Economics
The Open-Access, Open-Assessment E-Journal

Discussion Paper
No. 2019-31 | April 30, 2019 | http://www.economics-ejournal.org/economics/discussionpapers/2019-31

Market runs of hedge funds during financial crises

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Abstract

A hedge fund's capital structure is fragile because uninformed fund investors are highly loss sensitive and easily withdraw capital in response to bad news. Hedge fund managers, sharing common investors and interacting with each other through market price, sensitively react to other funds' investment decisions. In this environment, panic-based market runs can arise not because of systematic risk but because of the fear of runs. The authors find that when the market regime changes from a normal state to a "bad" state (in which runs are possible), hedge funds reduce investment prior to runs. In addition, the market runs are more likely to occur in a market where hedge funds hold greater market exposure and uninformed traders have greater sensitivity to past price movement.

JEL G01 G23
Keywords Market sustainability; market runs; hedge funds; limits of arbitrage; financial crises; synchronization risk

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Citation Sangwook Sung, Hoon Cho, and Doojin Ryu (2019). Market runs of hedge funds during financial crises. Economics Discussion Papers, No 2019-31, Kiel Institute for the World Economy.
http://www.economics-ejournal.org/economics/discussionpapers/2019-31

Received March 10, 2019 **Accepted as Economics Discussion Paper** April 29, 2019
Published April 30, 2019

1. Introduction

Around the global financial crisis of 2007–2009, hedge funds, characterized as informed and sophisticated arbitrageurs, simultaneously exited the market after the financial sector experienced an exceptionally large shock.[1] Ben-David et al. (2011, p.2), extensively examining research on hedge funds around the global financial crisis, state that "hedge funds exited the U.S. stock market en masse as the financial crisis evolved, primarily in response to the tightening of funding by investors and lenders." According to their empirical evidence, hedge funds reduced equity holdings by 6% each quarter during the Quant Meltdown of 2007 and by 15% on average (29% with compounding) during Lehman Brothers' bankruptcy of 2008 (see Figure 1).[2] He et al. (2010) find a similar result, that during the crisis (2007:Q4–2009:Q1) the amount of securitized assets owned by hedge funds was reduced by $800 billion. Moreover, Ang et al. (2011), investigating the leverage of hedge funds from December 2004 to October 2009, document that hedge funds decreased their leverage even before the financial crisis started in mid-2007. As major motivation for this massive asset liquidation, Ben-David et al. (2011) point out fund withdrawals that explain half of the decline in the equity holdings of hedge funds. In addition, by analyzing the types of stocks hedge funds sold off during the crisis, the authors show that high-volatility stocks were more likely to experience fire sales than low-volatility stocks were.

[Insert Figure 1 about here]

Regarding these issues, a large body of literature on the limits of arbitrage proposes theoretical explanations for the price divergence caused by the asset fire sales of constrained arbitrageurs. In pioneering work, Shleifer and Vishny (1997) argue that arbitrageurs could abandon an arbitrage strategy due to a performance-based compensation structure. If uninformed investors reward and punish in accordance with short-term fund performance, fund managers become myopic, leaving arbitrage opportunities in the market unexploited. Gromb and Vanyanos (2002), Liu and Longstaff (2004), and Brunnermeier and Pedersen (2009) provide a model in which arbitrageurs cannot pursue arbitrage profits because of margin constraints on collateralized assets. When arbitrageurs face constraints on collateral, they are limited in their arbitrage positions. Liu and Mello (2011) link the limits of arbitrage to liquidity risk. They conclude that fund managers who are subject to liquidity risk reduce their portion of risky assets in preparation of fund runs.

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[1] According to Hedge Fund Research Inc., 1,471 of hedge funds were liquidated in 2008, a historical high, and the next year, 1,023 hedge funds closed, a second historical high.

[2] Ben-David et al. (2011) report that a quarter of hedge funds sold off over 40% of their equity holdings during 2008:Q3–Q4.

We develop a market model that explains synchronized market runs by informed and rational fund managers. The market is subject to negative price shocks and funding liquidity shocks and uninformed fund investors who invest in funds prudently can request early withdrawal based on their own market views, created by aggregate numbers of managers who gave up their market investment. In this respect, these investors can be viewed as similar to the types of investors described by Shleifer and Vishny (1997), who determine investment decisions depending on past information about funds. In this model framework, we show that low levels of noise in a private signal about liquidity shock can lead to panic-based market runs, not because of liquidity risk per se but because of the fear of runs.

The possibility of synchronized runs in various financial sectors has historically been of great concern to market participants, policy makers, and researchers[3] and recent studies (Morris and Shin (2004), Goldstein and Pauzner (2005), Liu and Mello (2011), and Allen et al. (2014)) attempt to derive a unique equilibrium threshold of panic-based runs using a global game method. This approach allows for a unique threshold strategy in which runs may or may not occur, depending on economic conditions, and therefore calculates the probability of synchronized runs. However, most of the studies assume fixed short-term asset returns that are unrelated to agents' investment decisions,[4] so they fail to explain market crashes caused by investors who collectively liquidate risky assets in fear of price drops due to the liquidation of other investors.

In our model, short- and long-run market returns are endogenously determined by the optimal investment strategies of fund managers, who need to make two optimal decisions: asset allocation and whether to stay. The optimal decision rules of fund managers are simple: At t_1, managers decide the optimal asset allocation to maximize final investment returns and, at t_2, they optimally choose whether to stay in the market, depending on observed private signals. Fund managers interact with market returns so that, as they invest more capital, the market return rises and all managers can benefit and vice versa. When a market is in a "bad" state (in which runs are possible), fund managers can also interact with each other by deciding whether to stay, because investors request capital withdrawals in proportion to the number of exiting funds. Therefore, the investment decision of each fund affects the others' investment returns and obviously affects optimal decisions.

Fund managers should then take into account not only market conditions but also runs by other managers in deciding their own optimal strategy. As more fund managers decide to exit the market, the expected investment returns of the remaining funds diminish faster than their exit

[3] Diamond and Dybvig (1983), Postlewaite and Vives (1987), Green and Lin (2003), Peck and Shell (2003), Goldstein and Pauzner (2005), Ennis and Keister (2009a, 2009b), and Allen et al. (2014) discuss bank runs, Bernardo and Welch (2004) and Morris and Shin (2004) study market runs, and Liu and Mello (2011) investigate fund runs.

[4] Morris and Shin (2004) provide a model in which investors' decisions on whether to stay can affect the market price. However, those investors are not subject to early withdrawals, which is the core driver of market runs in our model; instead, they are constrained by loss limits.

investment returns, which could encourage these funds to run. We consider the situation in which all funds decide to exit due to the fear of runs, even if economic conditions are not that bad, that is, a panic-based market run. Using a global game method, we show that a unique threshold strategy exists in which, if the liquidity shock is below the threshold, no managers run but, otherwise, all managers run. Having established a unique threshold strategy, we compare the optimal levels of market exposure in two states, state I, in which runs are impossible, and state II, in which runs are possible. By comparing the two states, we find that when fund managers consider the possibility of runs, they optimally hold less market exposure. In addition, we calculate the ex ante probability of market runs, which is related to the distribution of funding liquidity shocks, by summing the probabilities over the possible ranges of market runs. Through this analysis, we discover that the likelihood of runs rises as both the market exposure of funds and the price sensitivity of trend followers increase.

Our findings help to explain some stylized facts on the behaviors of fund managers that were observed around the global financial crisis. Hedge funds, whose investors are highly sensitive to losses, are vulnerable to funding liquidity risk and thus fund managers are cautious, holding more cash prior to crises. Nevertheless, if an exceptionally devastating liquidity shock sweeps the market, fund investors may start to request capital withdrawals from their funds in response to the initial loss. Even if some fund managers know that the liquidity shock is not strong enough to make them exit the market, the fear of capital withdrawals and price deterioration due to runs by other funds induces them to run as well. In the worst case, hedge funds collectively exit the market, not because of risk itself, but because of fear, which explains the stock market exodus of hedge funds during the Quant Meltdown of 2007 and Lehman Brothers' bankruptcy of 2008. In the meantime, high-volatility stocks are more likely to experience greater fire sales by hedge funds than low-volatility stocks are, because high-volatility stocks respond sensitively to price movement and during a market downturn are more likely to experience price drops. In this sense, high-volatility shocks are riskier and are more likely to deteriorate fund asset values than low-volatility stocks are. Hence, high-volatility stocks have a high chance of suffering from synchronized runs by hedge funds.

Our study is related to the literature on panic-based crises using the global game technique developed by Carlsson and van Damme (1993). Goldstein and Pauzner (2004, p.152) define panic-based crises as "crises that occur just because agents believe they are going to occur," which is the common feature of most crises in several parts of the financial sector. Global game methods are useful in resolving many of the equilibrium problems of panic-based crises. In recent decades, several attempts using a global game method have been made to explain panic-based crises in financial sectors. Morris and Shin (1998) were among the first and describe the currency protection of the government against the currency attacks of speculators. Goldstein and Pauzner (2005) modify the bank run model of Diamond and Dybvig (1983) to show the existence of a unique bank run equilibrium and then calculate the probability of bank runs. Extending Goldstein and Pauzner's model, Allen et al. (2014) investigate the financial effects of government guarantees on the banking system. Abreu and Brunnermeier (2002, 2003) reveal long-lasting unexploited arbitrage opportunities in asset markets due to the synchronization failure of arbitrageurs. Goldstein and Pauzner (2004) explain crisis contagion between two financial markets with common investors as being due to the fear of a crisis, which can make investors withdraw capital from both markets.

The study of Liu and Mello (2011), who highlight withdrawal synchronization as the major driver of the massive asset liquidation by hedge funds during the global financial crisis, is closely related to ours. However, there are critical differences. First, Liu and Mello assume informed and sophisticated fund investors; however, according to the empirical evidence of Ben-David et al. (2011), during the global financial crisis, hedge fund investors very actively withdrew their capital from poorly performing funds, which implies that investors determine their investment decisions depending on the past information of funds rather than information regarding their future. Accordingly, our model assumes fund investors make investment decisions after observing the aggregate investment decisions of fund managers. Second, Liu and Mello consider an isolated fund that cannot interact with the market or other funds and focus on fund-specific rather than economy-wide characteristics. Therefore, the linkage between fund runs and price deterioration is weak and the influence of fund managers' decisions on the financial market is inferred indirectly. In contrast, we develop a market model in which fund managers interact with other funds through the market price. This approach allows us to identify a direct linkage between price deterioration and its impact on the decisions of fund managers.

