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The e-monetary theory

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Abstract
The author develops a dynamic model with two types of electronic money: reserves for transactions between bankers and zero-maturity deposits for transactions in the non-bank private sector. Using this model, he assesses the efficacy of unconventional monetary policy since the Great Recession. After quantitative easing, keeping the interest on reserves near zero too long might create deflation. The central bank can safely get out of the "low rate-cum-deflation" trap by "raising rate and raising money supply".

JEL E4 E5
Keywords Interest on reserves; quantitative easing; unwinding QE; e-money; excess reserves; raise rate raise money supply

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1 Introduction

Nowadays, money mostly exists in the electronic form. According to data from the Federal Reserve Bank of St. Louis, the total stock of M1 in Jun 2016 is around USD 3274 billion, consisting of USD 1381 billion in currency and USD 1850 billion in checkable deposits. However, as the world currency, most US dollar bills are held outside US. Judson (2012) estimates that 60 percent of US dollar bills are in foreign countries. If we exclude that number from M1 and M2, currency only accounts for 23 percent of M1, 5 percent of M2 and 4.2 percent of MZM[1]. In this paper, we focus on a popular group of e-money issued by commercial banks, including checkable deposits, saving deposits and money market deposit accounts. Together they account for 80 percent of M2. For convenience, we call this group as zero-maturity deposits (ZMDs) thereafter.

ZMDs are different from currency in two salient features. First, in nature, currency is an IOU issued by the central bank while ZMDs are IOUs issued by commercial banks. In the language of economics, currency is outside money while ZMDs are inside money. Second, in the households' perspective, unlike currency, ZMDs can earn nominal interest. Banks pay interest for saving accounts and money market deposit accounts for a long time, but the tricky parts are checking accounts. In a perfectly competitive banking market, the interest rate on checkable deposits should be positive and follow the federal funds rate[2]. However, before 2012, under the Regulation Q, banks in US were prohibited from paying interest on checking accounts. During this period, banks still implicitly paid the demand deposit rate under the form of NOW (negotiable order of withdrawal) accounts, giving gifts or reducing the cost of additional services to their customers, see Mitchell (1979), Startz (1979), Dotsey (1983). Becker (1975) estimates that the implicit demand deposit rate in US during 1960-1968 was around 2.64 percent to 3.74 percent.

Since 2012, the Regulation Q has been no longer valid, and most banks are now paying interest rate on checkable deposits. Data in September 2016 of Federal Deposit Insurance Cor-

[1]MZM (Money zero maturity) is equal to M2 less small-denomination time deposits plus institutional money funds.

[2]When the interbank rate is negative, the checkable deposits might earn negative nominal rates.

poration (FDIC) show that the national average interest on checkable account is 0.04 percent, on saving account is 0.06 percent. These rates are low as the federal funds rate is near zero. If the federal funds rate is around 4 percent, these rates are likely from 1 percent to 2 percent. As a result of that, in the era of electronic money, it is more natural to model money as an interest-earning asset that provides liquidity service.

This paper builds a dynamic general equilibrium model where currency does not exist (a cashless model). There are two forms of money in our model: ZMDs and reserves. ZMDs are inside money issued by commercial banks. They are used for settling transactions between every pair of agents in the private sector, except between bankers-bankers. In these interbank transactions, bankers have to use reserves- another type of e-money issued by the central bank. The amount of ZMDs that banks can issue is restricted by two constraints: the reserves requirement and the capital requirement. In our model, the central bank only controls the level of reserves while the level of the money supply (amount of ZMDs) depends on the interaction between the central bank, the commercial banks and the public (Mishkin, 2007).

We use our model to discuss unconventional monetary policy during and after the Great Recession. Here are some key results:

i. In normal times, when the central bank controls the federal funds rate by adjusting the level of reserves, the effect of cutting rates in our model is nearly identical to the one founded in the standard New Keynesian model. After the interbank rate goes down, the real rate goes down as price is sticky. Banks lend out more to households by creating more money (Sheard, 2013), stimulating inflation and investment.

ii. After a shock making banks' capital constraint binding, an interest rate policy following a Taylor rule is not sufficient to recover the economy quickly. Both output and inflation are lower than their steady state levels for a long time.

iii. A central bank's large scale asset purchase (LSAP) program, with the aim of directly injecting money into the economy, is very efficient at dealing with the situation when bankers cut loans. Inflation will go up immediately after this program. The byproduct of LSAP is a huge amount of excess reserves in the banking system (Keister and McAndrews, 2009);

the reserves requirement is no longer binding; and interest on reserves (IOR) becomes the main tool to control the federal funds rate.

iv. After LSAP, the longer the federal funds rate is committed at the lower bound, the higher is inflation in the short run. As loans have the longer maturity than deposits, commitment to keep the short-term rate near zero for a long time pushes down the loan rate stronger and pushes up the inflationary expectation. However, in the long run, it might create a persistent deflation due to the Neo-Fisherian's effect. The real short-term rate will slowly climb back to the long-run level. The endogenous money supply declines, and deflation realizes. This matches with the US data since the Great Recession (Figure 1).

v. It is not easy to safely escape from the "low rate-cum-deflation" trap. If the central bank raises rates (by raising IOR), the amount of banks' credits declines. The economy will suffer a short recession. Deflation is even more severe in the short run. Still, after a period of time, inflation will move back to the central bank's target in the long run. Therefore, the central bank falls into a dilemma between to raise or not to raise rates. Either way the outcome is not bright.

vi. When raising IOR, if the central bank simultaneously commits to target the growth rate of the money supply in response to inflation, the inflation and output path will be stabilized. With the new tool IOR, the central bank somehow can manipulate both interest rate and money supply at the same time. These tools should be utilized simultaneously so that the central bank can hit the inflation target better.

Related Literature

On the money supply side, our approach is similar to Bianchi and Bigio (2014) and Afonso and Lagos (2015) when the central bank can increase the level of the money supply and cut down the federal funds rate by injecting more reserves in the banking system. These papers explicitly model the search and matching process of heterogeneous agents in the interbank market while our model is frictionless with identical bankers. On the other hand, our model can connect the central bank policy to not only banks' balance sheet but also the production sector, which is

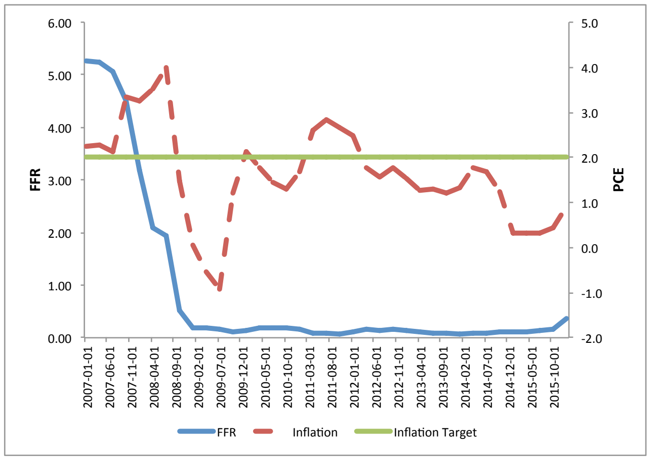


Figure 1: Federal Funds Rate and Personal Consumption Expenditure Index

missing in both Bianchi and Bigio (2014) and Afonso and Lagos (2015).

On the money demand side, our model follows the cash-in-advance approach in Lucas and Stokey (1987). As our model does not have currency, "cash" here should be interpreted as ZMDs. In Belongia and Ireland (2006, 2014), currency and deposits are bundled together and provide the liquidity service to households. We also extend the Clower constraint to investment (Stockman (1981), Abel (1985), Fuerst (1992), Wang and Wen (2006)). Indeed, most empirical research, for example Friedman (1959) and Mulligan and Sala-I-Martin (1997), usually uses the income, rather than the consumption alone, to estimate the money demand function.

Our model is still in the general New Keynesian framework with the crucial sticky price feature. The important role of financial frictions in the New Keynesian has been emphasized for a long time (see Bernanke, Gertler and Gilchrist (1999), Christiano, Motto and Rostagno (2004)). Recently, many New Keynesian research (Gertler and Kiyotaki (2010), Curdia and Woodford (2011), Gertler and Karadi (2011)) incorporates the banking sector to their models, aiming to answer what happened in the Great Recession and the role of the unconventional monetary policy. There is also a large literature that discusses interest on reserves, see Sargent and Wallace (1985), Goodfriend et al. (2002), Ireland (2014), Cochrane (2014), Keister (2016). Our paper differs mainly from this line of research in the money supply process. We can characterize the micro-foundation link between bank reserves, banks' balance sheets, money supply, interest rate and output. We emphasize the importance of both money supply and interest rate in monetary policy when the central bank adjusts the interbank rate by IOR.

Our approach also relates to Brunnermeier and Sannikov (2016) where macro shocks can affect strongly to the balance sheets of intermediaries and the amount of inside money. Both papers emphasize the importance of inside money in the deflation episode. However, two papers differ mainly in the role of money and the money supply process. They emphasize the money function as a store of value in a risky environment while our paper focus on the common role of money - medium of exchange - in a deterministic setting.

2 The Environment

2.1 Notation:

Let P_t be the price of the final good. We use lowercase letters to represent the real balance of a variable or its relative price to the price of the final good. For example, the real reserves balance $n_t = N_t/P_t$, or the relative price of the intermediate good to the final good is $p_t^m = P_t^m/P_t$. The timing notation follows this rule: if a variable is determined or chosen at time t, it will have the subscript t. The gross inflation rate is $\pi_t = P_t/P_{t-1}$.

2.2 Time, Demographics and Preferences

Time is discrete, indexed by t and continues forever. The model is in the deterministic setting and has five types of agents: bankers, households, wholesale firms, retail firms, and the consolidated government.

There is a measure one of identical infinitely lived bankers in the economy. Bankers discount the future with the discount factor β. Each period, they gain utility from consuming the final consumption good c_t. Their utility at the period t can be written as:

$$\sum_{i=0}^{\infty} \beta^i \log(c_{t+i})$$

There is also a measure one of identical infinitely lived households. Households discount the future with the discount factor $\widetilde{\beta}<\beta$, so they will borrow from bankers in the steady state.

Each period, households gain utility from consuming the final consumption good $\widetilde{c}_t$ and lose utility when providing labor l_t to their own production. Household's utility at the period t can be written as:

$$\sum_{i=0}^{\infty} \widetilde{\beta}^i \left(\log(\tilde{c}_{t+i}) - \chi \frac{l_{t+i}^{1+\nu}}{1+\nu} \right)$$

where ν is the inverse Frisch elasticity of labor supply.

Wholesale firms, retail firms are owned by households.

The consolidated government includes both the government and the central bank, so for convenience, we assume there is no independence between the government and the central bank.

2.3 Goods and Production Technology

There are three types of goods in the economy: final good y_t produced by retailers, wholesale goods $y_t(j)$ produced by wholesale firm j and intermediate good y_t^m produced by households.

Each period households self-employ their labor l_t and use the capital k_{t-1} to produce the homogeneous intermediate good y_t^m under the Cobb-Douglas production function:

$$y_t^m = k_{t-1}^{\alpha} l_t^{1-\alpha}$$

where α is the share of capital in the production function. Capital k depreciates with the rate δ_k. Households also own a technology to convert one unit of final good y_t to one unit of capital type k and vice versa. So each period they also make an investment $i_t = k_t - \delta_k k_{t-1}$. Households sell y_t^m to wholesale firms in the competitive market with price P_t^m.

