Referee report "A quantum framework for economic science: New directions" by Sudip Patra

In the article "A quantum framework for economic science: New directions' written by Sudip Patra, new areas of application of quantum theory to model different fields in economic science were listed and briefly discussed. Unfortunately, the paper is not written in a good language and the mathematical presentation, the equations and mathematical symbols, of the basic principles of the used quantum framework is not done well.

I cannot support the publication of the article in the present form and in the following I will summarize only my scientific remarks: A brief historical remark is missing in the introduction. The first articles of quantum game theory [6, 1] should be cited. On page the author writes: "*However, we don't want to build a physical theory of entanglement between cognitive states here, which may be a possibility for future research.*". This statement is not true, as the effects of entanglement have been studied extensively. The mathematical description of entangled states is mathematically formulated within the Eisert's representation of quantum game theory [1]. In the following the basic principles of quantum game theory within the Eisert's representation will be summarized:

In QGT, the measurable pure classical strategies (s_1 and s_2) correspond to the orthonormal unit basis vectors $|s_1\rangle$ and $|s_2\rangle$ of the two dimensional complex space, the so called Hilbert space $\mathcal{H}_i$ of player i ($i = A, B$). A quantum strategy of a player i is represented as a general unit vector $|\psi\rangle_i$ in his strategic Hilbert space $\mathcal{H}_i$. The whole quantum strategy space $\mathcal{H}$ is constructed with the use of the direct tensor product of the individual Hilbert spaces: $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$. The main difference between classical and quantum game theory is that in the Hilbert space $\mathcal{H}$ correlations between the players' individual quantum strategies are allowed, if the two quantum strategies $|\psi\rangle_A \in \mathcal{H}_A$ and $|\psi\rangle_B \in \mathcal{H}_B$ are entangled. The overall state of the system we are looking at is described as a 2-player quantum state $|\Psi\rangle \in \mathcal{H}$. We define the four basis vectors of the Hilbert space $\mathcal{H}$ as the classical game outcomes ($\left|s_1^A s_1^B\right\rangle := (1,0,0,0)$, $\left|s_1^A s_2^B\right\rangle := (0,-1,0,0)$, $\left|s_2^A s_1^B\right\rangle := (0,0,-1,0)$ and $\left|s_2^A s_2^B\right\rangle := (0,0,0,1)$).

The setup of the quantum game begins with the choice of the initial state $|\Psi_0\rangle$. We assume that both players are in the state $|s_1\rangle$. The initial state of the two players is given by

$$|\Psi_0\rangle \;=\; \widehat{\mathcal{J}}\left|s_1^A s_1^B\right\rangle \;=\; \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) \\ 0 \\ 0 \\ i\sin\left(\frac{\gamma}{2}\right) \end{pmatrix} \quad , \tag{1}$$

where the unitary operator $\hat{\mathcal{J}}$ (see equation (7)) is responsible for the possible entanglement of the 2-player system. The players' quantum decision (quantum strategy) is formulated with the use of a two parameter set of unitary 2×2 matrices:

$$\widehat{\mathcal{U}}(\theta, \varphi) := \begin{pmatrix} e^{i\varphi}\cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & e^{-i\varphi}\cos(\frac{\theta}{2}) \end{pmatrix} \tag{2}$$

$$\forall \quad \theta \in [0, \pi] \;\wedge\; \varphi \in [0, \tfrac{\pi}{2}]$$

By arranging the parameters θ and φ, a player chooses his quantum strategy. The classical strategy s_1 is selected by appointing $\theta = 0$ and $\varphi = 0$:

$$\widehat{s_1} := \hat{\mathcal{U}}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \tag{3}$$

whereas the strategy s_2 is selected by choosing $\theta = \pi$ and $\varphi = 0$:

$$\widehat{s_2} := \hat{\mathcal{U}}(\pi,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

In addition, the quantum strategy $\widehat{Q}$ is given by

$$\widehat{Q} := \hat{\mathcal{U}}(0,\pi/2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad . \tag{5}$$

$|\psi\rangle_A$, the decision state of player A, is formally constructed as a matrix-vector multiplication of the decision operator $\widehat{\mathcal{U}}(\theta_A, \varphi_A)$ acting on the initial state $\left|s_1^A\right\rangle$:

$$|\psi\rangle_A = \widehat{\mathcal{U}}(\theta_A, \varphi_A) \left|s_1^A\right\rangle = \begin{pmatrix} e^{i\,\varphi_A} \cos(\frac{\theta_A}{2}) \\ -\sin(\frac{\theta_A}{2}) \end{pmatrix} \tag{6}$$
$$\psi_1^A = e^{i\,\varphi_A} \cos(\frac{\theta_A}{2}) \,, \;\; \psi_2^A = \sin(\frac{\theta_A}{2})$$