The remainder of this article is organized as follows. Section 2 describes the model and Section 3 finds the equilibrium for state I (in which runs are impossible) and state II (in which runs are possible). Section 4 explores the effect of environmental changes on the decisions of fund managers. Section 5 summarizes and concludes the article.

2. Model

We consider a model with two assets, a risky asset and cash, during four periods (t_0, t_1, t_2, t_3). The risky asset market is subject to a negative price shock and three types of agents participate in the market: fund managers, trend followers, and fund investors.

Agents of the first type, or fund managers, are homogeneous and fully rational, with risk-neutral utility, and know the fundamental value, which is higher than the current price. Fund managers trade the risky asset to exploit arbitrage opportunities but funding liquidity risk prevents them from making maximum arbitrage profits. Since there exist infinitely many fund managers[5] and they compete with each other in the market, individual fund managers, in equilibrium, optimally allocate their own capital to the risky asset and to cash to maximize their final asset value in response to competitors' investment strategies.

Trend followers, the second type of agent, have unlimited capital but have no information on the fundamental values of risky assets, so they just follow market trends and their aggregate demand is therefore positively related to past market returns.

Fund investors, the last type of agent, are also uninformed but, unlike trend followers, do not directly invest in risky assets because these markets are highly specialized and they are too prudent to invest by themselves. Instead, fund investors give their capital to fund managers, distributing the capital among all funds equally to reduce risk, and observe the aggregate funds' investment decisions. Based on ex post aggregate information about funds' decisions, fund investors then decide the amount of capital withdrawal in proportion to the aggregate portion of exiting funds.

The brief time schedule of our model is as follows. At t_0, the market price is at the fundamental value and then diverges due to a negative price shock at t_1. At t_1, fund managers also receive capital from investors and determine their investment strategy based on market conditions. A funding shock occurs between t_1 and t_2 and, at t_2, each fund manager decides whether to stay in the market. Then, fund investors withdraw some of their capital from the remaining funds after they observe the aggregate portion of exiting funds. At t_3, trend followers finally realize the fundamental value and the market price converges to it. We provide more details below.

At t_1, fund investors provide aggregate capital f distributed equally among all funds. Fund i then allocates a portion x_i of capital to the risky asset and the remaining portion c_i to cash, where $x_i \in [0, 1]$. Cash pays a return of one unit, but the payoff of the risky asset is uncertain, in that a funding liquidity shock could deteriorate short-term market returns. After fund managers execute their investment strategies, the market price rises by the amount of aggregate capital they provided, which is $\frac{f}{n}\sum_{i=1}^{n} x_i$. For convenience, we normalize the market supply of t_1 to unity and express exogenous market factors such as market value, capital amounts, and price shocks relative to the price at t_1. In addition, without loss of generality, we assume that the market price at t_1 is one. Then, the market clearing condition at t_1 is

$$1 = R - s + \frac{f}{n}\sum_{i=1}^{n} x_i \tag{1}$$

[5] We assume a perfectly competitive market, that is, one with infinitely many funds. However, for ease of presentation, in this section, we describe our model as if there were n funds in the market. Later, n goes to infinity.

where s is the impact of the negative market shock on the market price. The market shock drops the market return by s below the fundamental return R, which is greater than one. The fundamental return is then determined to be

$$R = 1 + s - \frac{f}{n}\sum_{i=1}^{n} x_i \tag{2}$$

As we can see from Equation (2), an individual fund manager's investment decision affects everyone else's investment returns through the market return.

Between t_1 and t_2, trend followers trade in response to past market returns; that is, since the market return of the risky asset changes from R to one between t_0 and t_1, the market return between t_1 and t_2 again declines to $\frac{1}{R}/\tau$ due to the aggregate trading of trend followers, where $\tau \in [1/R, 1)$. The constant τ captures trend followers' sensitivity to past market trends and the boundaries on τ restrict the impact of trend followers on the market such that $\frac{1}{R} < \frac{1}{R}/\tau \leq 1$. The first inequality, $\frac{1}{R} < \frac{1}{R}/\tau$, means that the effect of trend followers' trading does not dominate the market so that the short-term market return does not fall below $1/R$. The second inequality, $\frac{1}{R}/\tau \leq 1$, limits the short-term return as one, to focus on only a distressed market.

In the meanwhile, a funding liquidity shock occurs so that a portion θ of funds are forced to go bankrupt, where θ is uniformly distributed on $[0,1]$. Fund managers who receive this shock should go bankrupt and they make no profits. Surviving fund managers, however, do not know the exact value of θ because fund managers cannot observe the status of other funds but, instead, receive noisy signals, such that the signal of a surviving manager of fund i is $\theta_i = \theta + \varepsilon_i$, where ε_i is an independent and identically uniform distribution on the interval $[-\varepsilon, \varepsilon]$ and ε is an arbitrarily small real number. Based on this signal, the manager of fund i forecasts the strategies of other managers and decides whether to stay in the market.

In the meantime, fund investors do not receive any signals. The only thing they can observe is the ex post aggregate portion λ of funds that exit the market, including both bankrupt funds and leaving funds among survivors, but they do not know which funds exit or stay. Therefore, they create their own market view based on only this portion λ and then request the withdrawal of a portion λ of capital from the remaining funds in the market because they conjecture that a higher λ means a riskier market situation. In the extreme case, if all fund managers exit the market and are just holding cash until t_3, investors forfeit all capital from funds at t_2 because holding cash requires no expertise. In this extreme case, fund managers make no profits.

Note that fund investors' cautious reaction to fund exits due to information asymmetry between fund managers and investors can stimulate runs of the remaining funds. Even a fund manager who knows that the funding liquidity shock is at a safe level can choose not to stay if he or she thinks that other managers will leave the market, because the exit of other funds not only induces withdrawals but also lowers market prices, deteriorating the investment return of the remaining funds. If the return deterioration seems severe enough, fund managers will decide to leave the market. After all, regardless of market fundamentals, market runs can be triggered by the fear of runs, which we call panic-based market runs.

Then, at t_2, the short-term market return r is determined as a function of the aggregate portion of exiting funds, as follows:

$$r(\lambda) = \begin{cases} \frac{1}{R\tau} - \frac{f}{n}\sum_{k=n-m+1}^{n} x_k + LI(\lambda) & 0 \le \lambda \le c_1 \\ \frac{1}{R\tau} - \frac{f}{n}\left((\lambda - c_1) + \sum_{k=n-m+1}^{n} x_k\right) + LI(\lambda) & c_1 \le \lambda \le c_2 \\ \vdots & \\ \frac{1}{R\tau} - \frac{f}{n}\left(\sum_{k=1}^{i}(\lambda - c_k) + \sum_{k=n-m+1}^{n} x_k\right) + LI(\lambda) & c_i \le \lambda \le c_{i+1} \\ \vdots & \\ \frac{1}{R\tau} - \frac{f}{n}\left(\sum_{k=1}^{n-m}(\lambda - c_k) + \sum_{k=n-m+1}^{n} x_k\right) + LI(\lambda) & c_{n-m} \le \lambda \le \pi_1 \\ \frac{1}{R\tau} - \frac{f}{n}\left(\sum_{k=2}^{n-m}(\lambda - c_k) + r x_1 + \sum_{k=n-m+1}^{n} x_k\right) + LI(\lambda) & \pi_1 \le \lambda \le \pi_2 \\ \vdots & \\ \frac{1}{R\tau} - \frac{f}{n}\left(\sum_{k=i+1}^{n-m}(\lambda - c_k) + r\sum_{k=1}^{i} x_k + \sum_{k=n-m+1}^{n} x_k\right) + LI(\lambda) & \pi_i \le \lambda \le \pi_{i+1} \\ \vdots & \\ \frac{1}{R\tau} - \frac{f}{n}\left(r\sum_{k=1}^{n-m} x_k + \sum_{k=n-m+1}^{n} x_k\right) + LI(\lambda) & \pi_{n-m} \le \lambda \le 1 \end{cases} \tag{3}$$

and π_i is the minimum value of fund i's short-term investment return before capital outflows.

For the sake of convenience, we index n funds such that the m highest are exiting funds, that is, $\lambda = \frac{m}{n}$, and the remaining $n - m$ lower-indexed funds are the remaining funds.[6] We also index the remaining funds in order of risky asset holdings, such that $c_1 \le \cdots \le c_{n-m} \le \pi_1 \le \cdots \le \pi_{n-m}$.[7] The term $LI(\lambda)$ represents new liquidity inflow into the risky asset market at t_2. To solve the model analytically, we assume that $LI(\lambda) = \frac{f}{n}\sum_{k=n-m+1}^{n} x_k$; however, this assumption does not change the implications of the model because the negative relation between liquidity outflow and market return is maintained. The short-term investment return before the capital outflow of fund i is $\pi_i(\lambda) = x_i r(\lambda) + c_i$ and, if $\pi_i(\lambda) < \lambda$, fund i should declare default. Therefore, π_i is the upper bound of θ at which fund i need not declare default and also the minimum value of $\pi_i(\lambda)$, defined as $\pi_i \coloneqq \pi_i(\pi_i) = x_i r(\pi_i) + c_i$. Then, $\pi_i = \frac{c_i\left(1 + \frac{f}{n}\sum_{k=1}^{i} x_k\right) + x_i\left(\frac{1}{R\tau} + \frac{f}{n}\sum_{k=i+1}^{n-m} c_k\right)}{1 + \frac{f}{n}\sum_{k=1}^{i} x_k + \frac{f x_i}{n}(n-m-i)}$.

[6] It seems that λ is a discrete variable, but it is continuous because n later goes to infinity.

[7] We assume that $c_{n-m} \le \pi_1$, because, in a non-cooperative game, all homogeneous agents optimally choose an identical strategy in equilibrium and a fund's cash holdings are always less than the investment return before capital outflows.

If λ is in the interval $[c_i, c_{i+1}]$ of Equation (3), capital outflows are greater than fund i's cash holdings, so the first i funds whose cash holdings are less than or equal to those of funds i are forced to liquidate some of their risky assets. Due to their liquidation, the short-term market return drops by $\frac{f}{n}\sum_{k=1}^{i}(\lambda - c_k)$. When λ is in the interval $[\pi_i, \pi_{i+1}]$, all remaining funds liquidate at least some of their risky assets and the first i funds liquidate all assets and declare default.

At t_3, trend followers realize the fundamental value and the market converges to it. All funds completely liquidate their risky asset and realize profits.

When the manager of fund i decides to exit the market, the fund can achieve an exit investment return Π_i^E such that

$$\Pi_i^E(x_i, \lambda) = \begin{cases} x_i r(\lambda) + c_i & 0 \le \lambda < \pi_i \\ 0 & \pi_i \le \lambda \le 1 \end{cases} \tag{4}$$

In Equation (4), Π_i^E is the short-term investment return before capital outflows $\pi_i(\lambda)$ if capital outflows are less than π_i. In the other range, even if the manager of fund i decides to liquidate early, the manager gains nothing.

