frac{\sigma}{1+\kappa\sigma\psi}\begin{bmatrix} 1 \\ \kappa \end{bmatrix}\upsilon_t \\
&\quad -\frac{\sigma}{(1+\kappa\sigma\psi)^2}\begin{bmatrix} 1+\kappa\sigma(1-\beta\psi) \\ 1+\beta+\kappa\sigma \end{bmatrix}\mu_t \\
&= -\frac{\sigma}{1+\kappa\sigma\psi}\begin{bmatrix} 1 \\ \kappa \end{bmatrix}\underbrace{(\upsilon_t+\mu_{t-1})}_{\varepsilon_t} \\
&\quad -\frac{\sigma}{(1+\kappa\sigma\psi)^2}\begin{bmatrix} 1+\kappa\sigma(1-\beta\psi) \\ 1+\beta+\kappa\sigma \end{bmatrix}\underbrace{\mu_t}_{E_t\varepsilon_{t+1}}
\end{aligned}
$$

The nominal interest rate then is equal to

$$
\begin{aligned}
R_t &= \psi\pi_t + \varepsilon_t = \\
&= \psi\left[-\kappa\frac{\sigma}{1+\kappa\sigma\psi}(\mu_{t-1}+\upsilon_t) - \frac{\sigma(1+\beta+\kappa\sigma)}{(1+\kappa\sigma\psi)^2}\mu_t\right] + \upsilon_t + \mu_{t-1} \\
&= \left(1-\psi\kappa\frac{\sigma}{1+\kappa\sigma\psi}\right)(\mu_{t-1}+\upsilon_t) - \psi\frac{\sigma(1+\beta+\kappa\sigma)}{(1+\kappa\sigma\psi)^2}\mu_t \\
&= \frac{1}{1+\kappa\sigma\psi}(\mu_{t-1}+\upsilon_t) - \psi\frac{\sigma(1+\beta+\kappa\sigma)}{(1+\kappa\sigma\psi)^2}\mu_t
\end{aligned}
$$

Indeterminacy

Under indeterminacy, there is only one unstable eigenvalue, λ_2. The submatrices of $J^{-1}\Psi^*$ and $J^{-1}\Pi^*$ take the forms

$$
\begin{aligned}
\left[J^{-1}\Psi^*\right]_{1\cdot} &= \frac{\kappa\sigma}{2\beta l_2}\begin{bmatrix} 0 & \frac{-\beta}{1+\kappa\sigma\psi}2l_2 \\ 1 & \frac{\beta}{1+\kappa\sigma\psi}\lambda_2 \end{bmatrix}, \\
\left[J^{-1}\Psi^*\right]_{2\cdot} &= \frac{\kappa\sigma}{2\beta l_2}\begin{bmatrix} -1 & \frac{-\beta}{1+\kappa\sigma\psi}\lambda_1 \end{bmatrix}
\end{aligned}
$$

and

$$\begin{aligned} \left[J^{-1}\Pi^*\right]_{1\cdot} &= \frac{1}{2\beta l_2}\begin{bmatrix} 0 & 0 \\ \kappa\lambda_1 & 1+\kappa\sigma\psi-\lambda_1 \end{bmatrix}, \\ \left[J^{-1}\Pi^*\right]_{2\cdot} &= \frac{1}{2\beta l_2}\begin{bmatrix} -\kappa\lambda_2 & -(1+\kappa\sigma\psi-\lambda_2) \end{bmatrix}. \end{aligned}$$

The stability condition gives only one restriction for determining two expectation errors η_t^x and η_t^π :

$$-\underbrace{\begin{bmatrix} -\kappa\lambda_2 & -(1+\kappa\sigma\psi-\lambda_2) \end{bmatrix}}_{\Pi_x^J}\begin{bmatrix} \eta_t^x \\ \eta_t^\pi \end{bmatrix} = \underbrace{\begin{bmatrix} -\kappa\sigma & \frac{-\beta}{1+\kappa\sigma\psi}\lambda_1 \end{bmatrix}}_{\Psi_x^J}\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix}$$