There is a continuum of wholesale firms indexed by $j \in [0,1]$. Each wholesale firm purchases the homogeneous intermediate good y_t^m from households and differentiates it into a distinctive wholesale goods $y_t(j)$ under the following technology:

$$y_t(j) = y_t^m$$

Then retail firms produce the final good y_t by aggregating a variety of differentiated wholesale goods $y_t(j)$:

$$y_t = \left(\int_0^1 y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

2.4 Assets

There are three main types of financial assets (excluding reserves and deposits): bank loans B_t^h, share of wholesale firms x_t and interbank loans B_t^f.

(a) **Bank loans to households (B_t^h)** : We follow Leland and Toft (1996) and Bianchi and Bigio (2014) to model the loan structure between bankers and households. The market for bank loan is perfectly competitive and the price of loan is q_t^L. When a household wants to borrow 1 dollar at time t, bankers will create an account for her and deposit q_t^L dollars to her account. In the exchange for that, this household promises to pay δ_b, δ_b^2, ..., δ_b^n, δ_b^{n+1}... dollars at time $t+1$, $t+2$,... $t+n$, $t+n+1$... where n runs to infinity (Table 1). Loans are illiquid and bankers cannot sell loans.

Let B_t^h be the nominal balance of loan stock in the period t, let S_t be the nominal flow of new loan issuance, we have:

$$B_t^h = \delta_b B_{t-1}^h + S_t$$

(b) **Shares of wholesale firms (x_t)** : are issued by the wholesale firms. Bankers are not allowed to hold this share, so they are only traded between households. Each share has a price υ_t and pays a real dividend w_t. The number of wholesale firms' shares is 1. In the LSAP, the central bank might purchase these shares to inject money into the market.

(c) **Interbank loan (B_t^f)**: Bankers can borrow reserves from other bankers in the federal funds market. The nominal gross interest rate in the federal funds market is the federal funds rate R_t^f.

Banker	
Loans: +S_t	Deposits: + $q_t^L S_t$
	Net worth: + $(1-q_t^L)S_t$

Banker	
Loans: -$(1-\delta_b)B_{t-1}^h$	Deposits: -$\delta_b B_{t-1}^h$
	Net worth: +$(2\delta_b - 1)B_{t-1}^h$

Table 1: Banker issues loans (left) and collects loans (right)

2.5 Money

There are two types of electronic money in our economy: reserves n_t and zero-maturity deposits m_t.

(a) **Reserve (n_t)**: is a type of e-money issued by the central bank. Only bankers have an account at the central bank, so only bankers have reserves. Each period, the central bank pays a gross interest rate R_t^n on these reserves. The rate R_t^n is decided solely by the central bank. Reserves are used for settling the transactions between bankers and bankers, bankers and central bank, bankers and government.

(b) **Zero maturity deposit (m_t)**: is a type of e-money issued by bankers. Each period, banks pay the interest rate R_t^m for these ZMDs which is determined by the perfectly competitive market. Money m_t is used for settling transactions in the non-bank private sector and the ones between households and bankers. These ZMDs are insured by the central bank, so they are totally safe. ZMDs and reserves have the same unit of account.

In the electronic payment system, there is a connection between the flows of reserves and deposits. For example, we assume that wholesale firm A (whose account at bank A) pays 1 dollar for household B (whose account at bank B). Then the flow of payment will follow Table (2).

For a transaction between the consolidated government and households, money still flows through banks, so we can think that this contains two sub-transactions. One is between the government and banks, which is settled by reserves. One is between banks and households, which is settled by ZMDs.

In the conventional monetary policy, the consolidated government targets the interbank rate by helicopter money or lump-sump tax on households. Each period, the central bank sends τ_t dollars in checks to households. It can be thought as a shortcut of the open market operation pro-

Wholesale firm A	
Deposit: -1	Payable:-1

Household B	
Deposit: +1	
Receivable:-1	

Bank A	
Reserve: -1	Deposit: -1

Bank B	
Reserve: +1	Deposit: +1

The Fed	
	Reserve (bank A): -1
	Reserve (bank B):+1

Table 2: Electronic Payment System

cess when the central bank purchases government bonds from the government (through banks). Then, the government transfers the payoffs to households (Table 3). In fractional reserve banking, the amount of τ_t needed to change the federal funds rate is extremely small in comparison to the total money supply.

The Fed		Banks		Public	
	Reserves:+τ_t	Reserves:+τ_t	Deposits: +τ_t	Deposits:+τ_t	Net worth: +τ_t
	Net worth:-τ_t				

Table 3: Helicopter Money / Lump-sum tax

2.6 Timing within one period

(i) Production takes place. Households sell goods to wholesalers, who, in turn, sell goods to retailers. All of the payments between households-wholesalers, wholesalers-retailer are delayed until the step (iv).

(ii) The loan market between bankers and households opens.

(iii) The final good market opens. Households need ZMD-in-advance to purchase the final good from retailers. Bankers create ZMD to purchase the final good from retailer.

(iv) Payments in the non-bank private sector are settled.

(v) The banking market opens. Banker can adjust the level of reserves by borrowing in the interbank market, receiving new deposits.

3 Agents' Problems

3.1 Bankers

There is a measure one of identical bankers in the economy. These bankers have to maintain a good balance sheet under the regulation of the central bank. There are three types of assets on a banker's balance sheet: reserves (n_t), loans to households (b_t^h), loan to other bankers (b_t^f). His liability side contains the zero-maturity deposits that households deposit here (m_t).

Banker	
Reserves: n_t	Zero Maturity Deposits: m_t
Loans to households: b_t^h	Net worth
Loans to other bankers: b_t^f	

Cost: We assume that the banker faces a cost of managing loan, which is θb_t^h in terms of final goods.

On the timing of the market, it is worth noting that he can adjust the level of his deposits and reserves after households and firms pay for each other. When the different parties in the economy pay each other, he can witness that the deposits and reserves outflow from or inflow to his bank. Let e_t be the net inflow of deposits and reserves go into his bank, he will treat e_t as exogenous. When the banking market opens, as the deposit market is perfectly competitive, he can choose any amount d_t of deposit inflows or outflows to his bank[3].

In each period, the banker treats all the prices as exogenous and choose $\{c_t, n_t, b_t^h, s_t, m_t, b_t^f, d_t\}$

[3] If d_t<0, the banker terminates contract with some customers. In a perfectly competitive market, customers can always find other banks to transfer money into.

to maximize his utility over a stream of consumptions:

$$\max \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

subject to

$$\frac{R^n_{t-1} n_{t-1}}{\pi_t} + \frac{R^f_{t-1} b^f_{t-1}}{\pi_t} + d_t + e_t + \tau_t = n_t + b^f_t \quad \text{(Reserve Flows)} \tag{1}$$

$$m_t = \frac{R^m_{t-1} m_{t-1}}{\pi_t} + q^L_t s_t + \theta b^h_t - \delta_b \frac{b^h_{t-1}}{\pi_t} + c_t + d_t + e_t + \tau_t \quad \text{(Deposit Flows)} \tag{2}$$

$$b^h_t = \delta_b \frac{b^h_{t-1}}{\pi_t} + s_t \quad \text{(Loan Flows)} \tag{3}$$

$$n_t \geq \varphi m_t \quad \text{(Reserves Requirement)} \tag{4}$$

$$n_t + b^f_t + b^h_t - m_t \geq \kappa_t b^h_t \quad \text{(Capital Requirement)} \tag{5}$$

Reserve Flows: in each period are shown in the equation (1). After receiving the interest on reserves, the previous reserve balance becomes $R^n_{t-1} n_{t-1}/\pi_t$. He also collects the payment from the interbank loan he lends out to other bankers in the previous period $R^f_{t-1} b^f_{t-1}/\pi_t$. He can also increase his reserves by taking more deposits d_t. When doing that, his reserves and his liability increase by the same amount d_t (Table 4). That is the reason we see d_t appear on both the equation (1) and (2). The similar effect can be found on τ_t when the central bank drops money. The banker treats τ_t as given. Then, he can leave reserves n_t at the central bank's account to earn IOR, or lend reserves to another bankers b^f_t with the rate R^f_t.

Deposit Flows: for the banker are shown in the equation (2). He makes loans to households by issuing deposits or creating ZMDs (Table 1). So when he makes a loan (s_t), the balance sheet expands. When he collects the payoffs from loans to households ($\delta_b b^h_{t-1}/\pi_t$), the balance sheet shrinks[4].

The banker also issues his own ZMDs to purchase the consumption good from retailers (c_t) and to pay for the cost (in terms of final goods) related to lending activities (θb^h_t) (Table 5). It is

[4]It is assumed that households have to pay loans from the account at the bank they borrow. So if they want to use money from account at bank B to pay for loans from bank A, they need to transfer deposits from bank B to bank A first. In fact, this assumption does not matter in equilibrium.

Banker	
Reserves: $+d_t$	Deposits: $+d_t$

Banker	
Reserves: - b_t^f	
Interbank loan: + b_t^f	

Table 4: The banker takes more deposits (left) and makes interbank loan (right)

Banker	
	Deposits: $+c_t + \theta b_t^h$
	Net worth: - $c_t - \theta b_t^h$

Retailers	
Deposits: $c_t + \theta b_t^h$	
Inventory: $-c_t - \theta b_t^h$	

Table 5: The banker buys the final good from retailers

noted that he cannot create infinite amount of money for himself to buy consumption goods as there exists the capital requirement and reserve requirement.

Reserve Requirement: At the end of each period, he has to hold enough reserves as a fraction of total deposits (Equation 4)[5]. This constraint should be interpreted more broadly than the the real life reserves requirement in the US because the total ZMDs here include not only checkable deposits but also saving deposits and money market deposit account.

Capital Requirement: The second constraint plays the key role in our model - the capital requirement constraint. The left hand side of the equation (5) is the banker's net worth (capital), which is equal to total assets minus total liabilities[6]. The constraint requires the banker to hold capital greater than a fraction of total loans in his balance sheet. We assume that κ_t is an exogenous variable reflecting the risk weight of the banker's asset. We later put the unexpected shock on this κ_t to reflect the shock in the Great Recession[7].

Let γ_t, μ_t^r and μ_t^c be respectively the Lagrangian multipliers attached to the reserves flows, reserves constraint and the capital constraint. The first order conditions of the banker's problem

[5]During one period, his reserve balance can go temporarily negative. But in the end of that period, it must be positive and satisfies the regulation.

[6]We use the book value B_t^h rather than the "market value" of loans $q_t^L B_t^h$ in the capital constraint. The reason is that illiquid bank loans should be treated differently from bonds. In reality, bank loans are often not revalued in the balance sheet when the interest rate changes.

[7]Clearly, it is a simplified way to reflect the increase in the bad loans and the aggregate risk during the Great Recession. Still, we can keep our model simple to illustrate our insight.

can be written as:

$$\gamma_t = \frac{1}{c_t} \tag{6}$$

$$\gamma_t = \frac{\beta R_t^f \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c \tag{7}$$

$$\gamma_t = \frac{\beta R_t^m \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c + \varphi \mu_t^r \tag{8}$$

$$\gamma_t = \frac{\beta R_t^n \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c + \mu_t^r \tag{9}$$

$$(q_t^L + \theta)\gamma_t = \frac{\beta[\delta_b + \delta_b q_{t+1}^L]\gamma_{t+1}}{\pi_{t+1}} + (1-\kappa_t)\mu_t^c \tag{10}$$

And two complementary slackness conditions:

$$\mu_t^r \geq 0, \quad n_t - \varphi m_t \geq 0, \quad \mu_t^r(n_t - \varphi m_t) = 0 \tag{11}$$

$$\mu_t^c \geq 0, \quad n_t + b_t^f + (1-\kappa_t)b_t^h - m_t \geq 0, \quad \mu_t^c \left(n_t + b_t^f + (1-\kappa_t)b_t^h - m_t \right) = 0 \tag{12}$$

3.2 Households

There is a measure one of identical households. These self-employed households produce the homogeneous intermediate good y^m to sell to the wholesale firms at the price P_t^m, or at the real relative price p_t^m. In each period, a household purchases the final good from the retail firms to consume ($\widetilde{c}_t$) and make her investment (i_t).