When the manager of fund i decides to stay in the market, the investment return Π_i^S of staying for fund i is

$$\Pi_i^S(x_i, \lambda) = \begin{cases} x_i R + c_i - \lambda & 0 \le \lambda \le c_i \\ x_i R - (\lambda - c_i)\frac{R}{r(\lambda)} & c_i \le \lambda \le \pi_i \\ 0 & \pi_i \le \lambda \le 1 \end{cases} \tag{5}$$

In Equation (5), if λ is less than c_i, fund i can cover all fund outflows with cash at t_2 and obtain an investment yield $x_i R$ in the risky asset at t_3. If λ is greater than c_i, the fund should liquidate as much risky asset as the shortfall amount at t_2 and thus suffer a loss of $(\lambda - c_i)\frac{R}{r}$ at t_3. If λ is large enough to exceed π_i, fund i declares default at t_2.

3. Equilibrium

The optimal strategy of risk-neutral fund managers is to maximize their expected final return and, since homogeneous fund managers, who know the optimal strategies of their peers, compete with each other in the market, all managers select the identical asset allocation strategy at equilibrium.[8] Using this identity condition, we can simplify the derivation of the optimal strategy.

In equilibrium, it is possible to assume that $x \coloneqq x_1 = x_2 = \cdots$ and then the long- and short-term market returns and the investment returns of exiting and remaining funds in Equations (2) to (5), respectively, can be rewritten as

[8] This is a Nash equilibrium.

$$\bar{R} \coloneqq \lim_{n\to\infty}[R]_{x=x_1=\cdots=x_n} = 1 + s - fx \quad (6)$$

$$\bar{r}(\lambda) \coloneqq \lim_{n\to\infty}[r]_{x=x_1=\cdots=x_n} = \begin{cases} \frac{1}{\bar{R}\tau} & 0 \le \lambda \le c \\ \frac{1}{\bar{R}\tau} - f(\lambda - c) & c \le \lambda \le \pi \\ \frac{1}{\bar{R}\tau(1+fx)} & \pi \le \lambda \le 1 \end{cases} \quad (7)$$

$$\overline{\Pi^E}(x,\lambda) \coloneqq \lim_{n\to\infty}\left[\Pi_i^E\right]_{x=x_1=\cdots=x_n} = \begin{cases} \frac{x}{\bar{R}\tau} + c & 0 \le \lambda \le c \\ \frac{x}{\bar{R}\tau} - fx(\lambda - c) + c & c \le \lambda < \pi \\ 0 & \pi \le \lambda \le 1 \end{cases} \quad (8)$$

$$\overline{\Pi^S}(x,\lambda) \coloneqq \lim_{n\to\infty}\left[\Pi_i^S\right]_{x=x_1=\cdots=x_n} = \begin{cases} x\bar{R} + c - \lambda & 0 \le \lambda \le c \\ x\bar{R} - (\lambda - c)\bar{R}/\left(\frac{1}{\bar{R}\tau} - f(\lambda - c)\right) & c \le \lambda \le \pi \\ 0 & \pi \le \lambda \le 1 \end{cases} \quad (9)$$

where $\pi = \frac{x}{\bar{R}\tau(1+fx)} + c$. In addition, by differentiating Π_i^E and Π_i^S with respect to x_i, we obtain the following equations:

$$\frac{\partial\overline{\Pi^E}}{\partial x}(x,\lambda) \coloneqq \lim_{n\to\infty}\left[\frac{\partial\Pi_i^E}{\partial x_i}\right]_{x=x_1=\cdots=x_n} = \begin{cases} \frac{1}{\bar{R}\tau} - 1 & 0 \le \lambda \le c \\ \frac{1}{\bar{R}\tau} - f(\lambda - c) - 1 & c \le \lambda < \pi \\ 0 & \pi \le \lambda \le 1 \end{cases} \quad (10)$$

$$\frac{\partial\overline{\Pi^S}}{\partial x}(x,\lambda) \coloneqq \lim_{n\to\infty}\left[\frac{\partial\Pi_i^S}{\partial x_i}\right]_{x=x_1=\cdots=x_n} = \begin{cases} \bar{R} - 1 & 0 \le \lambda \le c \\ \bar{R} - \bar{R}/\left(\frac{1}{\bar{R}\tau} - f(\lambda - c)\right) & c \le \lambda \le \pi \\ 0 & \pi \le \lambda \le 1 \end{cases} \quad (11)$$

3.1. State I: Runs are impossible

As in the benchmark case, we first consider an equilibrium when surviving fund managers cannot exit the market. Then, the only funds that exit the market are those that receive a liquidity shock. Therefore, λ equals θ. Under this condition, panic-based market runs are not possible because no surviving fund manager fears the runs of the other survivors. Therefore, the investment return of the surviving funds is the investment return Π_i^S of staying, while bankrupt funds gain nothing. Therefore, at t_1, the expected final return of fund i, denoted $E\left[\Pi_i^I\right]$, is

$$E\left[\Pi_i^I\right] = \int_0^1 \left[(1-\theta)E\left[\Pi_i^S\right]\right] d\theta = E\left[\Pi_i^S\right]\int_0^1 (1-\theta)\, d\theta = \frac{1}{2}E\left[\Pi_i^S\right] = \frac{1}{2}\int_0^{\pi_i(x_i)} \Pi_i^S \, d\theta \quad (12)$$

and the manager of fund i chooses optimal market exposure to maximize $E\left[\Pi_i^I\right]$, that is,

$$\max_{x_i} E\left[\Pi_i^I\right] \quad (13)$$

under the boundary conditions

$$0 \le x_i \le 1,\ 1/\tau \le R,\ f < s \quad (14)$$

The first boundary condition indicates short-selling and borrowing constraints on the investment. The second one is imposed by the restriction on τ and guarantees the existence of an arbitrage opportunity, because $1 < 1/\tau \leq R$ is a sufficient condition for $1 < R$. The third boundary condition excludes the possibility that a market shock can be canceled out only by a fund's own investment.

At equilibrium, all fund managers choose the same optimal point, which should satisfy the equation

$$\frac{\partial E[\Pi^I]}{\partial x} \coloneqq \lim_{n\to\infty}\left[\frac{\partial E[\Pi_i^I]}{\partial x_i}\right]_{x=x_1=\cdots=x_n} = \frac{1}{2}\lim_{n\to\infty}\left[\frac{\partial}{\partial x_i}\left(\int_0^{\pi_i(x_i)} \Pi_i^S(x_i,\theta)\,d\theta\right)\right]_{x=x_1=\cdots=x_n} = 0 \qquad (15)$$

Since $\Pi_i^S(x_i,\theta)$ is continuous over $[0,1]\times[0,1]$ and its partial derivative $\frac{\partial}{\partial x_i}\Pi_i^S(x_i,\theta)$ is continuous over $[0,1]\times[0,c_i]$ and $[0,1]\times[c_i,\pi_i]$, by the Leibniz integral rule,[9] $\frac{\partial}{\partial x_i}\left(\int_0^{\pi_i(x_i)} \Pi_i^S(x_i,\theta)\,d\theta\right)$ in Equation (15) can be expressed as

$$\begin{aligned}&\frac{\partial}{\partial x_i}\left(\int_0^{\pi_i(x_i)} \Pi_i^S(x_i,\theta)\,d\theta\right) = \frac{\partial}{\partial x_i}\left(\int_0^{c_i(x_i)} \Pi_i^S(x_i,\theta)\,d\theta\right) + \frac{\partial}{\partial x_i}\left(\int_{c_i(x_i)}^{\pi_i(x_i)} \Pi_i^S(x_i,\theta)\,d\theta\right) = \\ &\left[\int_0^{c_i(x_i)} \frac{\partial}{\partial x_i}\Pi_i^S(x_i,\theta)\,d\theta + \Pi_i^S\big(x_i,c_i(x_i)\big)\frac{\partial c_i}{\partial x_i} - \Pi_i^S(x_i,0)\frac{\partial 0}{\partial x_i}\right] + \left[\int_{c_i(x_i)}^{\pi_i(x_i)} \frac{\partial}{\partial x_i}\Pi_i^S(x_i,\theta)\,d\theta + \right. \\ &\left. \Pi_i^S\big(x_i,\pi_i(x_i)\big)\frac{\partial \pi_i}{\partial x_i} - \Pi_i^S\big(x_i,c_i(x_i)\big)\frac{\partial c_i}{\partial x_i}\right]\end{aligned} \qquad (16)$$

Then,

$$\begin{aligned}&\lim_{n\to\infty}\left[\frac{\partial}{\partial x_i}\left(\int_0^{\pi_i(x_i)} \Pi_i^S(x_i,\theta)\,d\theta\right)\right]_{x=x_1=\cdots=x_n} \\ &= \begin{bmatrix}\int_0^c \frac{\partial\overline{\Pi^S}}{\partial x}(x,\theta)\,d\theta \\ +\overline{\Pi^S}(x,c)\frac{\partial c}{\partial x} - \overline{\Pi^S}(x,0)\frac{\partial 0}{\partial x}\end{bmatrix} + \begin{bmatrix}\int_c^\pi \frac{\partial\overline{\Pi^S}}{\partial x}(x,\theta)\,d\theta \\ +\overline{\Pi^S}(x,\pi)\frac{\partial \pi}{\partial x} - \overline{\Pi^S}(x,c)\frac{\partial c}{\partial x}\end{bmatrix} = \int_0^\pi \frac{\partial\overline{\Pi^S}}{\partial x}(x,\theta)\,d\theta\end{aligned} \qquad (17)$$

Therefore, $\frac{\partial E[\Pi^I]}{\partial x}$ in Equation (15) can be easily calculated by integrating $\frac{\overline{\partial\Pi^S}}{\partial x}$ with respect to θ and the optimal solution x^{I*} can be found with the equation

$$\begin{aligned}\frac{\partial E[\Pi^I]}{\partial x} &= \frac{1}{2}\int_0^\pi \frac{\partial\overline{\Pi^S}}{\partial x}\,d\theta = \frac{1}{2}\left[\int_0^c (\bar{R}-1)\,d\theta + \int_c^\pi\left(\bar{R} - \frac{\bar{R}}{\frac{1}{\bar{R}\tau} - f(\theta - c)}\right)d\theta\right] \\ &= \frac{1}{2}\left[fx^2 - (f+s)x + s + \frac{x}{(1+fx)\tau} - \frac{R}{f}\ln(1+fx)\right] = 0\end{aligned} \qquad (18)$$

[9] If a function $g(y,z)$ and its partial derivative $g_y(y,z)$ are both continuous over $[y_0,y_1]\times[z_0,z_1]$, then $\frac{\partial}{\partial y}\left(\int_{z_0}^{z_1} g(y,z)\,dz\right) = \int_{z_0}^{z_1} g_y(y,z)\,dz + g(y,z_1)\frac{\partial z_1}{\partial y} - g(y,z_0)\frac{\partial z_0}{\partial y}$.