The full characterization of the solutions is found by computing the singular value decomposition of the matrix $\Pi_x^J = \begin{bmatrix} -\kappa\lambda_2 & -(1+\kappa\sigma\psi-\lambda_2) \end{bmatrix}$. Since exactly the same matrix arises in the model without news shocks, the singular value decomposition computed in the Technical Appendix to Lubik and Schorfheide (2004) can be used. By definition, the singular value decomposition of the matrix Π_x^J is

$$\Pi_x^J = \begin{bmatrix} U_{\cdot 1} & U_{\cdot 2} \end{bmatrix}\begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_{\cdot 1}' \\ V_{\cdot 2}' \end{bmatrix} = \underset{m\times m}{U}\ \underset{m\times k}{D}\ \underset{k\times k}{V'} = \underset{m\times r}{U_{\cdot 1}}\underset{r\times r}{D_{11}}\underset{r\times k}{V_{\cdot 1}'},$$

where the number of unstable roots $m = 1$, the number of restrictions $r = 1$, and the number of number of endogenous expectation errors $k = 2$. In this example

$$\begin{aligned} U_{\cdot 1} &= 1, \\ D_{11} &= d = \sqrt{(\kappa\lambda_2)^2 + (\lambda_2 - 1 - \kappa\sigma\psi)^2}, \\ V_{\cdot 1}' &= \frac{1}{d}\begin{bmatrix} -\kappa\lambda_2 & \lambda_2 - 1 - \kappa\sigma\psi \end{bmatrix}, \\ V_{\cdot 2}' &= \frac{1}{d}\begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi & \kappa\lambda_2 \end{bmatrix}. \end{aligned}$$

The endogenous expectation errors are linear functions of the unexpected monetary policy shock υ_t, the news shock μ_t and the reduced form sunspot shock ζ_t^*:

$$\eta_t = H \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + V_{\cdot 2} \left(\tilde{M} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \zeta_t^* \right), \tag{19}$$

where

$$\begin{aligned} H &\equiv -V_{\cdot 1} D_{11}^{-1} U_1' \Psi_x^J = \frac{\kappa\sigma}{d^2} \begin{bmatrix} -\kappa\lambda_2 & \frac{-\beta}{1+\kappa\sigma\psi} \kappa\lambda_2\lambda_1 \\ \lambda_2 - 1 - \kappa\sigma\psi & \frac{\beta}{1+\kappa\sigma\psi} (\lambda_2 - 1 - \kappa\sigma\psi)\lambda_1 \end{bmatrix}, \\ V_{\cdot 2} &= \frac{1}{d} \begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{bmatrix}, \\ \tilde{M} &= \begin{bmatrix} \tilde{m}_1 & \tilde{m}_2 \end{bmatrix} \end{aligned}$$

and values $\tilde{m}_1$ and $\tilde{m}_2$ are unrestricted.

The forecast errors are

$$\begin{aligned} \eta_t &= -\frac{\kappa\sigma}{d^2} \begin{bmatrix} \kappa\lambda_2 & \kappa\lambda_2 & \frac{\beta}{1+\kappa\sigma\psi} \kappa\lambda_2\lambda_1 \\ & -(\lambda_2 - 1 - \kappa\sigma\psi) & \frac{-\beta}{1+\kappa\sigma\psi} (\lambda_2 - 1 - \kappa\sigma\psi)\lambda_1 \end{bmatrix} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} \\ &\quad + \frac{1}{d} \begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{bmatrix} \left(\tilde{M} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \zeta_t^* \right) \end{aligned}$$

From the law of motion (15), the initial restrictions on w_0 and the values for η_t, the solution to the model in terms of the transformed variables w_t :

$$\begin{bmatrix} w_{1,t} \\ w_{2,t} \\ w_{3,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \\ w_{3,t-1} \end{bmatrix} + \begin{bmatrix} \left[J^{-1}\Psi^*\right]_{1\cdot} \\ \left[J^{-1}\Psi^*\right]_{2\cdot} \end{bmatrix} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \begin{bmatrix} \left[J^{-1}\Pi^*\right]_{1\cdot} \\ \left[J^{-1}\Pi^*\right]_{2\cdot} \end{bmatrix} \eta_t \tag{20}$$