Let $\widetilde{B}_t^h$ be the nominal debt stock that she borrows from bankers. Recalling from the section 2.4, each period she only pays a fraction δ_b of the old debts. We impose an exogenous borrowing constraint for households with the debt limit $\widetilde{b}_t^h \leq \overline{b^h}$.

After the loan market, she brings a_t amount of ZMDs into the final good market. Basically, she faces the "ZMD-in-advance" constraint when the good market opens. So the amount of loans that she gets from banks will affect her demand for the final goods.

In each period, she chooses $\{\widetilde{c}_t, \widetilde{m}_t, \widetilde{b}_t^h, \widetilde{s}_t, i_t, k_t, l_t, a_t\}$ to maximize her utility:

$$\max \sum_{t=0}^{\infty} \widetilde{\beta}^t \left(\log(\tilde{c}_t) - \chi \frac{l_t^{1+\nu}}{1+\nu} \right)$$

subject to

$$\text{Loan Market:} \quad a_t + \delta_b \frac{\widetilde{b}^h_{t-1}}{\pi_t} = \frac{R^m_{t-1}\widetilde{m}_{t-1}}{\pi_t} + q^L_t \widetilde{s}_t \tag{13}$$

$$\text{ZMD-in-advance:} \quad \widetilde{c}_t + i_t \le a_t \tag{14}$$

$$\text{Budget:} \quad \widetilde{m}_t + \widetilde{c}_t + i_t + \upsilon_t(\widetilde{x}_t - \widetilde{x}_{t-1}) = a_t + \tau_t + p^m_t y^m_t + w_t \widetilde{x}_{t-1} \tag{15}$$

$$\text{Investment:} \quad i_t = k_t - (1-\delta)k_{t-1} \tag{16}$$

$$\text{Production:} \quad y^m_t = k^{\alpha}_{t-1} l^{1-\alpha}_t \tag{17}$$

$$\text{Loan Flows:} \quad \widetilde{b}^h_t = \delta_b \frac{\widetilde{b}^h_{t-1}}{\pi_t} + \widetilde{s}_t \tag{18}$$

$$\text{Borrowing Constraint:} \quad \widetilde{b}^h_t \le \overline{b^h} \tag{19}$$

We assume that the household faces an exogenous borrowing constraint, rather than a collateral borrowing constraint like Kiyotaki and Moore (1997) and Iacoviello (2005). Our purpose is to emphasize that the mechanism of the shock transmission in our model is not related to the collateral constraint literature. Similar to the capital requirement, we impose the constraint on the face value of the loan.

Let η^z_t, η^b_t, λ^a_t be the Lagrangian for the cash-in-advance, borrowing constraint and budget constraint. Let λ^b_t be defined as the sum of η^z_t and λ^a_t. Let r^h_t be defined as the real short-term borrowing (lending) rate:

$$\frac{1}{\widetilde{c}_t} = \eta^z_t + \lambda^a_t = \lambda^b_t \tag{20}$$

$$\lambda^a_t = \frac{\widetilde{\beta} R^m_t \lambda^b_{t+1}}{\pi_{t+1}} \tag{21}$$

$$q^L_t \lambda^b_t = \frac{\widetilde{\beta}[\delta_b + \delta_b q^L_{t+1}]\lambda^b_{t+1}}{\pi_{t+1}} + \eta^b_t \tag{22}$$

$$\lambda^b_t = \widetilde{\beta}(1-\delta)\lambda^b_{t+1} + \widetilde{\beta}\alpha \frac{p^m_{t+1}\lambda^a_{t+1} y^m_{t+1}}{k_t} \tag{23}$$

$$\chi l^{\nu+1}_t = (1-\alpha) p^m_t y^m_t \lambda^a_t \tag{24}$$

$$\lambda^a_t \upsilon_t = \widetilde{\beta} \lambda^a_{t+1}(\upsilon_{t+1} + w_{t+1}) \tag{25}$$

$$r_t^h \equiv \frac{\delta_b + \delta_b q_{t+1}^L}{\pi_{t+1} q_t^L} \tag{26}$$

And two complementary slackness conditions:

$$\eta_t^z \geq 0, \quad a_t - c_t - i_t \geq 0, \quad \eta_t^z (a_t - c_t - i_t) = 0 \tag{27}$$

$$\eta_t^b \geq 0, \quad \overline{b^h} - \widetilde{b}_t^h \geq 0, \quad \eta_t^b (\overline{b^h} - \widetilde{b}_t^h) = 0 \tag{28}$$

As money plays the role of medium of exchange in our model, it's value contains the liquidity part. In the steady state, the rate of return on money has to be less than $1/\widetilde{\beta}$.

The equations (22) and (23) give us the marginal cost and the marginal benefit when the household borrows one more unit of loans from bankers and when she makes one more unit of investment. The equation (25) is a common asset pricing equation for the wholesalers' shares.

3.3 Retail Firms and Wholesale Firms

Follow Rotemberg pricing, we assume that each wholesale firm j faces a cost of adjusting prices, which is measured in terms of final good and given by:

$$\frac{\iota}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 y_t$$

where ι determines the degree of nominal price rigidity. The wholesale firm j discounts the profit in the future with rate $\widetilde{\beta}^i \lambda_{t+i}^a / \lambda_t^a$. Her real marginal cost is p_t^m.

In a symmetric equilibrium, all firms will choose the same price and produce the same quantity $P_t(j) = P_t$ and $y_t(j) = y_t = y_t^m$. The optimal pricing rule then implies that:

$$1 - \iota\,(\pi_t - 1)\,\pi_t + \iota \widetilde{\beta} \frac{\lambda_{t+1}^a}{\lambda_t^a} (\pi_{t+1} - 1)\,\pi_{t+1} \frac{y_{t+1}}{y_t} = (1 - p_t^m)\varepsilon \tag{29}$$

3.4 The Central Bank and Government

The consolidated government uses the payoffs from tax or their assets to pay for the interest on reserves, then injects (drains) $\hat{\tau}_t$ amount of reserves and deposits by helicopter money (lump-sum tax) to target the interbank rate. All transactions are conducted in the electronic system.

$$\tau_t = -\frac{(R^n_{t-1} - 1)n_{t-1}}{\pi_t} + \hat{\tau}_t \tag{30}$$

In the conventional monetary policy, we assume that the central bank follows the simple Taylor rule, fixing R^n_t at a constant level $\overline{R^n}$. To connect with the common New Keynesian literature, we assume there is a lower bound for R^f_t that is greater than $\bar{R}^n$, so there are no excess reserves[8]. Later, we relax the assumption and examine the situation when the banking system is awash of excess reserves and the central bank controls the federal funds rate by adjusting R^n_t. In this paper, we assume the inflation target in the long-term of the central bank $\bar{\pi} = 1$.

$$R^n_t = \overline{R^n} \tag{31}$$

$$R^f_t = \max\left\{\frac{\bar{\pi}}{\beta}\left(\frac{\pi_{t+1}}{\bar{\pi}}\right)^{\phi_\pi}, \quad \overline{R^n} + \varepsilon_f\right\} \tag{32}$$

4 Equilibrium

Definition: *A competitive equilibrium is a sequence of bankers' decision choice* $\{c_t, n_t, b^h_t, s_t, m_t, b^f_t, d_t\}$, *household's choice* $\{\widetilde{c}_t, \widetilde{b}^h_t, \widetilde{s}_t, \widetilde{m}_t, i_t, k_t, l_t, y^m_t, \widetilde{x}_t\}$, *firms' choice* $\{y_t\}$, *the central bank' choice* $\{\tau_t$, $R^n_t\}$, *and the market price* $\{q^L_t, R^f_t, \upsilon_t, \pi_t, p^m_t\}$ *such that:*

i *Given the market price and the central bank's choices, banker's choices solve the banker's problem, household's choices solve the household's problem, firm's choice solves the equation (29).*

ii *All markets clear:*

Net flows of reserves:	$d_t + e_t = 0$
The interbank market:	$b^f_t = 0$
Total ZMDs:	$m_t = \widetilde{m}_t$

[8] When the reserve requirement is no longer binding, a Taylor rule is not enough for the determinacy as we need a rule governing the motion of reserves.

Loan Market:	$b_t^h = \widetilde{b}_t^h$
Wholesalers' shares:	$\widetilde{x}_t = 1$
Good Market:	$y_t = c_t + \widetilde{c}_t + i_t + \theta b_t^h + \frac{\iota}{2}(\pi_t - 1)^2 y_t$

The list of equations in equilibrium could be found in the Appendix B. If we consider a model without currency where all banks are identical in the equilibrium, the net flows of reserves to the representative banker will be zero. We also make the following assumption to ensure that in the steady state households will borrow from bankers.

Assumption 1. *The discount factors of bankers and households satisfy:*

$$\frac{\beta \delta_b - \theta \overline{\pi}}{\overline{\pi} - \beta \delta_b} > \frac{\widetilde{\beta} \delta_b}{\overline{\pi} - \widetilde{\beta} \delta_b}$$

We also assume that in the long run, the inflation will be at the target level by restricting monetary policy in every regime to satisfy:

Assumption 2.

$$\lim_{t \to \infty} \frac{\hat{\tau}_t}{n_t} = \frac{\overline{\pi} - 1}{\overline{\pi}}$$

$$\lim_{t \to \infty} R_t^n = \overline{R^n}$$

$$\overline{R^n} + \varepsilon_f < \overline{\pi}/\beta$$

The relationship between the federal funds rate R_t^f, deposit rate R_t^m and interest on reserves R_t^n can be understood under the following theorem:

Theorem 1. *In equilibrium:*

i *The lower bound of the federal funds rate and the deposit rate is the interest on reserves. In all cases,* $R_t^n \leq R_t^m \leq R_t^f$

ii *When the constraint of reserve requirement is not binding,* $R_t^f = R_t^m = R_t^n$.

Proof: Please see the Appendix A for all proofs.

There are two benefits of holding reserves for bankers. First, bankers can earn the interest on reserves that central bank pays them. Second, it helps bankers satisfy reserve requirement. The cost of holding reserves is the federal funds rate that they give up when they do not lend reserves in the interbank market. When the banking system has a large amount of excess reserves, the second benefit vanishes and the federal funds rate must be equal to the interest on reserves.

In reality, the deposit rate of ZMDs might be lower than the interest on reserves due to the bankers' cost of providing liquidity services and market power. We ignore these factors in this model to present the main mechanism cleaner.

Theorem 2. *The total level of reserves in equilibrium is decided solely be the central bank:*

$$\frac{n_{t-1}}{\pi_t} + \hat{\tau}_t = n_t \tag{33}$$

Bankers themselves cannot change the total level of reserves in the banking system. Lending or not lending to households will not change the total level of reserves. The appearance of the huge amount of reserves after the large scale asset purchase is just a byproduct of the central bank's policy. Later we will examine this kind of policy.

5 The Steady State

We use a to denote the steady state value of a variable a_t.

Theorem 3. *Under the Assumption (1)-(2), in every steady state (if exists):*

i. The banker's reserves constraint (4), the household's borrowing constraint (19) and the ZMD-in-advance constraint (14) are binding.

ii. The banker's capital constraint (5) is not binding.