PROPOSITION 1. *If surviving fund managers cannot exit the market, a panic-based market run cannot occur and, in equilibrium, fund managers select the optimal market exposure* x^{I*} *that satisfies* $\frac{1}{2}\left[f{x^{I*}}^2-(f+s)x^{I*}+s+\frac{x^{I*}}{(1+fx^{I*})\tau}-\frac{1+s-fx^{I*}}{f}\ln(1+fx^{I*})\right]=0.$

Following Proposition 1, at t_1, all fund managers decide to invest a portion x^{I*} of their capital in the market and to hold the remaining portion $c^{I*}(=1-x^{I*})$ in cash. Then, the aggregate liquidity inflows from funds to the market are fx^{I*} and, accordingly, the long-run market return is determined to be $R^{I*}=1+s-fx^{I*}$.

At t_2, after a portion $\theta\ (=\lambda)$ of funds exit the market due to the liquidity shock, the short-term market return becomes

$$r^{I*}(\theta)=\begin{cases}\frac{1}{R^{I*}\tau} & 0\le\theta\le c^{I*}\\ \frac{1}{R^{I*}\tau}-f(\theta-c) & c^{I*}\le\theta\le\pi^{I*}\\ \frac{1}{R^{I*}\tau}-f\bar{r}x & \pi^{I*}\le\theta\le 1\end{cases}\tag{19}$$

where $\pi^{I*}=\frac{x^{I*}}{R^{I*}\tau(1+fx^{I*})}+c^{I*}$. In addition, the surviving funds can achieve an investment return Π^{I*} at t_3:

$$\Pi^{I*}(x^{I*},\theta)=\begin{cases}x^{I*}R^{I*}+c^{I*}-\theta & 0\le\theta\le c^{I*}\\ x^{I*}R^{I*}-(\theta-c^{I*})\frac{R^{I*}}{r^{I*}} & c^{I*}\le\theta<\pi^{I*}\\ 0 & \pi^{I*}\le\theta\le 1\end{cases}\tag{20}$$

3.2. State II: Run are possible

We now consider the case in which surviving funds can exit if it seems that the investment return of exiting the market is greater than the investment return of staying. In this case, the portion of exiting funds equals or is greater than the portion of bankrupt funds, so that $\lambda\ge\theta$.

In this section, we show how panic-based market runs can occur in a financial market as a result of the optimal decisions of rational fund managers. For this, we use a global game method.

For technical reasons, assume two dominance regions, where fund managers who believe θ is in one of these regions follow a dominance strategy, regardless of what other managers do. In detail, managers who believe that θ is lower than $\underline{\theta}$ choose to stay in the market. Here, $\underline{\theta}>2\varepsilon$ and $[0,\underline{\theta})$ is the lower dominance region. To guarantee the existence of this region, the fund manager's payoff structure requires a little adjustment. Suppose that, if fund managers stay when θ is in $[0,\underline{\theta})$, they can obtain a investment return $x\bar{R}+c$ instead of $\overline{\Pi^S}$. Then, whatever the others choose, these managers, who believe that θ is in $[0,\underline{\theta})$, would decide to stay, because their investment return, $x\bar{R}+c$, is independent of λ and greater than $\overline{\Pi^E}$. This result implies that when the liquidity shock is extremely weak, fund managers can be sure to obtain higher investment profits by maintaining their current market position, regardless of the behavior of other managers.

At the other extreme, if fund managers believe that θ is greater than $\overline{\theta}$, they will decide to exit the market, with $\overline{\theta} < \pi - 2\varepsilon$. Then, $(\overline{\theta}, 1]$ is the upper dominance region. First consider the clear case in which θ is in $[\pi, 1]$. Then, all funds should definitely exit the market because funds that stay will be in default. As in the second case, when θ is in the other region, $(\underline{\theta}, \pi)$, surviving funds decide whether to stay, depending on the situation. Here, fund managers compare investment returns $\overline{\Pi^E}$ and $\overline{\Pi^S}$ for each decision and the existence of a unique value λ in $[c, \pi]$ needs to be demonstrated for $\overline{\Pi^E}$ and $\overline{\Pi^S}$ to be equal. From Equations (8) and (9), we know $\overline{\Pi^E}(x, c) < \overline{\Pi^S}(x, c)$, $\lim_{\lambda \to \pi - 0} \overline{\Pi^E}(x, \lambda) > \overline{\Pi^S}(x, \pi) = 0$, and $\overline{\Pi^E}$ and $\overline{\Pi^S}$ decrease monotonically on λ. Then, by the intermediate value theorem, a unique λ that satisfies $\overline{\Pi^E} = \overline{\Pi^S}$ exists. If $\overline{\theta}$ is defined as this value of λ, fund managers know that, in the region $(\underline{\theta}, \pi)$, exiting the market is more profitable than staying, regardless of what other managers do.

Now, to determine a unique equilibrium strategy in the lower dominance region, consider a manager of fund i who receives a signal θ_i that is below $\underline{\theta} - \varepsilon$. This manager knows that θ is in the lower dominance region and decides to stay in the market. From this, we are assured that if θ is below $\underline{\theta} - 2\varepsilon$, all surviving fund managers receive signals in $[0, \underline{\theta} - \varepsilon)$; then no surviving manager will choose to exit. Therefore, $\lambda = \theta$ when $\theta < \underline{\theta} - 2\varepsilon$, which means that in this region there is only one equilibrium strategy. In region $[\underline{\theta} - 2\varepsilon, \underline{\theta}]$, as θ approaches $\underline{\theta}$, the portion of managers who receive signals below $\underline{\theta} - \varepsilon$ diminishes monotonically and becomes zero at $\theta = \underline{\theta}$.

Similarly, we can also demonstrate the existence of a unique equilibrium strategy in the upper dominance region; that is, fund manager i chooses to exit if $\theta_i > \overline{\theta} + \varepsilon$, so $\lambda = 1$ when $\theta > \overline{\theta} + 2\varepsilon$.

What would happen in the intermediate region, between the two dominance regions? A fund manager who receives a signal that is slightly higher than $\underline{\theta} - \varepsilon$ thinks that θ is more likely to be in the lower dominance region and could decide to stay. However, as the private signal approaches $\underline{\theta}$, the probability that θ is in $[0, \underline{\theta})$ decreases to zero. At $\theta_i = \underline{\theta}$, the manager of fund i knows that no manager receives a signal below $\underline{\theta} - \varepsilon$ and no managers can be sure that θ is in $[0, \underline{\theta})$. Nevertheless, the manager of fund i might decide to stay because he or she believes that other managers whose signals are lower than $\underline{\theta}$ might choose to stay. The next manager, whose signal is slightly higher than that of the manager of fund i, also believes that other managers may choose to stay and decides to stay as well. By applying this logic repeatedly to the next managers, we can extend the region in which fund managers decide to stay to far above $\underline{\theta}$. Similarly, the opposite situation, in which fund managers choose to exit because of their belief in the exit of other managers, can apply to the upper dominance region and the region in which fund managers decide to exit can thus be extended to far below $\overline{\theta}$. Then, the two regions can meet at a point θ^* in the middle of the intermediate region, where every fund manager follows the same strategy of staying if his or her signal is lower than θ^* and exiting otherwise. Therefore, for a given private signal θ_i, the threshold strategy of a manager of fund i at t_2 is

$$(\theta_i, x) \rightarrow \begin{cases} stay & \theta_i < \theta^*(x) \\ exit & \theta^*(x) \leq \theta_i \end{cases} \tag{21}$$

where the threshold $\theta^*(x)$ depends on the market exposure x the fund managers selected.

We now show that the threshold strategy in Equation (21) has a unique equilibrium. The approach is similar with that of Goldstein and Pauzner (2005). As a first step, assume that all surviving fund managers follow the threshold strategy. Then, for a given threshold $\theta'(x)$, the aggregate portion of exiting funds λ is determined to be the θ realized such that

$$\lambda(\theta, \theta'(x)) = \begin{cases} \theta & 0 \leq \theta \leq \theta'(x) - \varepsilon \\ \theta + (1-\theta)\frac{\theta - (\theta'(x) - \varepsilon)}{2\varepsilon} & \theta'(x) - \varepsilon \leq \theta \leq \theta'(x) + \varepsilon \\ 1 & \theta'(x) + \varepsilon \leq \theta \leq 1 \end{cases} \tag{22}$$

When $0 \leq \theta < \theta' - \varepsilon$, all surviving fund managers receive signals that are lower than θ', so they decide to stay. Similarly, when $\theta' + \varepsilon \leq \theta \leq 1$, all signals are greater than θ' and all managers choose to exit. When $\theta' - \varepsilon \leq \theta \leq \theta' + \varepsilon$, exiting funds are bankrupt funds corresponding to θ, plus leaving funds among survivors, which correspond to $(1-\theta)\frac{\theta - (\theta' - \varepsilon)}{2\varepsilon}$.