The value $w_{3,t} = 0$ guarantees the stability of the system. Using this stability restriction and expanding the matrices in (20) yields

$$
\begin{aligned}
w_{1,t} &= \frac{\kappa\sigma}{2\beta l_2}\left[\begin{array}{cc} 0 & \frac{-\beta}{1+\kappa\sigma\psi}2l_2 \end{array}\right]\left[\begin{array}{c} \upsilon_t \\ \mu_t \end{array}\right] = -\frac{\kappa\sigma}{1+\kappa\sigma\psi}\mu_t, \\
w_{2,t} &= \lambda_1 w_{2,t-1} + \frac{\kappa\sigma}{2\beta l_2}\left[\begin{array}{cc} 1 & \frac{\beta}{1+\kappa\sigma\psi}\lambda_2 \end{array}\right]\left[\begin{array}{c} \upsilon_t \\ \mu_t \end{array}\right] \\
&\quad + \frac{1}{2\beta l_2}\left[\begin{array}{cc} \kappa\lambda_1 & 1+\kappa\sigma\psi-\lambda_1 \end{array}\right]\eta_t, \\
w_{3,t} &= 0.
\end{aligned}
\tag{21}
$$

The coefficients of the matrix $\tilde{M}$ can be centered around the solution that replicates the responses of policy shocks under determinacy. In the terminology of Lubik and Schorfheide (2003), this is a solution obtained under the continuity assumption. Let $M^c = \left[\begin{array}{cc} \tilde{m}_1^c & \tilde{m}_2^c \end{array}\right]$ denote the matrix corresponding to this solution. The particular values for its coefficients are

$$
\begin{aligned}
\tilde{m}_1^c &= \frac{\sigma}{d}\left(1 - \frac{\lambda_2\left(1+\kappa^2\right)}{1+\kappa\sigma\psi}\right), \\
\tilde{m}_2^c &= \frac{\sigma}{d}\frac{1}{1+\kappa\sigma\psi}\left[1+\kappa\sigma\left(1-\beta\psi\right) - \lambda_2\frac{\left(1+\kappa^2\right)\left(1+\kappa\sigma\right)+\kappa\beta\left(\kappa-\sigma\psi\right)}{1+\kappa\sigma\psi}\right].
\end{aligned}
$$

If $M = M^c$, then the forecast errors and the forecasts take the following expressions:

$$
\eta_t^c = -\frac{\sigma}{1+\kappa\sigma\psi}\left[\begin{array}{cc} 1 & \frac{1+\kappa\beta(1-\beta\psi)}{1+\kappa\sigma\psi} \\ \kappa & \frac{\kappa(1+\beta+\kappa\sigma)}{1+\kappa\sigma\psi} \end{array}\right]\left[\begin{array}{c} \upsilon_t \\ \mu_t \end{array}\right] + \frac{1}{d}\left[\begin{array}{c} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{array}\right]\zeta_{t}^{*},
$$

$$
\begin{aligned}
w_{1,t} &= -\frac{\kappa\sigma}{1+\kappa\sigma\psi}\mu_t, \\
w_{2,t} &= \lambda_1 w_{2,t-1} + \frac{1}{d}\left[\begin{array}{c} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{array}\right]\zeta_t^*, \\
w_{3,t} &= 0.
\end{aligned}
$$

The general solution can be obtained by choosing the coefficients of $\tilde{M}$ centered around the continuity solution M^c

$$\begin{aligned}
\tilde{M} &= M^c + M, \\
M &= \begin{bmatrix} m_1 & m_2 \end{bmatrix}.
\end{aligned}$$

However, it is convenient to adopt the following normalization:

$$\begin{aligned}
\tilde{M} &= M^c + \frac{d}{1+\kappa\sigma\psi} M, \\
\zeta_t &= \frac{1+\kappa\sigma\psi}{d} \zeta_t^*.
\end{aligned}$$

Then the general solution for η_t and w_t can be written as follows:

$$\begin{aligned}
\eta_t &= -\frac{\sigma}{1+\kappa\sigma\psi} \begin{bmatrix} 1 & \frac{1+\kappa\beta(1-\beta\psi)}{1+\kappa\sigma\psi} \\ \kappa & \frac{\kappa(1+\beta+\kappa\sigma)}{1+\kappa\sigma\psi} \end{bmatrix} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} \\
&\quad + \frac{1}{1+\kappa\sigma\psi} \begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{bmatrix} \left(M \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \zeta_t \right), \\
w_{1,t} &= -\frac{\kappa\sigma}{1+\kappa\sigma\psi} \mu_t, \\
w_{2,t} &= \lambda_1 w_{2,t-1} + \frac{\kappa}{\beta} \left(M \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \zeta_t \right), \\
w_{3,t} &= 0.
\end{aligned}$$