Theorem 4. *Under the Assumption (1) and (2), the capital in every steady state (if exists)* *satisfies the following equation:*

$$\frac{1}{r^m \alpha_m k - \delta k + q^L s - \delta_b \frac{\overline{b}^h}{\overline{\pi}}} = \frac{\chi \alpha_l^{\nu+1} k^\nu}{(1-\alpha) p^m \alpha_y r^m} \tag{34}$$

where r^m, α_m, α_l, α_y are constants independent of k.

We make one more assumption to ensure that there exists a unique steady state. The uniqueness of the steady state will be very important as we mostly examine the global nonlinear dynamic of our model.

Assumption 3. *The parameters satisfy:*

$$\kappa < 1 - \frac{(1-\varphi)m}{\overline{b^h}}$$
$$\frac{(\beta\delta_b - \overline{\pi}\theta)(\overline{\pi} - \delta_b)}{\overline{\pi} - \beta\delta_b} > \delta_b$$
$$r^m \alpha_m - \delta > 0$$

where m is defined in (A.21), r^m and α_m are defined in (A.12) and (A.16).

The restriction on the parameter κ is to ensure that the capital constraint is not binding. The last two restrictions are to ensure that the equation (34) has a unique positive solution k^*.

Theorem 5. *Under the Assumption (1)-(3), there is a* ***unique*** *steady state.*

6 Quantitative Analysis

6.1 Calibration

For the bankers' parameters, we choose the discount factor $\beta = 0.99$ to match with the federal funds rate of 4% annually before the Great Recession. The reserves requirement is set as the ratio between reserves and the total ZMDs (including checking account, saving account and money market deposit account) before the financial crisis, which is around $\varphi = 0.002$. The

Table 6: Parameter values

Param.	Definition	Value
Bankers		
β	Banker's discount factor	0.99
φ	The reserves requirement	0.002
κ	The risk weight	0.22
θ	The monitoring cost	0.0005
δ_b	Loan amortization	0.5
Households		
$\widetilde{\beta}$	Household's discount factor	0.985
χ	Relative Utility Weight of Labor	0.586
ν	Inverse Frisch Elasticity of Labor Supply	0.5
$\overline{b^h}$	The borrowing limit	3.4
δ_k	Capital's depreciation rate	0.025
α	Capital share in production function	0.34
Firms		
ε	Elasticity of substitution of wholesale goods	4
ι	Cost of changing price	100
Central bank		
ϕ_π	Policy respond to inflation	1.25
$\overline{R^n}$	The constant IOR	1+0.25/400
$\overline{R^n}+\varepsilon_f$	The lower bound for FFR	1+0.5/400

monitoring cost θ and loan amortization δ_b are set exogenously. The risk weight κ is exogenously set so that 10 percent increase of κ from steady state will make the capital constraint binding. (Table 6)

Most of the households' parameters are standard in the literature. The only one that needs to be calibrated is the borrowing limit $\overline{b^h}$. We calibrate it to match with the ratio between total households' debts and households' income before the Great Recession - around 1.3 times. All other parameters are also in the range which is often seen in the macro literature. We tried using as few parameters as possible to illustrate the main mechanism of the model. The numerical method could be found in the Appendix C.

6.2 Federal funds rate shock

We examine an interest rate shock in the Taylor rule and compare the mechanism of this model to the standard one in the New Keynesian literature.

$$
\begin{aligned}
&R_t^f = \max\left\{\frac{1}{\beta}\left(\frac{\pi_{t+1}}{\bar{\pi}}\right)^{\phi_\pi}\exp(u_t^f), \quad \overline{R^n} + \varepsilon_f\right\} \\
&R_t^n = \overline{R^n} \\
&u_t^f = \rho_f u_{t-1}^f, \quad u_0^f \text{ is given}
\end{aligned}
\tag{P1}
$$

From the steady state, there is an unexpected shock at $t = 0$ with $u_0^f = -2/400$, then agents know that the shock will die slowly with $\rho_f = 0.6$.

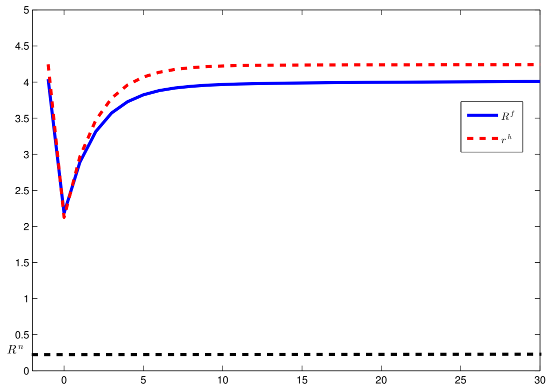
Similar to the standard New Keynesian model: As the price is sticky, when the central bank cut the federal funds rate, the real rate goes down and stimulates the economy in the short run. (Figure 2)[9].

Difference from the standard New Keynesian model:

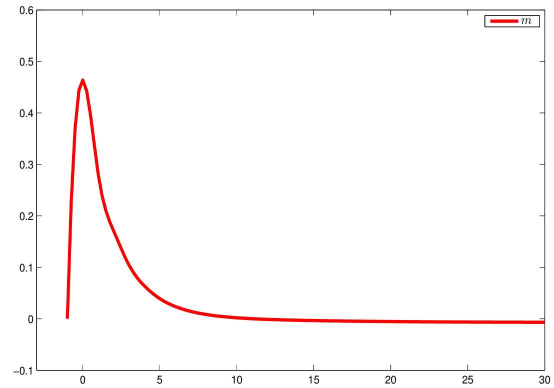
i Banks play an important role in creating money. After the interest rate shock, the real money balance increases by 0.45 percent. Most of that new money is created by banks when they increase loans. The amount of money that the central bank actually "drops" to the economy $\hat{\tau}$ to change the federal funds rate only accounts for 0.02 percent of this increase. So unlike the standard model in New Keynesian, our model focuses on the money creation process by commercial banks and the pass-through effect from the federal funds rate to the loan rate.

ii Without any adjustment cost functions, investment still well-behaves after the cut in the real interest rate. The constraint for a huge sudden jump of investment comes naturally from the ZMD-in-advance constraint and the borrowing constraint.

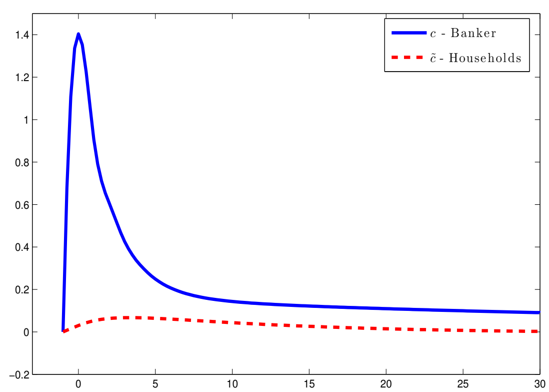
[9]Except the federal funds rate and the real borrowing rate are converted to the annual level, all other figures show the percentage deviation of a variable from its steady state value.



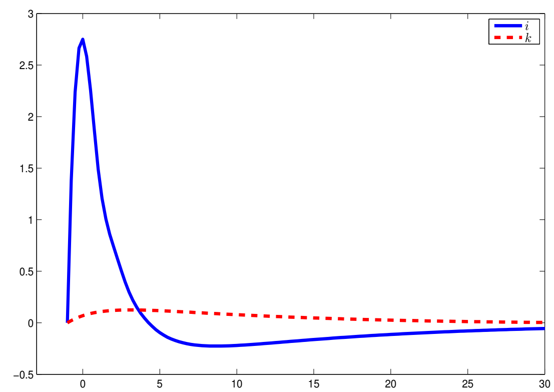
(a) Federal funds rate and real borrowing rate



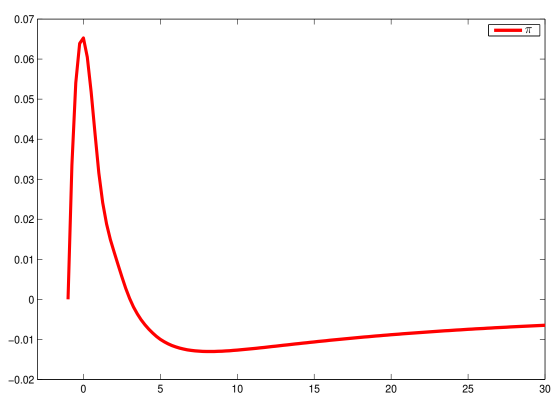
(b) Real balance of money supply (ZMDs)



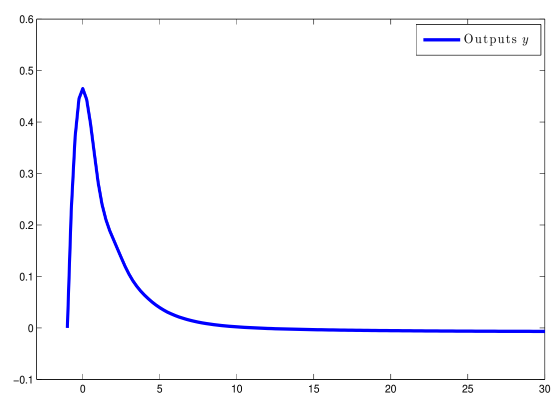
(c) Consumptions



(d) Investment and Capital



(e) Inflation



(f) Output

Figure 2: Impulse Response to Interest Rate Shock in (**P1**)

6.3 Financial Crisis - Taylor Rule Response

From the steady state, we illustrate a financial crisis by imposing an unexpected shock at κ_t in the capital constraint. This is a simplified way to reflect a sudden increase in the "potential" bad loans in the bankers' balance sheets. This paper does not try answering the cause of the Great Recession, so this reduced form is neat to assess different monetary policy rules. In this section, the conventional monetary policy still follows the Taylor rule in (31) and (32).

$$\begin{aligned} R_t^f &= \max\left\{\frac{1}{\beta}\left(\frac{\pi_{t+1}}{\bar{\pi}}\right)^{\phi_\pi}, \quad \overline{R^n}+\varepsilon_f\right\} \\ R_t^n &= \overline{R^n} \\ \kappa_t &= \rho_\kappa \kappa_{t-1} + (1-\rho_\kappa)\bar{\kappa}, \quad \kappa_0 \text{ is given} \end{aligned} \tag{P2}$$

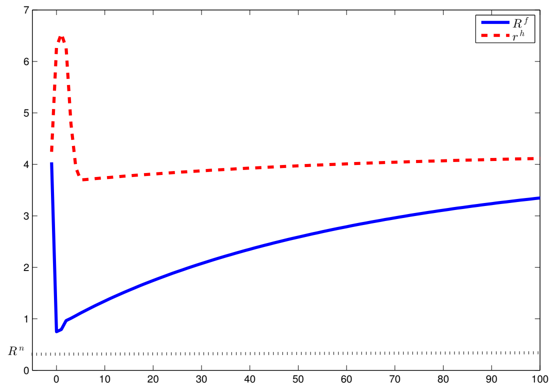
where $\rho_k = 0.95$ is the persistence of the shock and $\kappa_0 = 0.26$, which is 18 percent higher than the one in the steady state level. The capital requirement switches to the binding mode in the short run. The response of the economy is illustrated in the Figure 3.

The banking crisis is dangerous as it raises the spread between the prime rate and the federal funds rate. To satisfy the capital requirement (CR), bankers have to cut loans. Loan rate goes up even when the federal funds rate is cut down, as the shadow price of capital requirement μ_t^c is positive now.

