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The fallacy of the fiscal theory of the price level – one last time

Willem H. Buiter and Anne C. Sibert

Abstract
There have been attempts to resurrect the fiscal theory of the price level (FTPL). The original FTPL rests on a fundamental compounded fallacy: confusing the intertemporal budget constraint (IBC) of the State, holding with equality and with sovereign bonds priced at their contractual values, with a misspecified equilibrium nominal bond pricing equation, and the 'double use' of this IBC. This fallacy generates a number of internal inconsistencies and anomalies. The issue is not an empirical one. Neither does it concern the realism of the assumptions. It is about flawed internal logic. The issue is not just of academic interest. If fiscal policy authorities were to take the FTPL seriously, costly policy accidents, including sovereign default and hyperinflation, could be the outcome. Interpreting the FTPL as an equilibrium selection mechanism in models with multiple equilibria does not help. Attempts by Sims to extend the FTPL to models with nominal price rigidities fail. The attempted resurrection of the FTPL fails. It is time to bury it again – for the last time.

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Keywords Fiscal theory of the price level; intertemporal budget constraint; equilibrium bond pricing equation; monetary and fiscal policy coordination; equilibrium selection; fiscal dominance

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(1) Introduction

(1.A.) The original FTPL

The fiscal theory of the price level (FTPL) was developed in the 1990s and early 2000s by a number of distinguished economists, among them Leeper (1991), Sims (1994, 1999a), Woodford (1994, 1995, 1996, 1998a, b, c, 1999, 2001) and Cochrane (1999, 2001, 2005)). It was further discussed and developed by many others, e.g. Cushing (1999), Loyo (1999), Kocherlakota and Phelan (2000), Christiano and Fitzgerald (2000), Schmitt-Grohe and Uribe (2000) and McCallum (2001). The FTPL was quite popular for a number of years, with extensions to open-economy settings (see e.g. Sims (1999b, 2001), Bergin (2000), Dupor (2000) and Daniel (2001)), although there were few, if any, attempts at empirical verification of its observable implications.

The original FTPL proposed an alternative theory of the determination of the general price level in a dynamic monetary general equilibrium model with freely flexible nominal prices. This version of the FTPL was shown to be a fallacy by Buiter (1998, 1999, 2001, 2002 and 2005), Niepelt (2004) and Daniel (2007). We shall focus here on Buiter's arguments. The original FTPL was based on an elementary but fatal error: it confused a budget constraint with an equilibrium condition. Specifically, it confused the intertemporal budget constraint (IBC) of the State (the consolidated general government and central bank), with a misspecified equilibrium sovereign nominal bond pricing equation.

The original FTPL asserted that the IBC of the State, holding with equality and with government bonds priced at their *contractual* (i.e. free of default risk) values, determines the general price level. The equilibrium value of the general price level equates the real value of the outstanding stock(s) of nominal government bonds (priced at their contractual values) to the present discounted value of anticipated future augmented primary surpluses of the State.[1] The authors of the original FTPL did not recognize that this 'additional' equilibrium condition – that the IBC of the State holds with equality, with sovereign bonds priced at their contractual values – had already been used elsewhere in the model: it is an implication of the equilibrium real resource constraint and the IBC of the representative consumer, holding with equality and with bonds priced at their contractual values. This IBC holds with equality when household consumption and money demand are derived from the optimizing behavior of forward-looking households with rational expectations, when there is non-satiation in real money balances and/or consumption.

The aforementioned fatal fallacy was at times compounded with other confusions, including, first, the identification of the FTPL with "fiscal dominance" or "active fiscal policy and passive monetary policy"' in a game-theoretic view of the interaction of monetary and fiscal authorities (see e.g. Leeper (1991), and Bassetto (2002)); second, the interpretation of the FTPL as an equilibrium selection mechanism when there are multiple equilibria (Kocherlakota and Phelan (2000)); and, third, identifying the FTPL with the (perfectly coherent) view of the determination of the price level and the interaction of fiscal and monetary policy in the famous "Unpleasant Monetarist Arithmetic" model of Sargent and Wallace (1981). That model has a 'second policy regime' – 'fiscal dominance' - when the public debt to GDP ratio and the

[1] Primary surpluses are non-interest revenues net of non-interest expenditures. The augmented primary surplus of the State is the primary surplus of the State plus the value of the change in the stock of central bank money, minus any interest paid on central bank money.

primary surplus as a share of GDP are kept constant and base money issuance is endogenously determined to satisfy the budget constraint of the State.[2]

Buiter showed that the original flexible price level FTPL produces a handful of anomalies and one logical inconsistency or contradiction.