The optimal forecasts of output, inflation and the policy shock, using the values of ω defined by (21), are

$$\begin{aligned}
\xi_t &= J w_t = \begin{bmatrix} 1/\kappa \\ 1 \\ -\frac{1+\kappa\sigma\psi}{\kappa\sigma} \end{bmatrix} w_{1,t} + \begin{bmatrix} (\beta(\lambda_2 - 1) - \sigma\kappa)/\kappa \\ 1 \\ 0 \end{bmatrix} w_{2,t} \\
&= \frac{\sigma}{1+\kappa\sigma\psi} \begin{bmatrix} -1 \\ -\kappa \\ \frac{1+\kappa\sigma\psi}{\sigma} \end{bmatrix} \mu_t + \begin{bmatrix} (\beta(\lambda_2 - 1) - \sigma\kappa)/\kappa \\ 1 \\ 0 \end{bmatrix} w_{2,t}
\end{aligned}$$

The full set of rational expectations solution for output and inflation is

$$\begin{aligned}
\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \xi^x_{t-1} \\ \xi^\pi_{t-1} \end{bmatrix} + \begin{bmatrix} \eta^x_t \\ \eta^\pi_t \end{bmatrix} = \\
&= -\frac{\sigma}{1+\kappa\sigma\psi} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} \mu_{t-1} + \begin{bmatrix} \left(\beta\left(\lambda_2 - 1\right) - \sigma\kappa\right)/\kappa \\ 1 \end{bmatrix} w_{2,t-1} \\
&\quad -\frac{\sigma}{1+\kappa\sigma\psi} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} \upsilon_t - \frac{\sigma}{(1+\kappa\sigma\psi)^2} \begin{bmatrix} 1+\kappa\sigma\left(1-\beta\psi\right) \\ \kappa\left(1+\beta+\kappa\sigma\right) \end{bmatrix} \mu_t \\
&\quad +\frac{1}{d} \begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{bmatrix} \left(\begin{bmatrix} m_1 & m_2 \end{bmatrix} \begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + \zeta^*_t \right) \\
&= \frac{1}{1+\kappa\sigma\psi} \begin{bmatrix} -\sigma \\ -\kappa\sigma \end{bmatrix} \underbrace{\left(\upsilon_t + \mu_{t-1}\right)}_{\varepsilon_t} \\
&\quad -\frac{\sigma}{(1+\kappa\sigma\psi)^2} \begin{bmatrix} 1+\kappa\sigma\left(1-\beta\psi\right) \\ \kappa\left(1+\beta+\kappa\sigma\right) \end{bmatrix} \underbrace{\mu_t}_{E_t\varepsilon_{t+1}} \\
&\quad +\frac{1}{d} \begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{bmatrix} \left(m_1\upsilon_t + m_2\mu_t + \zeta_t\right) \\
&\quad + \begin{bmatrix} \left(\beta\left(\lambda_2 - 1\right) - \sigma\kappa\right)/\kappa \\ 1 \end{bmatrix} w_{2,t-1}
\end{aligned}$$

with

$$w_{2,t} = \lambda_1 w_{2,t-1} + \frac{\kappa\left(1+\kappa\sigma\psi\right)}{\beta d} \left(M \begin{bmatrix} v_t \\ \mu_t \end{bmatrix} + \zeta_t \right).$$

The interest rate is

$$\begin{aligned}
R_t &= \psi\pi_t + \varepsilon_t = \frac{1}{1+\kappa\sigma\psi}\varepsilon_t - \frac{\sigma}{(1+\kappa\sigma\psi)^2}\psi\kappa\left(1+\beta+\kappa\sigma\right)\mu_t \\
&\quad +\frac{1}{d}\psi\kappa\lambda_2\left(m_1\upsilon_t + m_2\mu_t + \zeta_t\right) + \psi w_{2,t-1}
\end{aligned}$$