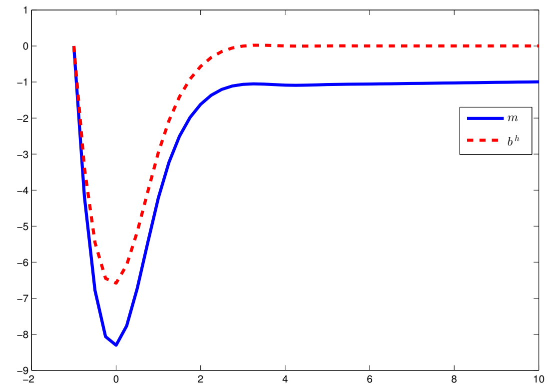
$$\gamma_t = \frac{\beta(\delta_b + \delta_b q_{t+1}^L)}{\pi_{t+1}(q_t^L+\theta)}\gamma_{t+1} + \underbrace{\frac{(1-\kappa_t)\mu_t^c}{q_t^L+\theta}}_{\text{Spread due to CR's binding}}$$

Money supply eventually drops due to the consequence of the debt deleveraging process. Deflation will be persistent under the Taylor rule as the conventional monetary policy only focuses on the pass through of federal funds rate to the prime rate, which will not work in this case.

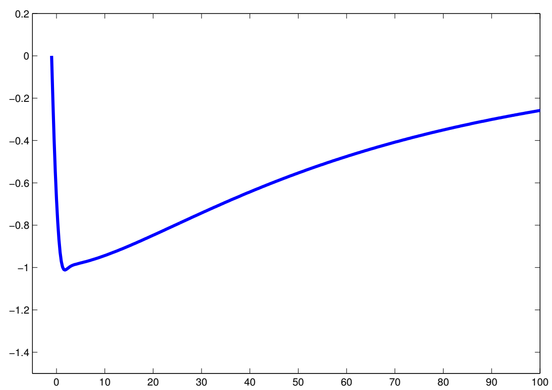
Standard New Keynesian model emphasizes the importance of monetary policy in correcting the deviation of real rate from its natural level due the the price stickiness. Under the framework where the banking sector is modeled clearly, there are two other inefficiencies that monetary policy can intervene to improve the social welfare. The first inefficiency arises from the binding



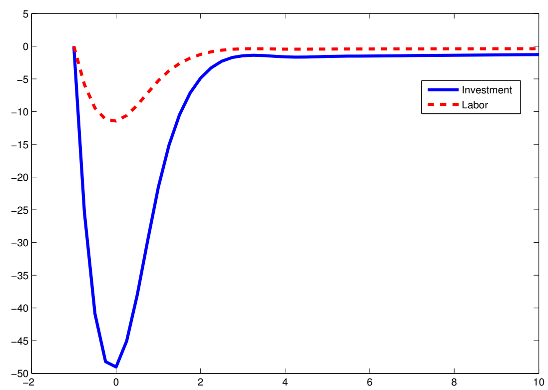
(a) Federal funds rate and real borrowing rate



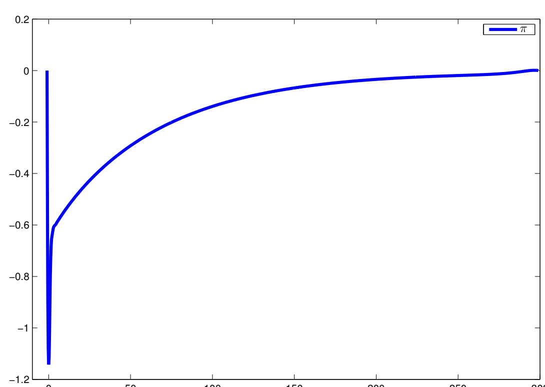
(b) Real balance of ZMDs and households' debts



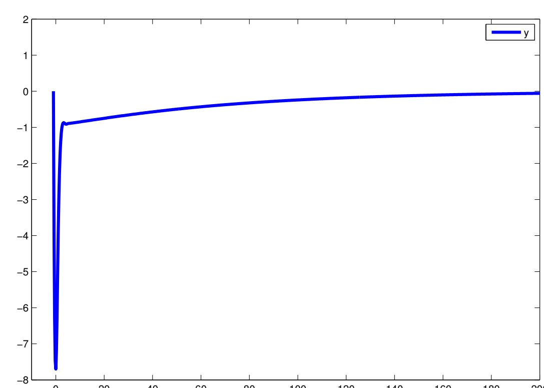
(c) Aggregate consumption



(d) Investment and Labor



(e) Inflation



(f) Outputs

Figure 3: Impulse Response to Capital Constraint Shock (**P2**)

of the capital constraints, which freezes the credit market between bankers and households. The second inefficiency comes from the households' borrowing constraint itself. Unconventional monetary policy focuses on the money supply and asset price might be a good remedy for this situation. We only focus on the money supply (liquidity provision) in this paper.

6.4 Financial Crisis - Large Scale Asset Purchase (LSAP)

Now, assume that central bank injects money directly into the market by purchasing the wholesale firms' shares. Let x_t be the number of shares that central bank decides to hold at time t and $\Delta x_t = x_t - x_{t-1}$ be the additional number of shares the central bank purchases at time t. Recall υ_t be the share's price and $\widetilde{x}_t$ is the number of shares that households hold.

$$x_t + \widetilde{x}_t = 1 \tag{35}$$

When the central bank makes transactions with households, in the electronic system, the flows of money will follow the Table 7:

The Fed		Bankers		Households	
Securities: $+\upsilon_t \Delta x_t$	Reserves: $+\upsilon_t \Delta x_t$	Reserves:$+\upsilon_t \Delta x_t$	Deposits: $+\upsilon_t \Delta x_t$	Deposits: $+\upsilon_t \Delta x_t$	
				Securities: - $\upsilon_t \Delta x_t$	

Table 7: Central Bank's Asset Purchase

Before time 0, $x_t = 0$. At time 0, there is an unexpected shock of purchasing assets from the central bank in response instantly to the unexpected shock on κ_t. The LSAP is conducted after the loan market. Then the central bank will slowly sell these assets back to the market. For the dividends earned from holding securities, we assume that the consolidated government will give them back to households under the form of lump-sum transfers. In equilibrium, the equations for reserve flows and deposit flows become:

$$\frac{n_{t-1}}{\pi_t} + \upsilon_t (x_t - x_{t-1}) + \hat{\tau}_t = n_t \tag{36}$$

$$m_t = \frac{R^m_{t-1} m_{t-1}}{\pi_t} + q^L_t s_t + \theta_t b^h_t - \delta_b \frac{b^h_{t-1}}{\pi_t} + c_t + \upsilon_t (x_t - x_{t-1}) + \hat{\tau}_t - (R^n_{t-1} - 1) \frac{n_{t-1}}{\pi_t} \tag{37}$$

The exogenous shock for κ_t and monetary policy rule are:

$$\begin{aligned}
&\kappa_t = \rho_\kappa \kappa_{t-1} + (1-\rho_\kappa)\bar{\kappa}, \quad \kappa_0 \text{ is given} \\
&x_t = \rho_x x_{t-1}, \quad x_0 \text{ is given} \\
&\hat{\tau}_t = \hat{\tau}, \quad R^n_t = \overline{R^n}, \quad \forall t \geq 0
\end{aligned} \tag{P3}$$

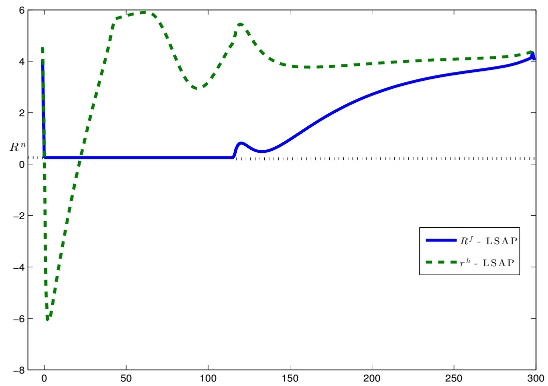
where $\rho_x = 0.98$ be the persistence of the asset purchasing shock and $x_0 = 0.0008$. We assume that the central bank does not follow the Taylor rule anymore. They still fix the interest on reserves at the constant level $\overline{R^n}$ and only use that asset purchase/sale program to adjust the money supply. Figure 4 shows the reaction of the economy to this monetary policy.

Here are some important remarks for LSAP's effect:

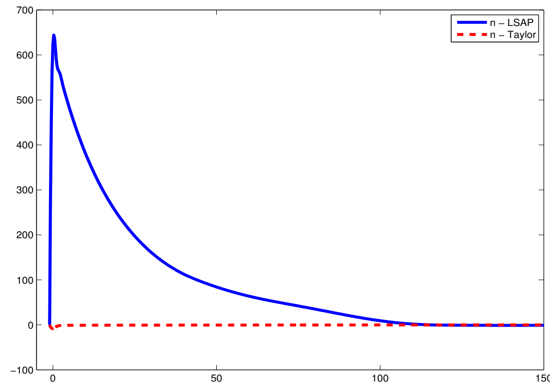
i *The excess reserves skyrockets and the long duration of the federal funds rate at the lower bound*: When the central bank purchases assets from the private sector, they inject simultaneously money supply into the market and banking reserves into the banking system. When the level of reserves increases by 700 percent, the reserve constraint is no longer binding, $\mu^r_t = 0$. As we assume that the central bank fixes IOR at a constant level, it is synonymous that the federal funds rate will be at the lower bound for a long time, around 25 years (100 quarters) in our model. After a long unwinding quantitative easing process, the reserve requirement will be binding again. The federal funds rate climbs back to its long run level. The whole transition process can take around 80 years in our model.

ii *Positive effect in the short-run*: The combination of the new money injection and the long duration of the federal funds rate at the lower bound steers the economy out of recession quickly, unlike the case with the Taylor rule. Due to the effect of forward guidance on the inflationary expectation, if the central bank commits to let the federal funds rate at the low level for a long time, the real lending rate will decline sharply. It combines with the relaxation of the liquidity constraint, stimulating the household's demand and pushing up inflation. This point in our research is identical to the New Keynesian literature.