As a second step, we need to calculate the expected net investment return of individual fund managers. We define $\Delta\overline{\Pi}(\lambda)$ as the net investment return between staying in the market and exiting:

$$\Delta\overline{\Pi}(\lambda) = \overline{\Pi^S} - \overline{\Pi^E} = \begin{cases} x\bar{R} - \frac{x}{\bar{R}\tau} - \lambda & 0 \leq \lambda \leq c \\ x\bar{R} + (\lambda - c)\left[fx - \bar{R}/\left(\frac{1}{\bar{R}\tau} - f(\lambda - c)\right)\right] - \frac{x}{\bar{R}\tau} - c & c \leq \lambda < \pi \\ 0 & \pi \leq \lambda \leq 1 \end{cases} \tag{23}$$

In the optimal investment decision of whether to stay, a manager of fund i considers the expected net investment return, denoted $E_{\theta_i,\theta'}[\Delta\overline{\Pi}]$. If $E_{\theta_i,\theta'}[\Delta\overline{\Pi}]$ is positive, the manager will prefer to stay and will exit otherwise. Since both θ and ε_i are uniformly distributed, from the perspective of the manager of fund i, whose signal is $\theta_i = \theta + \varepsilon_i$, θ is uniformly distributed over $[\theta_i - \varepsilon, \theta_i + \varepsilon]$. Therefore, $E_{\theta_i,\theta'}[\Delta\overline{\Pi}]$ can be calculated as

$$E_{\theta_i,\theta'}[\Delta\overline{\Pi}] = \frac{1}{2\varepsilon}\int_{\theta_i - \varepsilon}^{\theta_i + \varepsilon} \Delta\overline{\Pi}\big(\lambda(\theta, \theta')\big) d\theta \tag{24}$$

For a marginal fund manager whose signal equals the threshold θ', the expected net investment return should be zero, which means that the marginal fund manager has no preference between staying and exiting the market because the expected investment returns from both decisions are same. Then,

$$E_{\theta',\theta'}[\Delta\overline{\Pi}] = \frac{1}{2\varepsilon}\int_{\theta'-\varepsilon}^{\theta'+\varepsilon} \Delta\overline{\Pi}\big(\lambda(\theta,\theta')\big)d\theta = 0 \tag{25}$$

To prove the uniqueness of the threshold that satisfies Equation (25), two dominance regions should be established, as has already been done. As we have shown, in the lower dominance region, where $\theta' < \underline{\theta} - \varepsilon$, $E_{\theta',\theta'}[\Delta\overline{\Pi}]$ is positive because fund managers always choose to stay, irrespective of the decisions of the others. Similarly, in the upper dominance region, if $\theta' > \overline{\theta} + \varepsilon$, $E_{\theta',\theta'}[\Delta\overline{\Pi}]$ is negative. In addition, between the two extreme regions, $E_{\theta',\theta'}[\Delta\overline{\Pi}]$ is continuous and monotonically decreasing in θ'. The monotonic decrease indicates that, when both θ_i and θ' increase by the same amount, though the marginal manager's belief about the portion of exits among surviving funds is unchanged, the manager believes that the portion of bankrupt funds increases, leading to an increase in the aggregate portion of exiting funds. Then, the investment return of staying deteriorates faster than the investment return of exiting does. Therefore, $E_{\theta',\theta'}[\Delta\overline{\Pi}]$ is monotonically decreasing and there exists a unique point θ^* that satisfies $E_{\theta^*,\theta^*}[\Delta\overline{\Pi}] = 0$. Unlike Goldstein and Pauzner's (2005) model, in which the signal (economy fundamentals) and the number of exiting agents are independent, in our study the two are closely related, in that a higher signal indicates a stronger funding liquidity shock, which forces funds to go bankrupt and affects the investment return of surviving funds.

As the final step, we need to show that the threshold strategy in Equation (21) can be an equilibrium strategy. If this is true, $E_{\theta_i,\theta^*}[\Delta\overline{\Pi}]$ should be positive when $\theta_i < \theta^*$ and negative when $\theta_i > \theta^*$. By the uniqueness of θ^*, we know that there exists only one point $\theta_i = \theta^*$ that satisfies $E_{\theta_i,\theta^*}[\Delta\overline{\Pi}] = 0$, because θ^* is a unique solution of $E_{\theta^*,\theta^*}[\Delta\overline{\Pi}] = 0$. Suppose that a manager of fund i receives a signal θ_i that is lower than θ^*; then the integral range of $E_{\theta_i,\theta'}[\Delta\overline{\Pi}]$ is $[\theta_i - \varepsilon, \theta_i + \varepsilon]$, which is below $[\theta^* - \varepsilon, \theta^* + \varepsilon]$. From Equation (22), if $\theta < \theta^*$, $\lambda(\theta,\theta^*) < \lambda(\theta^*,\theta^*)$ implies that the integral of $\Delta\overline{\Pi}(\lambda(\theta,\theta^*))$ over the range $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ is greater than that over the range $[\theta^* - \varepsilon, \theta^* + \varepsilon]$, because $\Delta\overline{\Pi}(\lambda(\theta,\theta^*))$ is decreasing in λ. Thus, $E_{\theta_i,\theta^*}[\Delta\overline{\Pi}] > E_{\theta^*,\theta^*}[\Delta\overline{\Pi}] = 0$ when $\theta_i < \theta^*$, which means that the manager of fund i decides to stay in the market. Similar logic can apply to the case in which $\theta_i > \theta^*$.

THEOREM 1. *A unique threshold equilibrium exists in which, for a common threshold θ^*, fund managers decide to stay in the market if the private signal is lower than θ^* and run (exit) if the private signal is greater than θ^*.*

Having established the existence of a unique threshold equilibrium, we now need to find the threshold $\theta^*(x)$. Liu and Mello (2011, p.497) state that as "$\varepsilon \to 0$, the fundamental uncertainty disappears, while strategic uncertainty remains unchanged." Applying their argument to our model, we can say that $\lambda(\theta,\theta^*)$ in Equation (22) is uniformly distributed in the interval $[\theta^*,1]$ and $\lambda(\theta,\theta^*)$ becomes a straight line for $[\theta^*-\varepsilon,\theta^*+\varepsilon]$ as $\varepsilon\to 0$. Therefore, it is possible to find θ^* with the equation

$$\int_{\theta^*}^{1}\Delta\bar{\Pi}(\lambda)d\lambda=0 \tag{26}$$

which is the result of a transformation of variables in Equation (25) by the linearity of $\lambda(\theta,\theta^*)$. Figure 2 graphically specifies θ^*, which is determined at the point where the area of A (above the horizontal axis) is equal to that of B (below the horizontal axis). Using this result, we can find the value of $\theta^*(x)$ and the condition for $\theta^*(x)$ in the range (c,π).

[Insert Figure 2 about here]

COROLLARY 1. *If* $\frac{1}{f^2\tau}(fx-\ln f)>\frac{x}{(1+fx)\bar{R}^2\tau^2}\left[\frac{x}{\bar{R}\tau}+(1+fx)c-\frac{f}{2}\left(\frac{x}{(1+fx)\bar{R}\tau}+2c\right)\right]$, *then a unique threshold* $\theta^*(x)$ *exists in the interval* (c,π) *and* $\theta^*(x)$ *can be found by solving the equation* $\frac{1}{f^2\tau(\pi-\theta^*)}\ln[(1+fx)(1-f\bar{R}\tau(\theta^*-c)]=\frac{x}{\bar{R}\tau}-\frac{(1+fx)\bar{R}}{f}+(1+fx)c-\frac{f(\pi+\theta^*)}{2}$.[10]

As $\varepsilon\to 0$, the aggregate portion of exiting funds λ in Equation (22) becomes

$$\lambda(\theta,\theta^*(x))=\begin{cases}\theta & 0\le\theta\le\theta^*(x)\\ 1 & \theta^*(x)<\theta\le 1\end{cases} \tag{27}$$

When θ is below the threshold $\theta^*(x)$, all surviving fund managers decide to stay, but if θ is above $\theta^*(x)$, all funds exit the market. In contrast to state I, in state II all fund managers vanish from the market, even when θ is lower than π. If such a panic-based market run occurs, fund investors withdraw their entire capital from fund managers. Therefore, surviving fund managers cannot survive when $\theta\in(\theta^*,1]$.

As a general condition, our interest in this paper is the range $c<\theta^*<\pi$. Otherwise, if θ^* is below c, panic-based market runs can occur even though fund managers prudently manage their assets and hold enough cash in preparation of a run. Therefore, we exclude this case.

Fund managers facing the risk of panic-based market runs consider θ^* in deciding optimal market exposure at t_1. Therefore, the optimal asset allocation problem of fund manager i is

$$\max_{x_i}E\left[\Pi_i^{II}\right]=\max_{x_i}\frac{1}{2}\int_0^{\theta^*(x_i)}\Pi_i^S\,d\theta \tag{28}$$

In the optimization problem of Equation (28), the integral range is $[0,\theta^*(x_i)]$. This is because when $\theta^*(x)<\theta$, all fund managers choose to exit and investors withdraw their entire capital, so that fund managers obtain zero investment return. What would then happen if some fund managers

[10] Proofs are provided in Appendix.

remained in the market, even in the range $[\theta^*(x), 1]$? Those remaining managers certainly could obtain positive investment returns instead of zero, but other managers who chose to exit would make greater profits. Therefore, rational fund managers prefer to exit and consequently all fund managers choose to exit, even though they know that they will obtain zero investment return.

Now, let us return to t_1. Then, in equilibrium, all fund managers choose the same optimal exposure x^{II*}, which satisfies the equation

$$\frac{\partial E[\Pi^{II}]}{\partial x} \coloneqq \lim_{n\to\infty}\left[\frac{\partial E\left[\Pi_i^{II}\right]}{\partial x_i}\right]_{x=x_1=\cdots=x_n}$$
$$= \frac{1}{2}\lim_{n\to\infty}\left[\frac{\partial}{\partial x_i}\left(\int_0^{\theta^*(x_i)} \Pi_i^S(x_i,\theta)\,d\theta\right)\right]_{x=x_1=\cdots=x_n} = 0 \tag{29}$$

Once x^{II*} is determined at t_1, the threshold strategy is that if θ is below $\theta^*(x^{II*})$, all surviving funds will stay in the market, but, if not, all funds will exit the market. Moreover, we are able to calculate the ex ante probability of market runs. By the assumption that θ is uniformly distributed over the interval $[0,1]$, the probability of runs would be $p_{run} = 1 - \theta^*$.

In the remainder of this paper, we explore the effects of panic-based market runs on the asset allocation of fund managers and then investigate the changes in the probability of runs in response to an unexpected change in trend follower sensitivity.

4. Environmental Changes and Fund Manager Decisions

4.1. Effect of the possibility of a panic-based market run on fund asset allocation

When a panic-based market run can occur, the surviving range changes from $[0,\pi]$ to $[0,\theta^*]$. Then the optimal x^{I*} that maximizes $\frac{1}{2}\int_0^{\pi_i(x_i)} \Pi_i^S\,d\theta$ in state I is no longer optimal in the maximization problem of $\frac{1}{2}\int_0^{\theta^*(x_i)} \Pi_i^S\,d\theta$ in state II. If a run is possible, fund managers who are subject to panic-based market runs decide upon their optimal asset allocation by considering the surviving range $[0,\theta^*]$. Therefore, we investigate how fund managers' optimal decisions change with the introduction of a run.

Since x^{I*} and x^{II*} are the solutions of the maximization problem in states I and II, respectively, they should satisfy $\left[\frac{\partial E[\Pi^I]}{\partial x}\right]_{x=x^{I*}} = 0$ and $\left[\frac{\partial E[\Pi^{II}]}{\partial x}\right]_{x=x^{II*}} = 0$, respectively. Therefore, if $x^{II*} < x^{I*}$, $\left[\frac{\partial E[\Pi^{II}]}{\partial x}\right]_{x=x^{I*}}$ is negative but $x^{II*} > x^{I*}$, $\left[\frac{\partial E[\Pi^{II}]}{\partial x}\right]_{x=x^{I*}}$ is positive. In this section, we show that $x^{II*} < x^{I*}$ by proving that $\left[\frac{\partial E[\Pi^{II}]}{\partial x}\right]_{x=x^{I*}} < 0$ under the condition $\theta^* \in (c,\pi)$.