The final form of the general solution given in the main text is

$$\begin{aligned}
\begin{bmatrix} x_t \\ \pi_t \\ R_t \end{bmatrix} &= \frac{1}{1+\kappa\sigma\psi}\begin{bmatrix} -\sigma \\ -\kappa\sigma \\ 1 \end{bmatrix}\underbrace{(\upsilon_t+\mu_{t-1})}_{\varepsilon_t} \\
&\quad -\frac{\sigma}{(1+\kappa\sigma\psi)^2}\begin{bmatrix} 1+\kappa\sigma(1-\beta\psi) \\ \kappa(1+\beta+\kappa\sigma) \\ \psi\kappa(1+\beta+\kappa\sigma) \end{bmatrix}\underbrace{\mu_t}_{E_t\varepsilon_{t+1}} \\
&\quad +\frac{1}{d}\begin{bmatrix} \lambda_2-1-\kappa\sigma\psi \\ \kappa\lambda_2 \\ \psi\kappa\lambda_2 \end{bmatrix}(m_1\upsilon_t+m_2\mu_t+\zeta_t) \\
&\quad +\begin{bmatrix} (\beta(\lambda_2-1)-\sigma\kappa)/\kappa \\ 1 \\ \psi \end{bmatrix}\omega_{t-1},
\end{aligned}$$

where ω_t follows the AR(1) process

$$\omega_t = \lambda_1\omega_{t-1} + \frac{\kappa(1+\kappa\sigma\psi)}{\beta d}(m_1\upsilon_t+m_2\mu_t+\zeta_t).$$

It can be shown that $b<0$ and $1+\kappa\sigma\psi<\lambda_2$ for any values of $\kappa\geq 0$, $\sigma\geq 0$ and $\psi<1$.

5.2 Solving the New Keynesian Model Numerically

When the number of the anticipation periods is equal to three, the model can be written as follows:

$$\begin{aligned}
x_t &= \xi^x_{t-1} + \eta^x_t, \\
\pi_t &= \xi^\pi_{t-1} + \eta^\pi_t, \\
\xi^x_t + \sigma\xi^\pi_t - \sigma R_t &= \xi^x_{t-1} + \eta^x_t, \\
\beta\xi^\pi_t &= \xi^\pi_{t-1} - \kappa\xi^x_{t-1} + \eta^\pi_t - \kappa\eta^x_t, \\
R_t &= \psi\xi^\pi_{t-1} + \xi^1_{t-1} + \upsilon_t + \psi\eta^\pi_t, \\
\xi^1_t &= \xi^2_{t-1}, \\
\xi^2_t &= \xi^3_{t-1}, \\
\xi^3_t &= \mu_t,
\end{aligned}$$

where $\varepsilon_t = \xi^3_{t-1} + \upsilon_t$. This system is solved numerically using the method developed by Lubik and Schorfheide (2003).

5.3 Relation between Sunspots and Monetary Policy Shocks

This section explains why sunspot shocks and belief shocks triggered by these shocks can be treated as uncorrelated with the monetary policy shock.

Suppose that, in contrast, the reduced form sunspot shock ζ^*_t is correlated with current impulses to the monetary policy shock: $cov\left(\zeta^*_t v_t\right) = \phi_v$ and/or $cov\left(\zeta^*_t \mu_t\right) = \phi_\mu$. Then ζ^*_t can be represented as

$$\begin{aligned}
\zeta^*_t &= \frac{\phi_v}{\sigma^2_v} v_t + \frac{\phi_\mu}{\sigma^2_m}\mu_t + u_t, \qquad (22)\\
E_t u_{t+1} &= 0,\ E\left(v_t u_t\right) = 0,\ E\left(\mu_t u_t\right) = 0
\end{aligned}$$

Under indeterminacy, the equilibrium forecast errors are defined by (6). Using (10), the forecast errors can be written as

$$\eta_t = H\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + V_{\cdot 2}\left(\hat{M}\begin{bmatrix} \upsilon_t \\ \mu_t \end{bmatrix} + u_t\right), \qquad (23)$$

where $\hat{M} = \tilde{M} + \left[\begin{array}{cc} \frac{\phi_v}{\sigma_v^2} & \frac{\phi_\mu}{\sigma_m^2} \end{array} \right]$. Since there are no restrictions on the matrix $\hat{M}$, the forecast errors defined by (6) and (23) are observationally equivalent.