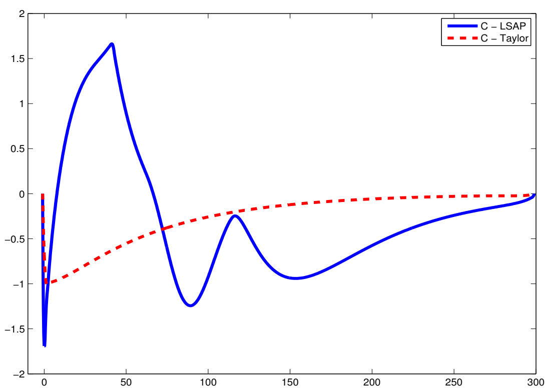
iii *Negative effect in the long-run*: After inflation jumps up in the short run, it starts declining,



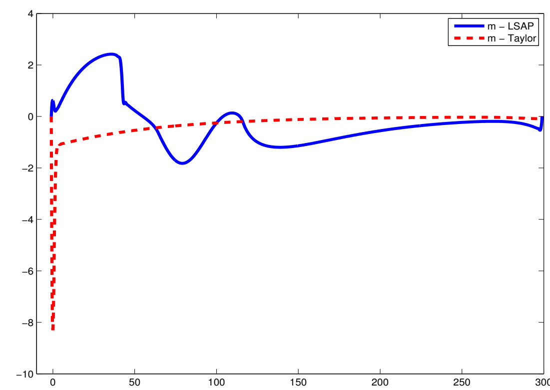
(a) Federal funds rate and real borrowing rate



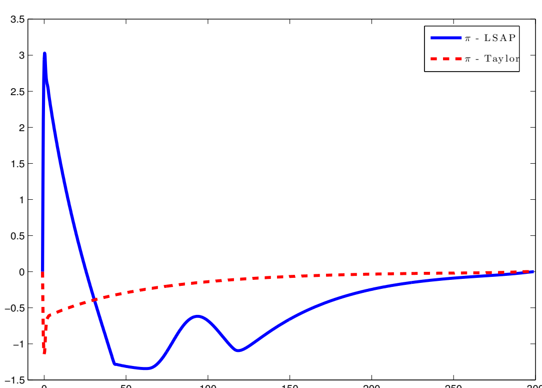
(b) Real balance of reserves



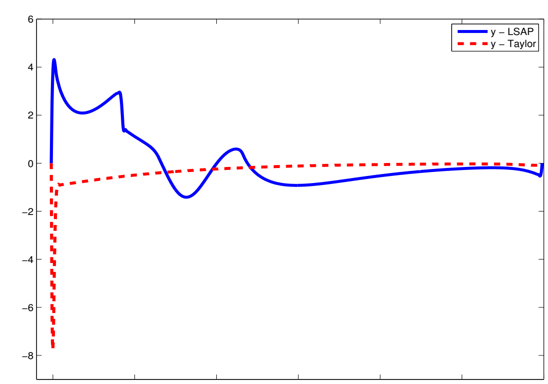
(c) Aggregate consumption



(d) Real balance of ZMDs



(e) Inflation



(f) Outputs

Figure 4: Response of economy to LSAP vs Taylor Rule

below the central bank's target in the medium run. This phenomenon can be explained by the Neo-Fisherian's idea. In the long run, the real short-term rate will be back to the long-term level. As $R_t^f = \overline{R^n}$, the deflation must realize to increase R_t^f/π_{t+1}.

iv *Intuitive Explanation*: When the central bank keeps the interbank rate at 25 basis points, the rate of saving account R^m will be at 25 basis points as deposits and interbank loans have the same short-term maturity. However, the real return on capital in the long-run recovers to the pre-crisis level. In equilibrium, the real return on deposits (plus the liquidity premium) must follow the real return on capital. The endogenous money supply declines gradually. Deflation must realize to ensure this condition.

6.5 Interest on Reserves (IOR) as Monetary Policy Tool

6.5.1 IOR: To raise or not to raise?

In the section 6.4, we know that after the LSAP program without adjusting R_t^n, the inflation - the central bank's main target - is high in the short run but below the target in the long-run. How long should the central bank keep the federal funds rate at the zero lower bound? And if the central bank decides to raise rate, what is a good strategy for the central bank?

In this section, we still conduct the experiment similar to the previous section with one twist. We assume that after T_u periods, the central bank will raise IOR. Then IOR will be brought back to the steady state level after T_d period. We choose the different level for T_u at 20, 40 and 80 quarters to see the effect of the prolonged low interest rate environment on output and inflation in the short run and long run. T_d is chosen at 200 quarters.

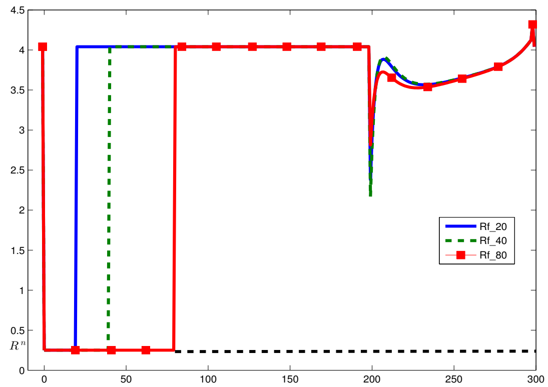
$$\begin{aligned}
&\kappa_t = \rho_\kappa \kappa_{t-1} + (1-\rho_\kappa)\bar{\kappa}, \quad \kappa_0 \text{ is given} \\
&x_t = \rho_x x_{t-1}, \quad x_0 \text{ is given} \\
&\hat{\tau}_t = \hat{\tau}, \quad \forall t \geq 0 \\
&R_t^n = \begin{cases} \overline{R^n} & \text{if } t < T_u \\ 1/\beta & \text{if } T_u \leq t \leq T_d \\ \overline{R^n} & \text{if } t > T_d \end{cases}
\end{aligned} \tag{P4}$$

Here are some remarks from our experiment: (Figure 5)

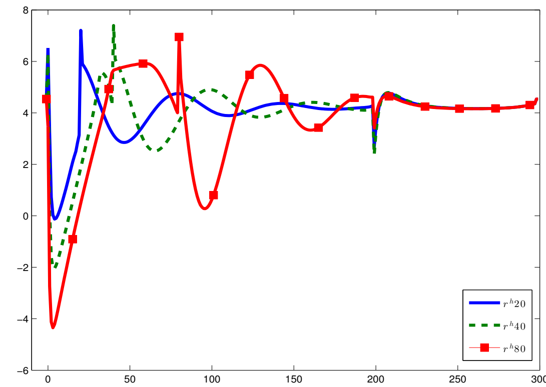
i. The longer is the duration of the federal funds rate at the lower bound, the higher is inflation in the short run. This forward guidance effect is well-documented in the New Keynesian literature when the central bank commits to set the short-term at the zero lower bound for a long time (Eggertsson and Woodford (2003)). However, the hyperinflation never happens in our model even with 20 years that rate is set at the lower bound. Due to the household's borrowing constraint and banker's capital constraint, the amount of the money supply is restricted even with the huge amount of excess reserves in the banking system.

ii. The longer is the duration of the federal funds rate at the lower bound, the bigger is the negative effect on output and deflation in the long run. It emphasizes that our model is Keynesian in the short run, but Neo-Fisherian in the long run.

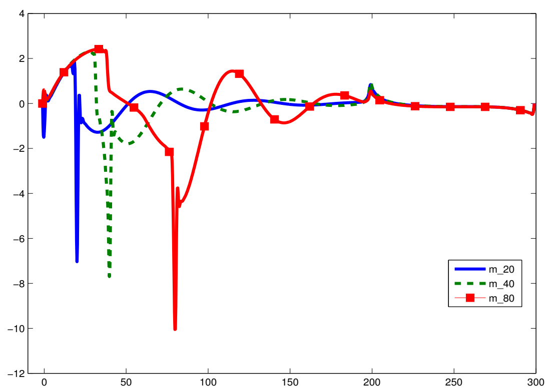
iii. The endogenous money supply drops sharply when the central bank raises rates. As price is sticky, the real fed funds rate and real lending rate must go up after this rate hike. Hence, the total of amount of bank credits declines, also implying a huge fall in the money supply. However, after some periods, the neo-Fisherian effect dominates the Keynesian effect, stabilizing inflation at the target level. After all, the central bank still needs to pay a big cost for a rate hike in the short run.



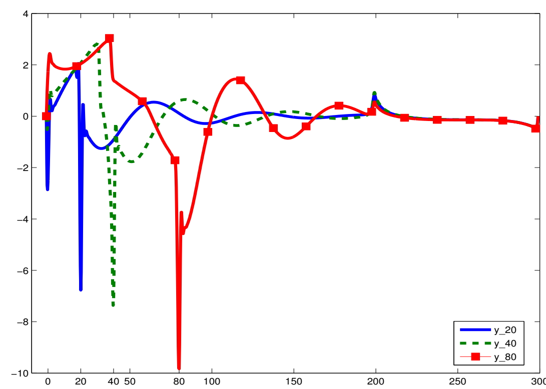
(a) Federal funds rate



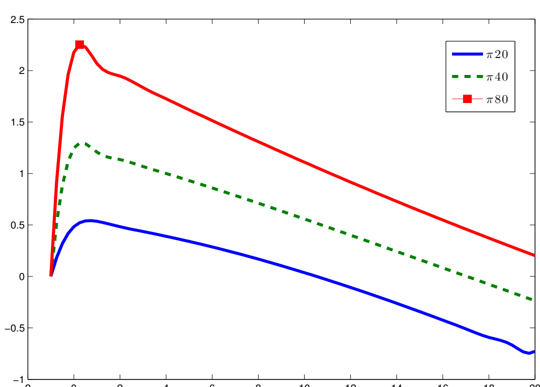
(b) Real rate of borrowing



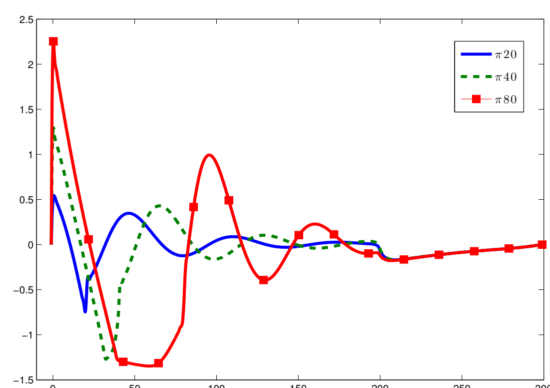
(c) Real balance of money supply



(d) Outputs



(e) Inflation - Short run



(f) Inflation - Long run

Figure 5: Raise IOR at different time horizons: after 20, 40, 80 quarters

The last point implies an important hint for monetary policy when the central bank decides to raise rate. The central bank can still stabilize inflation and the aggregate demand if it commits to a rule of targeting the money supply at the time of raising rates. The appearances of interest on reserves and electronic payment system allow the central bank to manipulate both the money supply and interest rate at the short run, which is very different from New Keynesian focusing only on the short-term rate. In this sense, our research is near to Monetarism in the long run. The growth rate of the money supply always decides the inflation path in the long run.

6.5.2 Raise rate and raise money supply - Money Supply Rule

We do an experiment similar to (**P4**) but at the time of raising IOR, the central bank also commits to a money supply rule (massive helicopter money if necessary) to target the inflation rate. The money supply rule simply responds to the deviation of the inflation rate from its target:

$$\frac{M_t}{M_{t-1}} = \left(\frac{\bar{\pi}}{\pi_t}\right)^{\rho_m} \tag{38}$$

where $\rho_m = 0.5$ is the coefficient showing how much the central bank will change the growth rate of the money supply in response to inflation.

To create the same interbank rate path as the one in the previous section, we assume this money supply rule only applies since the time the central bank decides to raise rates. The complete list of exogenous shocks and monetary policy for this experiment can be written as follows:

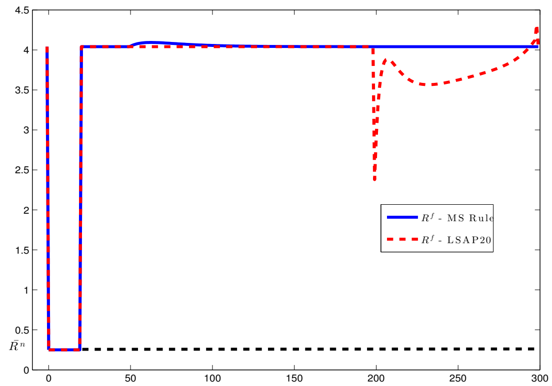
$$\begin{aligned}
&\kappa_t = \rho_\kappa \kappa_{t-1} + (1-\rho_\kappa)\bar{\kappa}, \quad \kappa_0 \text{ is given} \\
&x_t = \rho_x x_{t-1}, \quad x_0 \text{ is given} \\
&\begin{cases} \hat{\tau}_t = \hat{\tau} & \text{if } t < T_u \\ \log(m_t) - \log(m_{t-1}) = -(1+\rho_m)\log(\pi_t) & \text{if } t \geq T_u \end{cases} \\
&R^n_t = \begin{cases} \overline{R^n} & \text{if } t < T_u \\ 1/\beta & \text{if } T_u \leq t \leq T_d \\ \overline{R^n} & \text{if } t > T_d \end{cases}
\end{aligned} \tag{P5}$$

Figure 6, by comparing (**P5**) to (**P4**), shows the effectiveness of combining the rate hike with money supply targeting:

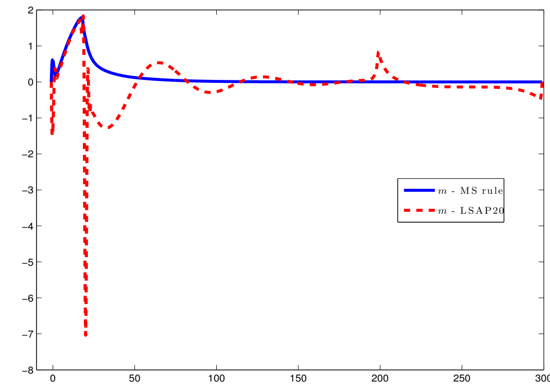
i. Even though the federal funds rate paths are nearly identical in the first 200 periods in our experiments, the dynamics of output and inflation are very different. It implies that interest rate path does not give enough information for the stance of monetary policy when central bank use IOR as the main tool. When there is no excess reserves, the federal funds rate path conveys all information about monetary policy. It is not this case with the current situation, when the central bank can manipulate both the money supply and the interest rate.

ii. Money supply targeting is extremely efficient in stabilizing inflation and output. The inflation is anchored at the target since the time the central bank targets the growth rate of the money supply in our model.