As a first step, we need to compare the magnitudes of the two integrals $\int_{\theta^*}^{\pi} \frac{\partial \overline{\Pi^E}}{\partial x} d\lambda$ and $\int_{\theta^*}^{\pi} \frac{\partial \overline{\Pi^S}}{\partial x} d\lambda$. From Equations (10) and (11), we know that, in the interval $c \leq \lambda < \pi$, $\frac{\partial \overline{\Pi^E}}{\partial x} = \bar{r} - 1$ and $\frac{\partial \overline{\Pi^S}}{\partial x} = \frac{\bar{R}}{\bar{r}}(\bar{r} - 1)$. Since $\bar{R} > 1 > \bar{r}$, $\frac{\partial \overline{\Pi^S}}{\partial x} < \frac{\partial \overline{\Pi^E}}{\partial x} < 0$. Then, the integrals of both formulas over the range $[\theta^*, \pi]$ also obey this dominance relation, so that $\int_{\theta^*}^{\pi} \frac{\partial \overline{\Pi^S}}{\partial x} d\lambda < \int_{\theta^*}^{\pi} \frac{\partial \overline{\Pi^E}}{\partial x} d\lambda < 0$.

LEMMA 1. $\int_{\theta^*}^{\pi} \frac{\partial \overline{\Pi^S}}{\partial x} d\lambda < \int_{\theta^*}^{\pi} \frac{\partial \overline{\Pi^E}}{\partial x} d\lambda < 0$.

As mentioned above, in the interval $c \leq \lambda < \pi$, we know that $\frac{\partial \overline{\Pi^S}}{\partial x} < \frac{\partial \overline{\Pi^E}}{\partial x} < 0$ and this inequality means that, for the same amount of increase in investment, the loss in returns of staying in the market is greater than the loss of exiting. Then, Lemma 1 implies that, on the interval $[\theta^*, \pi]$, for the same amount of increase in investment, the sum of the marginal losses of funds that stay is larger than that of exiting funds when λ is expected in the ranges of market runs.

From Equation (26), we know that $\int_{\theta^*}^{\pi} \overline{\Pi^E} d\lambda = \int_{\theta^*}^{\pi} \overline{\Pi^S} d\lambda$ and, by differentiating both sides with respect to x, we obtain the following lemma.

LEMMA 2. $\frac{\partial \theta^*(x)}{\partial x} = \frac{\int_{\theta^*}^{\pi} \frac{\partial \overline{\Pi^S}}{\partial x} d\lambda - \int_{\theta^*}^{\pi} \frac{\partial \overline{\Pi^E}}{\partial x} d\lambda}{\overline{\Pi^S}(x,\theta^*) - \overline{\Pi^E}(x,\theta^*)}$

The numerator on the right-hand side of Lemma 2 is negative, due to Lemma 1, and the denominator, $\overline{\Pi^S}(x, \theta^*) - \overline{\Pi^E}(x, \theta^*)$, is positive, because, at $\lambda = \theta^*$, the investment return of staying is greater than the investment return of exiting, leading to the result $\frac{\partial \theta^*(x)}{\partial x} < 0$.

THEOREM 2. *Threshold $\theta^*(x)$ is a decreasing function of market investment x under the general condition $c < \theta^*(x) < \pi$. Then, the probability of market runs, p_{run}, increases as fund managers invest more in the market.*

Since the probability of market runs is $p_{run}(x) = 1 - \theta^*(x)$, Theorem 2 states that p_{run} rises as x increases. When fund managers hold a large amount of risky asset, their investment returns are closely related to each other, so funds sensitively respond to each other's runs, since fund exits deteriorate short-term market returns and stimulate fund withdrawals. In fact, fund exits worsen the investment returns of both staying and exiting, but the remaining fnds then become more fragile than exiting funds, since the return of staying is rapidly diminished by short-term market return deterioration and fund withdrawals affect only remaining funds.[11] Thus, fund managers, if they hold a large amount of risky asset, could choose to exit the market after observing a private signal; otherwise, they could choose to stay. In other words, funds behave more prudently when they hold more risky assets, which means that θ^* decreases as x increases. Hence, the probability of market runs rises along with fund investment.

Using Lemmas 1 and 2, we can derive the following lemma.

LEMMA 3. $x^{II*} < x^{I*}$ if and only if $\left[\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)}\right]_{x=x^{I*}} < \left[\frac{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda}\right]_{x=x^{I*}}$

Lemma 3 explains the condition for $x^{II*} < x^{I*}$. Following Lemma 3, if $\left[\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)}\right]_{x=x^{I*}} < \left[\frac{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda}\right]_{x=x^{I*}}$, fund managers optimally reduce market exposure when they take into account the possibility of a run. Under this condition, the left-hand side, $\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)}$, is the investment return of staying relative to that of exiting at $\lambda = \theta^*$ and the right-hand side, $\frac{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda}$, is the relative value of the sums of marginal loss for an investment increase between staying and exiting conditional on λ being in the interval $[\theta^*,\pi]$. Then, $\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)} < \frac{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda}$ can be interpreted such that the relative value of the sum of marginal losses on $[\theta^*,\pi]$ is greater than the relative investment return at the threshold θ^*. Therefore, Lemma 3 predicts that, when $\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)} < \frac{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda}$, fund managers will lose more money if they do not decrease their investment after the regime has changed from state I to state II. Therefore, fund managers prefer to hold cash when there is a risk of market runs.

From Equations (8) and (11) and Lemma 3, we can derive the following theorem.

THEOREM 3. *Fund managers decrease their market investment when they consider market runs in their optimal investment strategy; that is,* $x^{II*} < x^{I*}$.

In our model, $\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)}$ is bounded above by $\bar{R}^2\tau$, because $\frac{\overline{\Pi^S}(x,c)}{\overline{\Pi^E}(x,c)}$ is $\bar{R}^2\tau$ at $\lambda = c$, and monotonically decreases as λ increases. On the other hand, $\frac{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda}$ is bounded below by $\bar{R}^2\tau$,

[11] Fund withdrawal indirectly affects the investment return of exits through short-term price deterioration.

because $\left|\frac{\partial \overline{\Pi^S}}{\partial x}\right|$ is greater than $\bar{R}^2\tau \cdot \left|\frac{\partial \overline{\Pi^E}}{\partial x}\right|$ on $[\theta^*, \pi]$. Therefore, our model satisfies the condition in Lemma 3 and, when the regime changes from state I to state II, fund managers can maximize their expected asset value by reducing market exposure up to x^{II*}, so that $x^{II*} < x^{I*}$.

Indeed, around the global financial crisis, the regime suddenly changed from state I to state II. Hedge fund managers who perceived this regime change anticipated that market runs might aggravate the illiquidity problem and quickly reduced market exposures, even prior to the crisis. Nevertheless, as the crisis evolved, fund investors started to withdraw and, in response to withdrawals, hedge funds also began to liquidate, not because of systematic risk per se, but because of the fear of the runs of others. Ultimately, as fear peaked following huge financial events (i.e., the Quant Meltdown and Lehman Brothers' bankruptcy), a mass exodus of hedge funds took place; that is, synchronized runs occurred due to the extreme fear of runs.

4.2. Effect of unexpected changes in price sensitivity on the probability of market runs

So far, we have explored the behavior of fund managers. Now, in this section we investigate the effect of trend followers on market stability by examining the relation between trend sensitivity and the probability of market runs.

In our model, after fund managers' asset allocation at t_1, the market price is depreciated by the trading of trend followers. Their sensitivity τ is known to fund managers, whose asset allocation takes into account this information. But what if trend followers suddenly become more risk averse between t_1 and t_2? This situation is quite likely when the market is experiencing severe price deterioration.

A change in τ affects the short-term market price, as well as funds' investment returns. Therefore, if an unexpected change in τ occurs, for a given x, fund managers would adjust their existing threshold strategy to reflect the new τ. In turn, this adjustment shifts the probability of market runs, defined as $p_{run} = 1 - \theta^*$.

We calculate the ex ante probability p_{run} as a function of τ. For a variety of sets of parameter values, p_{run} shows clear upward tendencies in τ. Figure 3 illustrates this tendency for different levels of market exposure x. In this figure, the parameter values are $f = 4$ and $s = 8$. The horizontal axis represents τ from 0.2 to 0.9 and the vertical axis denotes p_{run}. Since the boundary condition restricts τ to the range $[1/\bar{R}, 1)$, we exclude both upper and lower extreme values of τ. Each curve indicates p_{run} as a function of τ for different x values, which increase from 0.2 to 0.9 by increments of 0.1.

[Insert Figure 3 about here]

In Figure 3, all the curves are monotonically increasing, which indicates the fragility of a highly trend-sensitive market. In a financial market where investors respond with sensitivity to past price trends, even moderate levels of market shock can have a large impact on the market price, leading to a huge price drop. Price deterioration is more detrimental to funds that stay than to exiting funds, so a highly sensitive market is more prone to suffer from market runs. To put it concretely, an increase in sensitivity τ decreases the short-term market return $\bar{r}$ and, accordingly, $\overline{\Pi^S}$ falls more sharply than $\overline{\Pi^E}$ does. Since it decreases the net investment return function, $\Delta\overline{\Pi}(\lambda) = \overline{\Pi^S} - \overline{\Pi^E}$, the threshold θ^* shifts to a lower value because θ^* satisfies $\int_{\theta^*}^{1} \Delta\overline{\Pi}(\lambda) d\lambda = 0$ (see Figure 2). Therefore, in response to an unexpectedly high trend sensitivity τ, fund managers lower the threshold θ^* and the probability of market runs, p_{run}, increases.

The above results suggest a possible price stabilization process for a financial market experiencing severe price drops. When fund managers perceive that the regime is changing from state I to state II, they reduce market exposure. Meanwhile, as prices diverge more, trend followers become more risk averse and respond more sensitively to past price drops. Then, the probability of market runs is affected in two opposite ways: reducing x decreases p_{run} and increasing τ increases p_{run}. If the effect of a rising τ dominates the effect of a decreasing x, p_{run} increases, so that the financial market becomes more vulnerable to a funding liquidity shock. At the beginning of a crisis, the effect of the increased τ dominates the effect of the diminished x and market runs become highly probable. However, in the meantime, as the irrational fad wanes and correct information is diffused throughout the market, the trading forces of trend followers weaken and price sensitivity gradually diminishes. Finally, when the effect of τ equals the effect of x, p_{run} begins to decrease. It can rebound after reaching its lowest price.