Suppose that sunspots are correlated with past realizations of v_t or μ_t. Then values of ζ_{t+1}^* are predictable from the values of monetary policy shocks, which violates the requirement that sunspot shocks are martingale difference sequences with respect to the period t information set.

Finally, suppose that the sunspot shock is correlated with some future realization of impulses for some values $j > 0$, $k > 0$,

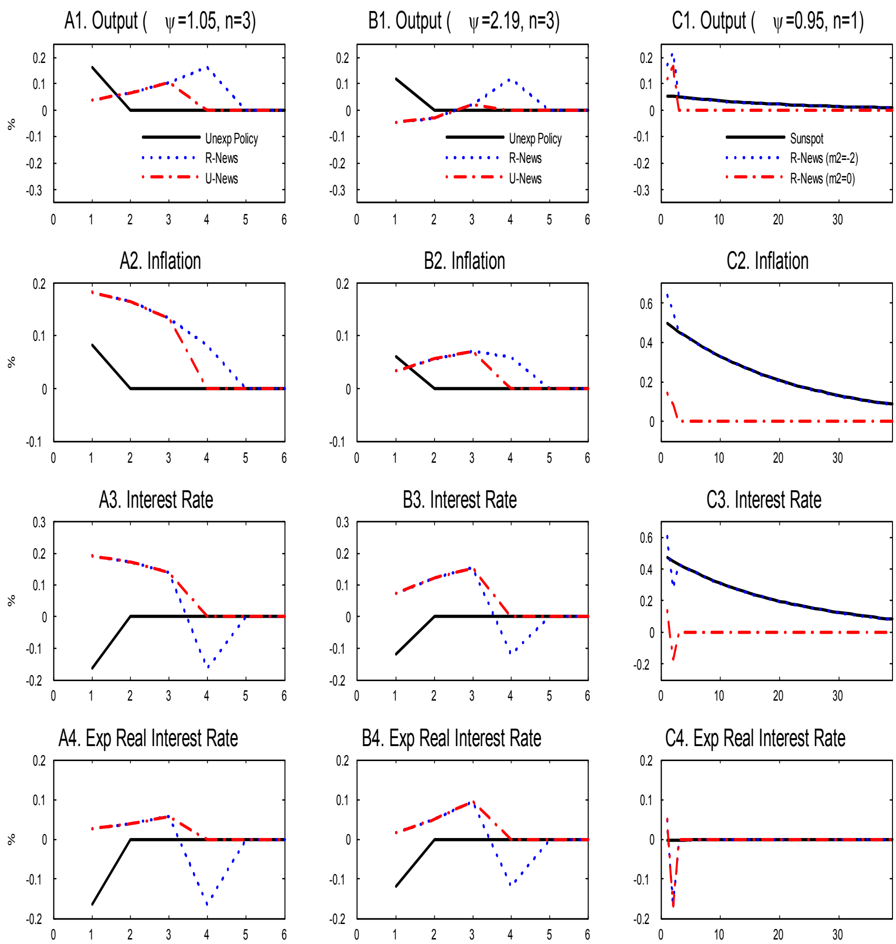
$$cov\,(\zeta_t^* v_{t+j}) \neq 0 \text{ and/or } cov\,(\zeta_t^* \mu_{t+k}) \neq 0.$$

If the impulses to the monetary policy shock are treated as unpredictable in the model solution, this would violate the rationality of expectations. The values of ζ_t^* would help to predict future realizations of ε.

Lubik and Schorfheide (2003) in section 4.2 (p. 279) show how sunspot shocks can trigger belief shocks that lead to forecast revisions of output and inflation. One can verify that these belief shocks can be modelled as uncorrelated with monetary policy shocks, following the arguments presented above.

6 Appendix 2

Figure 1: Responses of Output, Inflation and Interest Rates to Unexpected Policy Change, News and Sunspot Shocks



Notes: this Figure plots impulse responses to an unexpected 25-basis-point interest rate cut (panels A and B, solid lines), a reduced form sunspot shock of 0.5% (panel C, solid lines) and news shocks about future monetary expansion. News shocks correspond to a belief, formed in period 1 that in period 1+n the interest rate will be cut by 25 basis points. This belief is validated for R-News (dotted and dashed-dotted lines), but followed by no policy change for U-News (dashed lines). The other parameters are β=0.99, κ=0.5 and σ=1.

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