Note that neither inconsistency with the empirical evidence nor the lack of realism of its assumptions were reasons for the refutation of the FTPL by Buiter, Niepelt and Daniel. A logically inconsistent theory has no empirical implications, and the realism of its assumptions is irrelevant.

(1.B.) The resurrection of the FTPL

The refutations of the original FTPL by Buiter, Niepelt and Daniel were never disputed, let alone shown to be incorrect, in scholarly publications or other scientifically reputable media or fora. It is therefore surprising indeed that a theory exposed as a fallacy is making a comeback. This resurrection has both a scholarly and an economic policy dimension.

As regards the scholarly revival, on 1 April, 2016, a conference with as its theme "Next Steps for the Fiscal Theory of the Price Level" was held at the Becker Friedman Institute for Research on Economics at the University of Chicago.[3] Three of the four originators of the FTPL, Christopher Sims, John Cochrane and Eric Leeper, participated and asserted its continued validity and relevance (see e.g. Sims (2016b), Cochrane (2016b, c) and Leeper (2015), Jacobson, Leeper and Preston (2016)). Only Michael Woodford was missing.

The attempted scholarly revival takes two forms. One is an unreconstructed restatement of the original FTPL in a world with flexible prices. No new arguments are offered and because repeated assertion is not yet accepted in scholarly circles as an alternative mode of proof to induction and deduction, we dismiss it in what follows using the familiar earlier arguments of Buiter and Niepelt.

(1.B.a.) Sims's new FTPL: the FTLEA

The second attempted scholarly resurrection of the FTPL, due to Sims (2011, 2013, 2016a, b, c) uses dynamic monetary general equilibrium models with sticky nominal prices. There is both a New Keynesian and an old-Keynesian variant. Sims now accepts that the Old-Keynesian variant is not an example of his 'new' FTPL; we deal with it in Section 4.C.. The New-Keynesian variant also turns out to be quite unlike the traditional FTPL in that it does not use the IBC of the State twice (holding with equality and with sovereign bonds priced at their contractual values). However, it also does not produce the result, insisted on by Sims (2011, 2013), that sovereign solvency is guaranteed, in equilibrium, by the appropriate response of consumption and of nominal and real discount factors to the introduction of a 'non-Ricardian' budgetary policy. Instead it produces logical inconsistencies that are similar to but not identical to those that sank the original FTPL. This New-Keynesian FTPL (or FTLEA – for fiscal theory of the level of economic activity) turns out to be a perfectly conventional macroeconomic model for which non-Ricardian budgetary rules may be, but are not guaranteed to be, consistent with government solvency in equilibrium.

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[2] Sargent and Wallace (1981) and Sargent (1987) also show that in the 'game of chicken' between the monetary and fiscal authorities, both outcomes produce outcomes for the general price level that are perfectly consistent with conventional monetary theory. Since all non-monetary government debt in the "Unpleasant Monetarist Arithmetic" model is index-linked or real, the FTPL cannot even get to first base in this model.

[3] For the program and links to the presentations see https://bfi.uchicago.edu/events/next-steps-fiscal-theory-price-level. The only critical noise at the conference came from Harald Uhlig (2016).

(1.B.b.) Why the fallacy of the FTPL matters in the real world

The main reason we are worried about this attempted resurrection of the FTPL is that earlier this year, the FTPL cropped up twice in the economic policy arena. In Japan, Katsushiko Aiba and Kiichi Murashima noted - referring to the 'new' FTPL (in the version developed by Christopher Sims (2011, 2013, 2016a, b, c) - that *"… the Nikkei and other media have recently reported his prescription for achieving the inflation target based on the FTPL. We should keep a close eye on this theory because PM Abe's economic advisor Koichi Hamada is a believer, meaning that it might be adopted in Japan's future macroeconomic policies"*. (Aiba and Murashima (2017, page 1). In Brazil, André Lara Resende (2017) argued in a contribution to Valor Econômico, a Brazilian financial newspaper, that high real interest rates in Brazil are simply the result of high nominal interest rates. His analysis is based on the analysis of John Cochrane in Cochrane (2016a), which has the FTPL as one of its key building blocks.