To sustain market stability against market runs, we highlight an information cost. In the model, the trend-following strategy of uninformed investors delivers a past price shock to the future market and, as trend sensitivity increases, the probability of market runs rises. Goldstein and Pauzner (2005) and Liu and Mello (2011) predict the similar result that short-term price deterioration raises the possibility of synchronized runs. However, their results are derived from the simple assumption of a constant short-term return, because their models do not consider a price determination mechanism. In contrast, we develop a market model in which short-term market return is endogenously determined by several factors and, among them, we note price sensitivity is an important source that affects the likelihood of market runs. Since price sensitivity is positively related to the probability of market runs, we conclude that lowering the information cost that makes trend followers become informed could mitigate the synchronization problem in a distressed market, by enhancing market stability.

5. Conclusion

This paper develops a market model that illustrates the synchronized market runs of informed and rational fund managers. Using a global game technique, we show that the possibility of runs induces panic-based market runs, not because of systematic risk itself but because of the fear of runs. In addition, we find that when the market regime changes from a normal state (in which runs are impossible) to a bad state (in which runs are possible), fund managers quickly reduce risky asset exposure prior to the occurrence of runs. Through analyzing the ex ante probability of market runs, we also suggest that a market in which fund managers have high market exposure and trend followers are highly sensitive is more likely to experience synchronized runs.

On the basis of the above findings, we explain some stylized facts that were observed around the global financial crisis. Hedge funds, whose investors are highly loss sensitive, behaved cautiously, holding more cash prior to the crisis. Nevertheless, as the financial crisis evolved, fund investors started to withdraw capital from their funds in response to the initial loss. Even though some fund managers who had sufficiently reduced risky asset exposure in advance knew that the liquidity shock was not strong enough to pull them out of the market, the growing fear of capital withdrawals and price deterioration caused by the runs of other funds could ultimately induce them to run. In the worst case, hedge funds can collectively exit the market not because of risk itself, but because of fear, as occurred during the Quant Meltdown of 2007 and Lehman Brothers' bankruptcy of 2008. In the meantime, high-volatility stocks are more likely to experience stronger fire sales than low-volatility stocks are, because high-volatility stocks respond sensitively to price movement and, during a market downturn, are more likely to experience price drops.

Our findings also suggest a possible price stabilization process during a financial crisis. At the very first sign of crisis, fund managers reduce market exposure after perceiving that the regime has changed from a normal to a bad state. As the crisis proceeds, trend followers become more risk averse and respond more sensitively to past price drops. Then, two opposite effects arise: The reduced exposure decreases the probability of market runs and the increased sensitivity raises it. If the latter effect dominates, the probability of market runs rises so that the financial market becomes more vulnerable to a funding liquidity shock. At the beginning of a crisis, the former dominates and the market price is more likely to decline. However, as time passes, price sensitivity gradually diminishes, as does the probability of market runs. Ultimately, the market reaches the lowest price and then rebounds.

In work similar to ours, Goldstein and Pauzner (2005) and Liu and Mello (2011) develop models for panic-based runs and predict that a decrease in the short-term return increases the possibility of synchronized runs. However, they fail to suggest ways of stabilizing the market, because their models assume the short-term return to be a given value. In contrast, we develop a market model in which the short-term return is endogenously determined by market factors, among which we note price sensitivity to be an important factor that is positively related to the probability of market runs. Therefore, we suggest increasing market stability by decreasing the cost of the information that informs trend followers to mitigate synchronization risk.

Appendix

A.1. Proof of Corollary 1

The threshold θ^* satisfies $0 = \int_{\theta^*}^{1} \Delta\bar{\Pi}(\lambda) d\lambda = \int_{\theta^*}^{\pi} \Delta\bar{\Pi}(\lambda) d\lambda$. Then, there exists a unique threshold θ^* in the interval (c, π) if and only if $\int_{c}^{\pi} \Delta\bar{\Pi}(\lambda) d\lambda > 0$, because $\Delta\bar{\Pi}(\lambda)$ is positive for $\lambda < \theta^*$. The inequality $\int_{c}^{\pi} \Delta\bar{\Pi}(\lambda) d\lambda > 0$ can be rewritten as $\frac{1}{f^2\tau}(fx - \ln f) > \frac{x}{(1+fx)\bar{R}^2\tau^2}\left[\frac{x}{\bar{R}\tau} + (1+fx)c - \frac{f}{2}\left(\frac{x}{(1+fx)\bar{R}\tau} + 2c\right)\right]$. Thus, if this condition is satisfied, the existence of a unique θ^* in the interval (c, π) can be assured.

When $c < \theta^* < \pi$ is guaranteed, we can find θ^* by solving the equation $\int_{\theta^*}^{\pi} \Delta\bar{\Pi}(\lambda) d\lambda = 0$. We obtain $\int_{\theta^*}^{\pi} \Delta\bar{\Pi}(\lambda) d\lambda = \int_{\theta^*}^{\pi} \overline{\Pi^S} d\lambda - \int_{\theta^*}^{\pi} \overline{\Pi^E} d\lambda = \int_{\theta^*}^{\pi} \left[x\bar{R} - (\lambda - c)\bar{R}/\left(\frac{1}{\bar{R}\tau} - f(\lambda - c)\right)\right] d\lambda - \int_{\theta^*}^{\pi} \left[\frac{x}{\bar{R}\tau} - fx(\lambda - c) + c\right] d\lambda$ for $\theta^* \in (c, \pi)$, which is calculated as $\int_{\theta^*}^{\pi} \Delta\bar{\Pi}(\lambda) d\lambda = \bar{R}\left[\frac{1+fx}{f}(\pi - \theta^*) + \frac{\ln[(1+fx)(1-f\bar{R}\tau(\theta^*-c)]}{f^2\bar{R}\tau}\right] - (\pi - \theta^*)\left[\frac{x}{\bar{R}\tau} + (1+fx)c - \frac{f}{2}(\pi + \theta^*)\right] = 0$. After rearranging the equation, we have $\frac{1}{f^2\tau(\pi-\theta^*)}\ln[(1+fx)(1 - f\bar{R}\tau(\theta^* - c)] = \frac{x}{\bar{R}\tau} - \frac{(1+fx)\bar{R}}{f} + (1+fx)c - \frac{f(\pi+\theta^*)}{2}$, from which we can calculate the threshold θ^*.

A.2. Proof of Lemma 2

From Equation (26), we know that $0 = \int_{\theta^*}^{1} \Delta\bar{\Pi}(\lambda) d\lambda = \int_{\theta^*}^{\pi} \overline{\Pi^S} d\lambda - \int_{\theta^*}^{\pi} \overline{\Pi^E} d\lambda$. By differentiating both sides with respect to x, we obtain $\frac{\partial}{\partial x}\int_{\theta^*}^{\pi} \overline{\Pi^E} d\lambda = \frac{\partial}{\partial x}\int_{\theta^*}^{\pi} \overline{\Pi^S} d\lambda$. Using the Leibniz integral rule, we can rewrite the equation as $\int_{\theta^*}^{\pi} \frac{\partial\overline{\Pi^E}}{\partial x} d\lambda - \overline{\Pi^E}(x, \theta^*)\frac{\partial\theta^*}{\partial x} = \int_{\theta^*}^{\pi} \frac{\partial\overline{\Pi^S}}{\partial x} d\lambda - \overline{\Pi^S}(x, \theta^*)\frac{\partial\theta^*}{\partial x}$. Therefore, we obtain $\frac{\partial\theta^*(x)}{\partial x} = \frac{\int_{\theta^*}^{\pi} \frac{\partial\overline{\Pi^S}}{\partial x} d\lambda - \int_{\theta^*}^{\pi} \frac{\partial\overline{\Pi^E}}{\partial x} d\lambda}{\overline{\Pi^S}(x,\theta^*) - \overline{\Pi^E}(x,\theta^*)}$.

A.3. Proof of Lemma 3

The term $\frac{\partial E[\Pi^{II}]}{\partial x} - \frac{\partial E[\Pi^{I}]}{\partial x}$ can be rewritten as $\left[\int_0^{\theta^*} \frac{\partial \overline{\Pi^S}}{\partial x} d\lambda + \overline{\Pi^S}(x,\theta^*)\frac{\partial \theta^*}{\partial x} - \overline{\Pi^S}(x,0)\frac{\partial 0}{\partial x}\right] - \left[\int_0^{\pi} \frac{\partial \overline{\Pi^S}}{\partial x} d\lambda + \overline{\Pi^S}(x,\pi)\frac{\partial \pi}{\partial x} - \overline{\Pi^S}(x,0)\frac{\partial 0}{\partial x}\right] = -\int_{\theta^*}^{\pi} \frac{\partial \overline{\Pi^S}}{\partial x} d\lambda + \overline{\Pi^S}(x,\theta^*)\frac{\partial \theta^*}{\partial x}$ by the Leibniz integral rule. After substituting $\frac{\partial \theta^*}{\partial x} = \frac{\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^S}}{\partial x}d\lambda - \int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^E}}{\partial x}d\lambda}{\overline{\Pi^S}(x,\theta^*) - \overline{\Pi^E}(x,\theta^*)}$ and arranging the formula, we obtain $\frac{\partial E[\Pi^{II}]}{\partial x} - \frac{\partial E[\Pi^{I}]}{\partial x} = \frac{\overline{\Pi^E}(x,\theta^*)\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^S}}{\partial x}d\lambda - \overline{\Pi^S}(x,\theta^*)\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^E}}{\partial x}d\lambda}{\overline{\Pi^S}(x,\theta^*) - \overline{\Pi^E}(x,\theta^*)}$. Since $\overline{\Pi^S}(x,\theta^*) - \overline{\Pi^E}(x,\theta^*)$ is positive and both $\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^S}}{\partial x}d\lambda$ and $\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^E}}{\partial x}d\lambda$ are negative, we conclude that $\frac{\partial E[\Pi^{II}]}{\partial x} < \frac{\partial E[\Pi^{I}]}{\partial x}$ if and only if $\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)} < \frac{\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^E}}{\partial x}d\lambda}$. Substituting $x = x^{I*}$ into $\frac{\partial E[\Pi^{II}]}{\partial x} < \frac{\partial E[\Pi^{I}]}{\partial x}$ shows that $\left[\frac{\partial E[\Pi^{II}]}{\partial x}\right]_{x=x^{I*}} < 0$, which means $x^{II*} < x^{I*}$.