The set of classical mixed strategies of player A ($\tilde{\mathcal{S}}^A = \left\{\tilde{s}_1^A, \tilde{s}_2^A\right\}$) is a subset of the Hilbertspace $\mathcal{H}_A$ (angle φ_A is identical zero):

The imaginary part of the state $|\psi\rangle_A$ is zero for $\varphi_A \equiv 0$ and as a result the different classical mixed strategies can be obtained by arranging the angle $\theta \in [0, \pi]$. However for $\varphi_A > 0$ the imaginary part of the first component ψ_1^A of the spinor $|\psi\rangle_A$ is not zero and these kind of quantum strategies cannot be found in the theory of classical games. As the imaginary part of the state $|\psi\rangle_A$ is only present within its first component ψ_1^A, these quantum strategies are named s_1-quantum strategies. By exchanging the basis s_1 to s_2 it is possible to describe also s_1-quantum strategies.

After the two players have chosen their individual quantum strategies ($\hat{\mathcal{U}}_A := \hat{\mathcal{U}}(\theta_A, \varphi_A)$ and $\hat{\mathcal{U}}_B := \hat{\mathcal{U}}(\theta_B, \varphi_B)$) the disentangling operator $\widehat{\mathcal{J}}^\dagger$ is acting to prepare the measurement of the players' state. The entangling and disentangling operator ($\widehat{\mathcal{J}}, \widehat{\mathcal{J}}^\dagger$; with $\hat{\mathcal{J}} \equiv \hat{\mathcal{J}}^\dagger$) depends on one additional single parameter γ which measures the strength of the entanglement of the system:

$$\widehat{\mathcal{J}} := e^{i\,\frac{\gamma}{2}(\widehat{s_1} \otimes \widehat{s_1})} \;, \quad \gamma \in [0, \frac{\pi}{2}] \tag{7}$$

In the used representation, the entangling operator $\widehat{\mathcal{J}}$ has the following explicit structure:

$$\widehat{\mathcal{J}} := \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) & 0 & 0 & i\sin\left(\frac{\gamma}{2}\right) \\ 0 & \cos\left(\frac{\gamma}{2}\right) & -i\sin\left(\frac{\gamma}{2}\right) & 0 \\ 0 & -i\sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) & 0 \\ i\sin\left(\frac{\gamma}{2}\right) & 0 & 0 & \cos\left(\frac{\gamma}{2}\right) \end{pmatrix} \tag{8}$$

Finally, the state prior to detection can therefore be formulated as follows:

$$|\Psi\rangle = \hat{\mathcal{J}}^{\dagger} \left(\hat{\mathcal{U}}_A \otimes \hat{\mathcal{U}}_B \right) \hat{\mathcal{J}} \, \left| s_1^A s_1^B \right\rangle \tag{9}$$

The expected payoff within a quantum version of a general 2-player game depends on the payoff matrix and on the joint probability to observe the four observable outcomes P_{11}, P_{12}, P_{21} and P_{22} of the game

$$\begin{aligned} \$_A \quad &= \$_{11}^A \, P_{11} + \$_{12}^A \, P_{12} + \$_{21}^A \, P_{21} + \$_{22}^A \, P_{22} \\ \$_B \quad &= \$_{11}^B \, P_{11} + \$_{12}^B \, P_{12} + \$_{21}^B \, P_{21} + \$_{22}^B \, P_{22} \\ \text{with:} \quad P_{\sigma\sigma'} &= |\, \langle \sigma\sigma' | \Psi \rangle \,|^2 \;\; , \quad \sigma, \sigma' = \{s_1, s_2\} \quad . \end{aligned} \tag{10}$$

For $\gamma \to 0$ quantum game theory migrates to classical game theory, however for $\gamma > 0$, dilemma situations found in classical (2 player)-(2 strategy) normal form games could be resolved (for a detailed analysis see [2]). Several different applications, including "Quantum Game Theory and Open Access Publishing" [3], "Evolutionary Quantum Game Theory and Scientific Communication" [4] and a "Quantum game theory-based analysis of financial crises" [5] have been discussed in the literature in the context of entanglement. These articles, which deal with the impact of entanglement, should be mentioned within a revised version of the manuscript.

Finally, the whole paper has a large amounts of typos and the style of writing should be improved. Sentences like "*Let's have a few examples.*" (see page 2), "*(there are so many examples)*" (see page 2) and "*(and also crazy!)*" (see page 10) are unusual for a scientific article and the use of brackets "(...)" should be reduced in a revised version of the article.

After so much criticism, I want to mention that I like the aim of the article and I think that it is important to summarize the new directions within quantum economic science. However, in summary, I cannot support the article in the present version.

Yours sincerely, Referee

Literatur

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