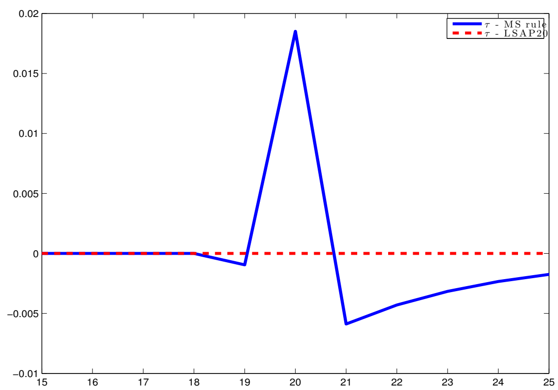
iii. At the time of raising rate (after 20 quarters), to stabilize the inflation and avoid a severe short recession, money supply targeting implies that the central bank should conduct a massive helicopter money. With this commitment, the central bank can increase the household's expectation about inflation path when raising rates. As a result of that, the real interest rate does not change.



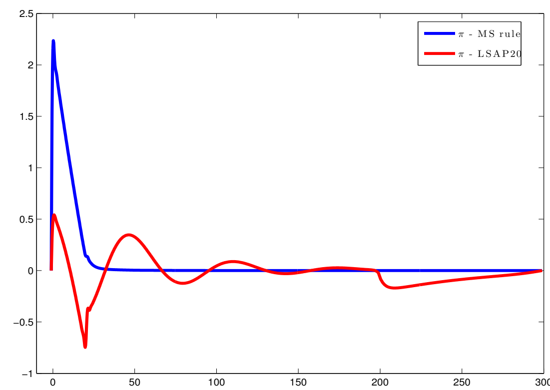
(a) Federal funds rate



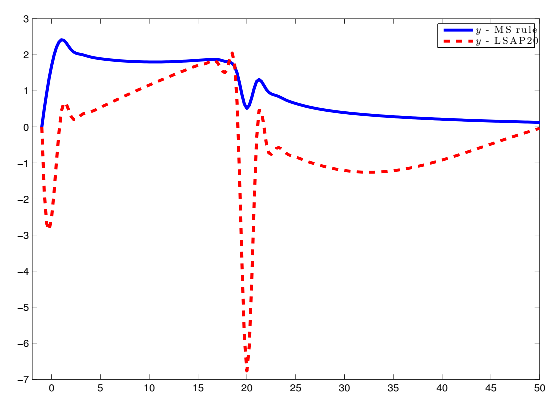
(b) Real balance of ZMDs



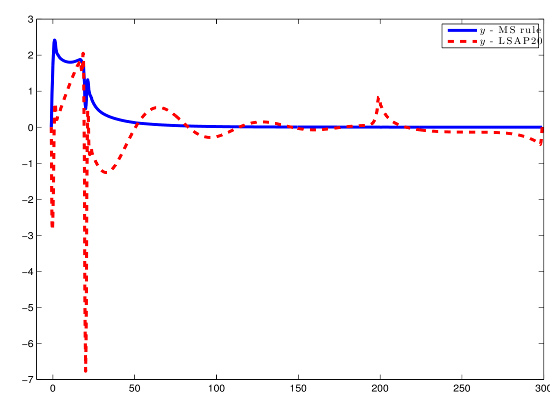
(c) The amount of money drops (level)



(d) Inflation



(e) Output - Short run



(f) Output - Long run

Figure 6: Raise IOR with and without targeting the growth rate of the money supply

7 Conclusion

Our research shows that, when the central bank controls the federal funds rate by adjusting interest on reserves, the path of the interbank rate alone does not provide full information on the stance of monetary policy. The endogenous money supply can complete go down when the federal funds rate is near zero for a long time. However, if the central bank simply raises rates, the economy will fall into a short recession. Deflation will be worse in the short run after the interest rate is lifted. Basically, the central bank falls into a dilemma to raise or not to raise rate, where outcome is not bright in either way.

One feasible solution for the central bank is to target the growth rate of the money supply in response to inflation when they raise rates. The key insight of our paper is that the central bank can adjust simultaneously the interbank rate and the money supply in this era. With that, they can completely avoid the negative short term effect of raising rates and do a better job at hitting the inflation target.

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A Mathematical Appendix

Proof for Theorem 1:

From the first order condition of bankers' problem, we have:

$$\gamma_t = \frac{\beta R_t^f \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c \tag{A.1}$$

$$\gamma_t = \frac{\beta R_t^m \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c + \varphi \mu_t^r \tag{A.2}$$

$$\gamma_t = \frac{\beta R_t^n \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c + \mu_t^r \tag{A.3}$$

As μ_t^c and μ_t^r are non-negative shadow price of capital constraint and reserve constraint, $\gamma_t > 0$ as $c_t > 0$, we have $R_t^n \leq R_t^m \leq R_t^f$.

The $'' ='' $ happens when $\mu_t^r = 0$, or when the reserver requirement is no longer binding.

Proof for Theorem 2:

The equation for reserves flow (1) is:

$$\frac{R_{t-1}^n n_{t-1}}{\pi_t} + \frac{R_{t-1}^f b_{t-1}^f}{\pi_t} + d_t + e_t + \tau_t = n_t + b_t^f$$

In equilibrium, $b_t^f = 0$, $d_t + e_t = 0$ and from (30):

$$\tau_t = -\frac{(R_{t-1}^n - 1) n_{t-1}}{\pi_t} + \hat{\tau}_t$$

Substitute that into the reserves flow:

$$\frac{n_{t-1}}{\pi_t} + \hat{\tau}_t = n_t$$

So the total level of reserves only depend on $\hat{\tau}$, which is decided solely be the central bank.

Proof for Theorem 3:

We use a to denote the steady state value of a variable a_t. From the Theorem 2, in every steady state:

$$\pi = \frac{1}{1-\tau/n} \tag{A.4}$$

Under the Assumption (2):

$$\frac{\tau}{n} = \frac{\overline{\pi}-1}{\overline{\pi}} \tag{A.5}$$

(A.4) and (A.5) $\rightarrow \pi = \overline{\pi}$. Money supply rule ensures that inflation reaches to its target in the steady state.

From (32), we have:

$$R^f = \max\{\overline{\pi}/\beta, \overline{R^n} + \varepsilon_f\} \tag{A.6}$$

Under the assumption (2): $\overline{R^n} + \varepsilon_f < \overline{\pi}/\beta$, we get $R^f = \overline{\pi}/\beta$. The equation (A.1) can be rewritten in the steady state as:

$$\gamma = \frac{\beta R^f \gamma}{\overline{\pi}} + \mu^c$$

When $R^f = \overline{\pi}/\beta$, we get $\mu^c = 0$, the capital constraint is not binding (if steady state exists). As $R^f > \overline{R^n}$, from the Theorem 1, $\mu^r > 0$, or the reserve requirement is binding.

When $\mu^c = 0$, from (10), at the steady state:

$$q^L = \frac{\beta\delta_b - \theta\overline{\pi}}{\overline{\pi} - \beta\delta_b} \tag{A.7}$$

Under the Assumption (1) and (22), at the steady state, $\eta_b > 0$, so the borrowing constraint is binding.

As $\mu^r > 0$, we get $R^m < R^f = \overline{\pi}/\beta$. From (20) and (21), at the steady state, $\eta^z > 0$, so the ZMD-in-advance constraint is binding.

Proof for Theorem 4:

Let r denote the gross real rate such that $r = R/\pi$. In the steady state, we have:

$$1 = \beta r^f + \frac{\mu^c}{\gamma} \tag{A.8}$$

$$1 = \beta r^m + \frac{\mu^c}{\gamma} + \frac{\varphi \mu^r}{\gamma} \tag{A.9}$$

$$1 = \beta r^n + \frac{\mu^c}{\gamma} + \frac{\mu^r}{\gamma} \tag{A.10}$$

As $r^f = 1/\beta$ and $\mu^c = 0$, we have:

$$\frac{\mu^r}{\gamma} = 1 - \beta r^n \tag{A.11}$$

$$r^m = \frac{1 - \varphi(1 - \beta r^n)}{\beta} \tag{A.12}$$

Besides that, it is easy to see that:

$$q^L = \frac{\beta \delta_b - \overline{\pi}\theta}{\overline{\pi} - \beta \delta_b}; \quad p^m = \frac{\varepsilon - 1}{\varepsilon}; \quad b^h = \overline{b}^h; \quad s = (1 - \frac{\delta_b}{\pi}) b^h$$

Substitute r^m into the equation (21) showing the liquidity premium of ZMDs:

$$\frac{\lambda^a}{\lambda^b} = \widetilde{\beta} r^m \tag{A.13}$$

Use (A.13) to substitute into (23), then define α_y as:

$$\frac{y}{k} = \frac{1 - \widetilde{\beta}(1-\delta)}{\widetilde{\beta} \alpha p^m \left(\frac{\lambda^a}{\lambda^b}\right)} = \frac{1 - \widetilde{\beta}(1-\delta)}{\widetilde{\beta} \alpha p^m \widetilde{\beta} r^m} \equiv \alpha_y \tag{A.14}$$

Use (A.14) to substitute into the production function, then define α_l as:

$$\frac{l}{k} = \left(\frac{y}{k}\right)^{1/(1-\alpha)} = \left(\frac{1 - \widetilde{\beta}(1-\delta)}{\widetilde{\beta} \alpha p^m \widetilde{\beta} r^m}\right)^{1/(1-\alpha)} \equiv \frac{1}{\alpha_l} \tag{A.15}$$

From the banker's deposit flows:

$$\begin{aligned}
&m = r^m m + q^L s - \delta_b \frac{b^h}{\pi} + \theta b^h + c + \hat{\tau} - (R^n - 1)\frac{n}{\pi} \\
&m = c + \tilde{c} + i + \theta b^h + \varphi m \left(1 - \frac{1}{\pi}\right) - \frac{(R^n - 1)\varphi m}{\pi} \quad \text{(Use ZMD in advance)} \\
&\left[1 - \varphi\left(1 - \frac{1}{\pi}\right) + \frac{(R^n - 1)\varphi}{\pi}\right] m = y = \alpha_y k
\end{aligned}$$