A.4. Proof of Theorem 3

Using Lemma 3, we show $x^{II*} < x^{I*}$ by proving $\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)} < \bar{R}^2\tau < \frac{\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^E}}{\partial x}d\lambda}$. The relation on the left-hand side is justified first, followed by the relation on the left-hand side. From Equations (8) and (9), we know that $\frac{\overline{\Pi^S}(x,c)}{\overline{\Pi^E}(x,c)} = \frac{x\bar{R}}{\frac{x}{\bar{R}\tau}+c}$ and we can determine that $\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)} < \frac{\overline{\Pi^S}(x,c)}{\overline{\Pi^E}(x,c)}$, because $\overline{\Pi^S}(x,\lambda)$ decreases more steeply than $\overline{\Pi^E}(x,\lambda)$ for $\lambda \in [c,\pi]$. A simple calculation leads to the result $\frac{\overline{\Pi^S}(x,c)}{\overline{\Pi^E}(x,c)} = \frac{x\bar{R}}{\frac{x}{\bar{R}\tau}+c} \leq \frac{x\bar{R}}{\frac{x}{\bar{R}\tau}} = \bar{R}^2\tau$. Therefore, the derivation of the left-hand relation is complete.

In Equations (10) and (11), it is obvious that $\frac{\frac{\partial \overline{\Pi^S}}{\partial x}(x,c)}{\frac{\partial \overline{\Pi^E}}{\partial x}(x,c)} = \bar{R}^2\tau$ and $\frac{\frac{\partial \overline{\Pi^S}}{\partial x}(x,\pi)}{\frac{\partial \overline{\Pi^E}}{\partial x}(x,\pi)} = \bar{R}^2\tau(1+fx)$. In addition, we can determine that $\frac{\frac{\partial \overline{\Pi^S}}{\partial x}(x,\lambda)}{\frac{\partial \overline{\Pi^E}}{\partial x}(x,\lambda)}$ is a monotonically increasing function of λ on $[c,\pi]$, so that $\frac{\frac{\partial \overline{\Pi^S}}{\partial x}(x,\lambda)}{\frac{\partial \overline{\Pi^E}}{\partial x}(x,\lambda)} > \bar{R}^2\tau$. Since $\frac{\partial \overline{\Pi^E}}{\partial x}(x,\lambda)$ is negative, $\frac{\partial \overline{\Pi^S}}{\partial x}(x,\lambda) \leq \bar{R}^2\tau\frac{\partial \overline{\Pi^E}}{\partial x}(x,\lambda)$. After integrating both sides over the range $[\theta^*,\pi]$ and dividing by $\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^E}}{\partial x}d\lambda$, we obtain $\frac{\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^E}}{\partial x}d\lambda} > \frac{\int_{\theta^*}^{\pi}\bar{R}^2\tau\frac{\partial \overline{\Pi^E}}{\partial x}(x,\lambda)d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial \overline{\Pi^E}}{\partial x}d\lambda}$.

When dividing, be sure that $\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda$ is negative. Since $\bar{R}^2\tau$ is irrelevant to λ, we can derive $\frac{\int_{\theta^*}^{\pi}\bar{R}^2\tau\frac{\partial\overline{\Pi^E}}{\partial x}(x,\lambda)d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda}=\bar{R}^2\tau\frac{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}(x,\lambda)d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda}=\bar{R}^2\tau$. The proof of the right-hand relation is complete.

By combining the two results, we obtain $\frac{\overline{\Pi^S}(x,\theta^*)}{\overline{\Pi^E}(x,\theta^*)}<\bar{R}^2\tau<\frac{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^S}}{\partial x}d\lambda}{\int_{\theta^*}^{\pi}\frac{\partial\overline{\Pi^E}}{\partial x}d\lambda}$. Thus, from Lemma 3, $x^{II*}<x^{I*}$.

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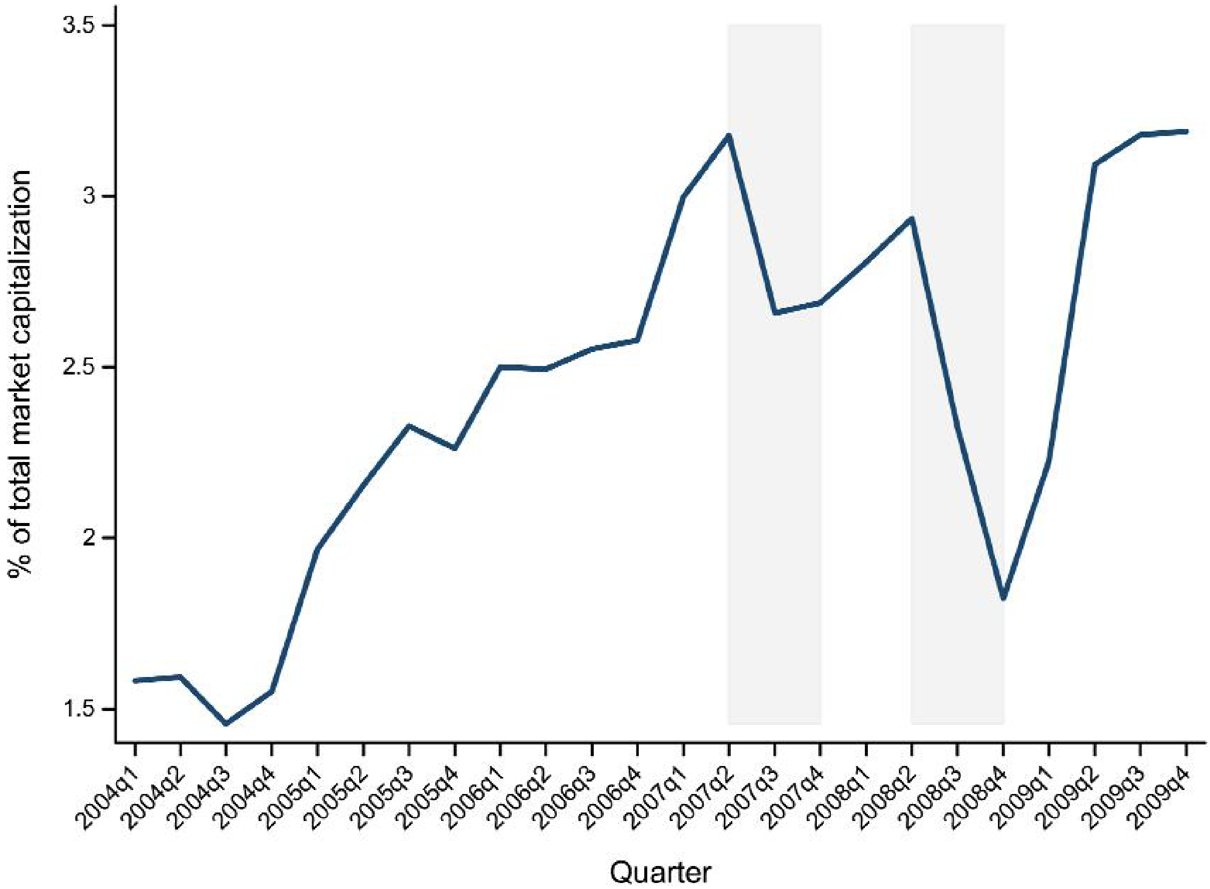
Figure 1. The fraction of U.S. stock market capitalization held by hedge funds

The shaded areas indicate the quarters around the Quant Meltdown (2007:Q3–Q4) and Lehman Brothers' bankruptcy (2008:Q3–Q4).



Source: Ben-David, Franzoni, and Moussawi (2011).

Figure 2. The net investment return between staying and exiting the market.

The net investment return between staying and exiting the market t, $\Delta\overline{\Pi}$, as a function of the aggregate portion of exiting funds, λ. Here, $f = 4$, $s = 8$, $\tau = 0.3$, and $x = 0.71644$.

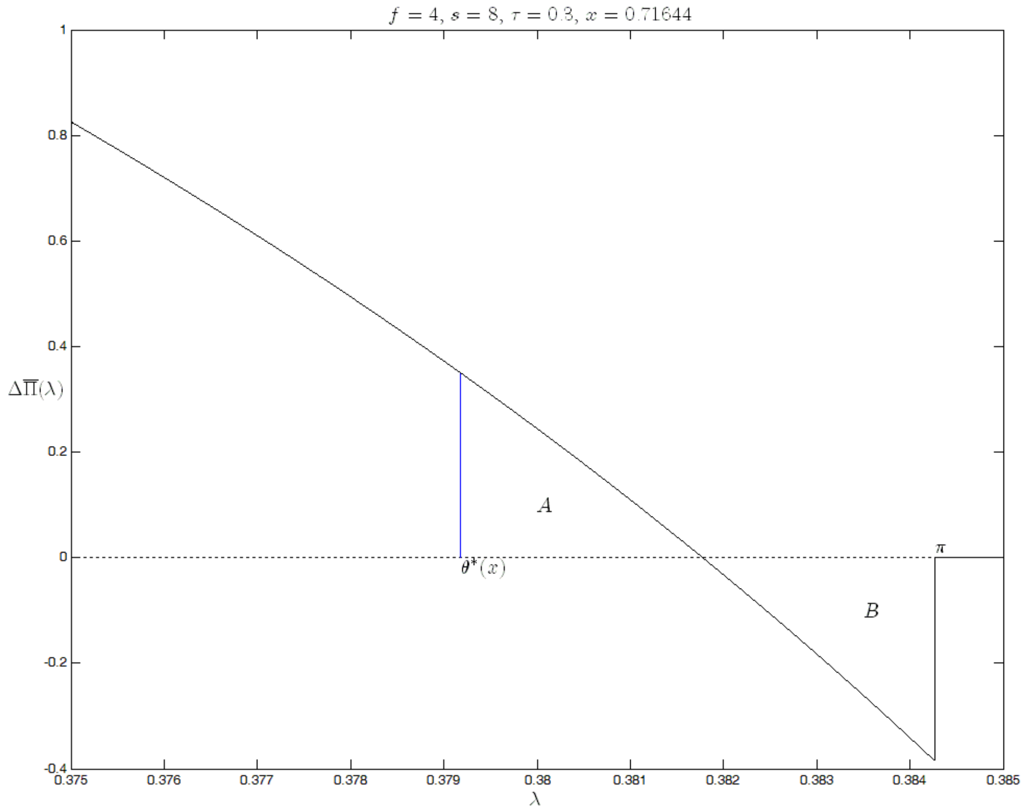


Figure 3. The probability of market runs

The probability of market runs, p_{run}, as a function of the price sensitivity of trend followers, τ. Each curve indicates the function for a different level of market exposure x, from 0.2 to 0.9. Here, $f = 4$ and $s = 8$.