There are material real-world policy risks associated with the FTPL: policy disasters could happen if fiscal and monetary policy makers were to become convinced that the FTPL is the appropriate way to consider the interaction of monetary and fiscal policy in driving inflation, aggregate demand, real economic activity and sovereign default risk.

An implication of the FTPL is that monetary and fiscal policy makers – either acting in a cooperative and coordinated manner or acting in an independent and uncoordinated manner – can choose just about any paths or rules for real public spending on goods and services, real taxes net of transfers, policy interest rates and/or monetary issuance, now and in the future, without having to be concerned about meeting their contractual debt obligations. Somehow, the general price level (in the classic FTPL) or real aggregate demand (in the FTLEA version of Sims) is guaranteed to take on the value required to ensure that the real contractual value of the outstanding stock of nominal non-monetary public debt outstanding is always equal to the PDV of the current and future real augmented primary surpluses of the State.

Because this is manifestly incorrect (as shown in Sections 3 and 4) it could be extremely dangerous if taken seriously and acted upon by monetary and fiscal policy makers, as pointed out in Buiter (2017a, b, c). After all, what could be more appealing to a politician anxious to curry favor with the electorate through public spending increases and tax cuts, than the reassurance provided by the FTPL, that solvency of the State is never a problem? Regardless of the outstanding stocks of State assets and liabilities, the State can specify arbitrary paths or (contingent) rules for public spending, taxation, monetary issuance and/or nominal policy interest rates. Explosive sovereign bond trajectories will never threaten sovereign solvency. The general price level or real economic activity, working through the nominal and real discount factors, will do whatever it takes to make the real contractual value of the outstanding stock of nominal government bonds consistent with solvency of the State for arbitrary, non-Ricardian budgetary rules. If some misguided government were to take this delusional theory seriously and were to act upon it, the result, when reality belatedly dawns, could be painful fiscal tightening, government default, excessive recourse to inflationary financing and even hyperinflation.

(1.C.) What the FTPL is not

(1.C.a.) Unanticipated inflation and/or financial repression can reduce the real value of nominal public debt

In standard/conventional monetary economics, a change in the general price level changes the real value of the outstanding stock of nominal bonds. Indeed, when faced with imminent default on its debt, a government may opt for monetary financing of its deficits. Inflation that was unanticipated at the time that fixed-rate nominal debt was issued can cause the realized real interest rate to be lower than was expected when the debt was issued. Financial repression (keeping nominal interest rates artificially low) can result in a reduction in the real value of current and future nominal debt service even if the inflation is anticipated, because it stops nominal interest rates from rising with expected inflation. This accounted for a sizeable part of the reduction in debt-to-GDP ratios after World War II in the United Kingdom, the United States and in many other countries. But, this has nothing to do with the FTPL.

In Japan today, the monetary authorities *target the yield curve* – they set a minus 0.1 percent, interest rate on Policy-Rate Balances in financial institutions' deposit accounts at the Bank of Japan and target the yield on ten-year government bonds at near zero percent. The ten-year government bond market is one of the most liquid markets in Japan and assuming that risk premia are relatively small, the ten-year rate should be close to the average expected short-term policy rate over the next ten years. Thus, the overnight rate is expected to average near zero over the next ten years. If the markets expect a successful attainment of the two percent inflation target starting, say, two years from now, this would mean the average term premium over the next ten years is expected to be about *minus* 1.6 percent. However, term premia in liquid markets cannot be manipulated in such a significant and persistent manner just by varying the net supplies. Indeed, in Sims (2011) the strict expectations hypothesis of the term structure of interest rates links the price of a nominal perpetuity to the expected future path of instantaneous policy rate. This looks like classic financial repression.

(1.C.b) The fiscal theory of seigniorage: monetary policy has an unavoidable fiscal dimension

The size and composition of the balance sheet of the central bank have unavoidable fiscal implications. The fiscal theory of seigniorage is the right way of thinking about the inherent fiscal dimension of monetary policy (see Buiter (2003, 2007, 2014, 2017b, c).

Monetary policy has an inevitable fiscal dimension - one that has nothing to do with the FTPL/FTLEA. Central bank money is irredeemable and, except at the ELB, is willingly held even though it is pecuniary-rate-of-return dominated. Central banking therefore should be profitable, not only away from the ELB but even at the ELB. The fiscal theory of seigniorage recognizes that the national Treasury is the beneficial owner of the central bank and that, consequently, a monetized balance sheet expansion by the central bank increases fiscal space – relaxes the intertemporal budget constraint of the government. This fiscal space can be filled with tax cuts or higher public spending. Helicopter money is the parable of the fiscal dimension of monetary policy (see Friedman (1969) and Buiter (2003, 2007, 2014)).