So we can write:

$$m = \alpha_m k \quad \text{where } \alpha_m = \frac{\alpha_y}{1 - \varphi\left(1 - \frac{1}{\pi}\right) + \frac{(R^n - 1)\varphi}{\pi}} \tag{A.16}$$

From the ZMD-in-advance constraint:

$$\tilde{c} = r^m \alpha_m k - \delta k + q^L s - \delta_b \frac{\overline{b}^h}{\pi} \tag{A.17}$$

From the household's foc w.r.t labor:

$$\lambda^a = \frac{\chi l^{\nu+1}}{(1-\alpha) p^m y} = \frac{\chi (\alpha_l k)^{\nu+1}}{(1-\alpha) p^m \alpha_y k} = \frac{\chi \alpha_l^{\nu+1} k^\nu}{(1-\alpha) p^m \alpha_y} \tag{A.18}$$

So we have:

$$\lambda^b = \frac{\lambda^a}{\widetilde{\beta} r^m} = \frac{\chi \alpha_l^{\nu+1} k^\nu}{(1-\alpha) p^m \alpha_y r^m} \tag{A.19}$$

So we have an equation with a single variable k:

$$\frac{1}{r^m \alpha_m k - \delta k + q^L s - \delta_b \frac{\overline{b}^h}{\pi}} = \frac{\chi \alpha_l^{\nu+1} k^\nu}{(1-\alpha) p^m \alpha_y r^m} \tag{A.20}$$

Proof for Theorem 5

Consider the following function:

$$f(k) = \frac{1}{r^m \alpha_m k - \delta k + q^L s - \delta_b \frac{\overline{b}^h}{\pi}} - \frac{\chi \alpha_l^{\nu+1} k^\nu}{(1-\alpha) p^m \alpha_y r^m}$$

Under the Assumption (3), it is clear that $f(k)$ is decreasing with k when $k > 0$. Moreove under this assumption, we have:

$$f(0) = \frac{1}{q^L s - \delta_b \frac{\overline{b^h}}{\overline{\pi}}} > 0; \quad \lim_{k \to +\infty} f(k) = -\infty$$

So $f(k) = 0$ has a unique positive root $k^* > 0$. It is equivalent that (A.20) has a unique solution $k^* > 0$. The steady state value of m is:

$$m = \alpha_m k^* \tag{A.21}$$

We still need to ensure that the capital constraint at this steady state is not binding. That's why we need the restriction on κ in the Assumption (3).

B System of Equations in Equilibrium

B.1 Conventional Monetary Policy: Taylor Rule

We have a system of 29 equations for 29 variables

Bankers:

$$\gamma_t = \frac{1}{c_t} \tag{B.1}$$

$$\gamma_t = \frac{\beta R_t^f \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c \tag{B.2}$$

$$\gamma_t = \frac{\beta R_t^m \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c + \varphi \mu_t^r \tag{B.3}$$

$$\gamma_t = \frac{\beta R_t^n \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c + \mu_t^r \tag{B.4}$$

$$(q_t^L + \theta)\gamma_t = \frac{\beta[\delta_b + \delta_b q_{t+1}^L]\gamma_{t+1}}{\pi_{t+1}} + (1 - \kappa_t)\mu_t^c \tag{B.5}$$

$$\frac{n_{t-1}}{\pi_t} + \hat{\tau}_t = n_t \tag{B.6}$$

$$m_t = \frac{R_{t-1}^m m_{t-1}}{\pi_t} + q_t^L s_t + \theta_t b_t^h - \delta_b \frac{b_{t-1}^h}{\pi_t} + c_t + \hat{\tau}_t - (R_{t-1}^n - 1)\frac{n_{t-1}}{\pi_t} \tag{B.7}$$

$$\mu_t^r \geq 0, \quad n_t - \varphi m_t \geq 0, \quad \mu_t^r (n_t - \varphi m_t) = 0 \tag{B.8}$$

$$\mu_t^c \geq 0, \quad n_t + (1 - \kappa_t) b_t^h - m_t \geq 0, \quad \mu_t^c \left(n_t + (1 - \kappa_t) b_t^h - m_t\right) = 0 \tag{B.9}$$

$$b_t^h = \delta_b \frac{b_{t-1}^h}{\pi} + s_t \tag{B.10}$$

Households:

$$\frac{1}{\widetilde{c}_t} = \eta_t^z + \lambda_t^a \tag{B.11}$$

$$\frac{1}{\widetilde{c}_t} = \lambda_t^b \tag{B.12}$$

$$\lambda_t^a = \frac{\widetilde{\beta} R_t^m \lambda_{t+1}^b}{\pi_{t+1}} \tag{B.13}$$

$$q_t^L \lambda_t^b = \frac{\widetilde{\beta}[\delta_b + \delta_b q_{t+1}^L]\lambda_{t+1}^b}{\pi_{t+1}} + \eta_t^b \tag{B.14}$$

$$\lambda_t^b = \widetilde{\beta}(1-\delta)\lambda_{t+1}^b + \widetilde{\beta}\alpha \frac{p_{t+1}^m \lambda_{t+1}^a y_{t+1}}{k_t} \tag{B.15}$$

$$\chi l_t^{\nu+1} = (1-\alpha) p_t^m y_t \lambda_t^a \tag{B.16}$$

$$\lambda_t^a \upsilon_t = \widetilde{\beta}\lambda_{t+1}^a(\upsilon_{t+1} + w_{t+1}) \tag{B.17}$$

$$a_t + \delta_b \frac{b_{t-1}^h}{\pi_t} = \frac{R_{t-1}^m m_{t-1}}{\pi_t} + q_t^L s_t \tag{B.18}$$

$$\eta_t^z \geq 0, \quad a_t - c_t - i_t \geq 0, \quad \eta_t^z(a_t - c_t - i_t) = 0 \tag{B.19}$$

$$\eta_t^b \geq 0, \quad \overline{b^h} - b_t^h \geq 0, \quad \eta_t^b(\overline{b^h} - b_t^h) = 0 \tag{B.20}$$

Firms:

$$1 - \iota\left(\pi_t - 1\right)\pi_t + \iota\widetilde{\beta}\frac{\lambda_{t+1}^a}{\lambda_t^a}\left(\pi_{t+1} - 1\right)\pi_{t+1}\frac{y_{t+1}}{y_t} = (1 - p_t^m)\varepsilon \tag{B.21}$$

$$y_t = k_{t-1}^\alpha l_t^{1-\alpha} \tag{B.22}$$

$$w_t = (1 - p_t^m)y_t - \frac{\iota}{2}\left(\pi_t - 1\right)^2 y_t \tag{B.23}$$

Central bank:

$$R_t^f = \max\left\{\overline{R^f}\left(\frac{\pi_{t+1}}{\overline{\pi}}\right)^{\phi_\pi}, \quad \overline{R^n} + \varepsilon_f\right\} \tag{B.24}$$

$$R_t^n = \overline{R^n} \tag{B.25}$$

Markets Clear:

$$y_t = c_t + \widetilde{c}_t + i_t + \theta b_t^h + \frac{\iota}{2}\left(\pi_t - 1\right)^2 y_t \tag{B.26}$$

$$k_t = (1-\delta)k_{t-1} + i_t \tag{B.27}$$

$$\widetilde{x}_t = 1 \tag{B.28}$$

Shock:

$$\kappa_t = \rho_\kappa \kappa_{t-1} + (1-\rho_\kappa)\overline{\kappa} \tag{B.29}$$

B.2 Unconventional Monetary Policy: LSAP

In comparison to the conventional monetary policy, we have one more variable x_t - the number of wholesale firms' shares held by the central bank. In timing within one period, it is assumed that asset purchases happen after the credit market between bankers and households. Therefore, only banker's reserves flows (B.6) and deposits flows (B.7) are modified:

$$\frac{n_{t-1}}{\pi_t} + \upsilon_t(x_t - x_{t-1}) + \hat{\tau}_t = n_t$$
$$m_t = \frac{R^m_{t-1}m_{t-1}}{\pi_t} + q^L_t s_t + \theta_t b^h_t - \delta_b \frac{b^h_{t-1}}{\pi_t} + c_t + \upsilon_t(x_t - x_{t-1}) + \hat{\tau}_t - (R^n_{t-1} - 1)\frac{n_{t-1}}{\pi_t}$$

Equation (B.28) is replaced by:

$$x_t + \widetilde{x}_t = 1$$

The Taylor Rule (B.24) is replaced by:

$$\hat{\tau} = 0$$

One more equation for the evolution of x:

$$x_t = \rho_x x_{t-1}$$

C Numerical Method

C.1 Inequality Constraints

There are 5 occasionally binding inequality constraints in our model: the reserve requirement, the capital requirement, the ZMD-in-advance, the household's borrowing constraint and the Taylor rule of the central bank.

For the reserve requirement and the ZMD-in-advance, we apply the method in Zangwill and Garcia (1981) and Schmedders, Judd and Kubler (2002) to transform the inequality constraints into the equality constraints. Here is an example for the reserve requirement:

$$
\begin{aligned}
n_t - \varphi m_t &= \max\{-\underline{\mu}_t^r, 0\}^2 \\
\gamma_t &= \frac{\beta R_t^n \gamma_{t+1}}{\pi_{t+1}} + \mu_t^c + \max\{\underline{\mu}_t^r, 0\}^2
\end{aligned}
$$

For the capital requirement and the household's borrowing constraint, we apply the penalty method in McGrattan (1996) to avoid the ill-conditioned of the system and deal with occasionally binding constraints. So the utility of banker and the capital constraint will be changed as:

$$
\begin{aligned}
U &= \log c_t - \frac{\rho_e}{3} \max\{\underline{\mu}_t^c, 0\}^3 \\
n_t + b_t^f + (1-\kappa_t) b_t^h - m_t &= -\underline{\mu}_t^c
\end{aligned}
$$

where $\rho_e = 1000$ is the penalty coefficient. When the capital constraint is violated, banker will lose the utility. However, when they get positive net worth, they do not get reward for that. The household's utility also is changed to deal with the borrowing constraint.

For the Taylor rule of the central bank, we use the soft max constraint to deal with the lower bound on $R_{min}^f = \overline{R^n} + \varepsilon_f$ so we can still take derivative to solve the system of equations:

$$
\begin{aligned}
u_t &= \overline{R^f} \left(\frac{\pi_{t+1}}{\overline{\pi}}\right)^{\phi_\pi} \\
R_t^f &= \begin{cases} u_t + \frac{\log(1+\exp(s_{max}(R_{min}^f - u_t)))}{s_{max}}, & \text{if } u_t \geq R_{min}^f \\ R_{min}^f + \frac{\log(1+\exp(s_{max}(u_t - R_{min}^f)))}{s_{max}}, & \text{if } u_t < R_{min}^f \end{cases}
\end{aligned}
$$

When $s_{max} \to \infty$, the soft max constraint converges to the hard max constraint. We choose the coefficient $s_{max} = 10^4$.

C.2 Dynamics of Economy

We solve the perfect foresight equilibrium with the unexpected shock by assuming that after $T = 300$ quarters, the economy will converge back to the initial steady state. The initial position before the unexpected shocks is the steady state. Basically, we need to solve a large system of equations to determine the dynamic path of the economy. The transformation of the occasionally inequality constraints in the previous section ensures that every equation is continuous and differentiable.

For every application, we use homotopy method (by gradually increasing the size of shocks) for solving this large system of equations, with the initial point starting from the steady state or the previous result. We use Ipopt written by Wachter and Biegler (2006) with the linear solver HSL[10] to conduct homotopy.

[10]HSL. A collection of Fortran codes for large scale scientific computation. http://www.hsl.rl.ac.uk/

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