The outline of the rest of the paper is as follows. Section 2 contains a rigorous but non-technical exposition of the main issues addressed in the paper. Section 3 provides a rigorous technical analysis of the original FTPL, which addressed a world with flexible nominal prices. We identify one inconsistency and six unacceptable anomalies. We also address the merits of treating the FTPL as an equilibrium selection mechanism when there are multiple equilibria. We extend the FTPL to an economy at the effective lower bound (ELB) and show that the most of the flaws of the FTPL are present also when the economy is at the ELB. In Section 4, we review recent attempts by Sims to make the case for non-Ricardian budgetary rules in a world with nominal price rigidities. We determine that the appropriate use by Sims of the IBC of the State in a New-Keynesian model does nothing to make a case for non-Ricardian budgetary rules. What we call Sims's fiscal theory of the level of economic activity (FTLEA) turns out to be a conventional analysis of fiscal stimulus and financial repression. The non-Ricardian budgetary rules proposed by Sims do not robustly rule out sovereign insolvency risk. The same applies to Sims's analysis of an Old-Keynesian model.

(2) The FTPL: a non-technical presentation

(2.A.) Key concepts and relationships

Consider a very simple closed endowment economy with two sectors: the household sector (represented by a representative household) and the State. The State (or the sovereign) is the consolidated general government and central bank. We will henceforth refer to the general government sector as the *Treasury* or the fiscal authority. There are no firms, because of the simple endowment technology and no financial institutions other than the central bank that is part of the State. There are four financial instruments: central bank money, which can only be issued by the State, a short (one-period or, in our continuous time formal model, instantaneous or zero-duration) nominal bond, a nominal consol (perpetuity) and a short index-linked bond. The formal model assumes there is no default and no default risk for both households and the State, so the bonds of the State and the bonds issued by the households are perfect substitutes.

We need the following terminology. The *contractual value* of a bond is the present discounted value (PDV) of its current and future contractual (or legally committed) debt service (interest payments or coupon payments plus repayment(s) of principal) discounted using default-risk-free discount factors. In principle, there could be other sources of risk in the economy (e.g. random shocks to the endowment, to the policy rules or to household preferences) that cause asset prices, interest rates and other endogenous variables to be different from what they were expected to be when contractual agreements were entered into. Even in the absence of uncertainty, the contractual value of longer-duration debt instruments can vary over time (as expected), as the exogenous drivers of the discount factors can be time-varying.

Formally we deal with a deterministic model without any risk or uncertainty. We do, however, adopt the common approach of considering a completely unexpected shock to one or more policy instruments. Let time, which we take to be continuous, be denoted by t. From the initial date, $t = t_0 = 0$, say, households expect the State to follow a fiscal-financial-monetary and interest rate program (henceforth a *budgetary rule*) under which solvency of the State is always guaranteed, in and out of equilibrium. This rule can, in principle, have time-varying parameters. Households hold their expectations concerning this rule with complete certainty and expect the budgetary rule to be followed forever. Then at some later date, $t = t_1 > 0$, there is a completely unexpected change to another budgetary rule. Households immediately expect that this new budgetary rule will be implemented forever. All endogenous variables in the economy will adjust to reflect the unexpected, permanent introduction of the new budgetary rule and the matching new set of firmly held beliefs held by the households.

Let g denote the real value of public spending on goods and services (exhaustive public spending), τ real taxes net of transfers, M the nominal stock of central bank money, i^M the interest rate on central bank money or base money, P the general price level, $\dot{M}(t) \equiv \frac{d}{dt}M(t)$ the instantaneous rate of change of the nominal stock of central bank money, $\sigma_1 = \left(\dot{M} - i^M M\right)/P$ the real value of a measure of flow 'seigniorage'- central bank monetary issuance net of interest paid on central bank money, and i the short (instantaneous) nominal interest rate on non-monetary financial claims (one-period or instantaneous risk-free nominal bonds). A *Ricardian* budgetary rule is a set of rules for real public spending on goods and services, real taxes net of transfers, the real value of seigniorage and policy rates that ensures that the State always satisfies its intertemporal budget constraint (IBC), for all values of the policy instruments and endogenous variables (current and anticipated future discount rates, the price level, the level of real economic variables and any exogenous parameters) that enter into the intertemporal budget constraint of the State, and with sovereign debt priced at its contractual value. Under a Ricardian budgetary rule, sovereign debt will always trade at its contractual value because, by construction, the State always satisfies its IBC, using default-risk-free discount factors and with sovereign bonds priced at their contractual values.

The condition for State bonds to be priced at their contractual value is that the present discounted value (using default-risk-free discount factors) of its current and future real primary surpluses, $s = \tau - g$, plus the PDV of the real value its current and future monetary issuance (net of any interest paid on money), σ_1 is greater than or equal to the real value of its net bond debt, when bonds are priced at their contractual values. The a*ugmented primary surplus,* $\hat{s} = \tau - g + \sigma_1$, is the sum of the primary surplus and monetary issuance minus interest paid on the outstanding stock of money. Note that we make the assumption that central bank monetary debt is free of default risk. Because it is an irredeemable 'liability' of the State, this is a fair assumption.

Any budgetary rule that does not ensure that the State always satisfies its IBC with sovereign bonds priced at their contractual values is called a *non-Ricardian* budgetary rule. It is of course possible that a non-Ricardian budgetary rule (or arbitrary budgetary rule) allows the State to satisfy its IBC, with sovereign bonds priced at their contractual values, *in equilibrium,* or for some other *subset* of the possible values that can be assumed by the exogenous, predetermined and endogenous variables that enter into the IBC of the State. Ricardian budgetary rules are *always* consistent with the State satisfying its IBC with its bonds priced at their contractual values, in and out of equilibrium. It is, however, easy to come up with examples of non-Ricardian budgetary rules that, for certain values of the parameters of the model and exogenous and the predetermined variables of the model, are consistent with the State satisfying its IBC in equilibrium, with State bonds priced at their contractual values. We don't, however, have a set of necessary and sufficient conditions for a budgetary rule to satisfy the IBC of the State. It is *sufficient* for the State to always satisfy its IBC in equilibrium, that the budgetary rule be Ricardian, because Ricardian rules ensure the solvency of the State both in and out of equilibrium. It is not necessary, however.

Any model that prices sovereign debt at its contractual value but imposes a non-Ricardian budgetary rule should be subjected to a consistency check in the form of a *counterfactual analysis*: assuming that current and future discount factors are free of default risk, is the IBC of the State indeed satisfied *in equilibrium* with the sovereign debt priced at its equilibrium value? If the answer is 'yes', the analysis can proceed. If the answer is 'no', the bonds of the State cannot be priced at their contractual values and the assumption that default-risk-free discount factors can be used is falsified. The model is not fit for purpose. Sovereign default risk and sovereign insolvency have to be considered and modeled explicitly. The terms on which a sovereign that is in default or at risk of default has access to the bond markets have to be made explicit etc.

The consistency check for any non-Ricardian rule is performed by using the IBC of the State as a counterfactual sovereign bond pricing equilibrium condition. Let $\hat{s}(t)$ be the real value of the augmented primary surplus of the State at time *t* (as defined above). Let $PDV(x;\ r,\ t)$ denote the present discounted value, at time t, of all current and future values of a real variable x over an infinite horizon when default-risk-free real discount factors are used to discount the future values of x; $PDV(y;\ i,\ t)$ is the present discounted value of a nominal variable y over an infinite horizon when default-risk-free nominal discount factors are used. $B(t)$ is the number of one period (instantaneous or zero-duration in continuous time) nominal government bonds outstanding at the beginning of period *t* (at time *t* in continuous time). The contractual value of such an instantaneous nominal bond is 1 unit of money. $B^{\ell}(t)$ is the number of nominal consols (perpetuities) outstanding at the beginning of period t (at time *t*); it is a promise to pay 1 unit of money in each period (at each point in time) forever. Its nominal contractual value at time t, denoted $P^{\ell}(t)$, is given by $P^{\ell}(t) = PDV(1;\, i,\, t)$; $b(t)$ is the number of one-period (instantaneous) index-linked government bonds outstanding at the beginning of period *t* (at time *t*): its nominal contractual value at time *t* is $P(t)$, the general price level in period *t* (at time *t*) – the reciprocal of the price of money in terms of the endowment commodity.

The intertemporal budget constraint of the State at time *t* is the requirement that the PDV of current and future augmented primary surpluses of the State is equal to or greater than the contractual value of the outstanding non-monetary debt of the State. It can be written as follows:

$$P(t)^{-1}\left(B(t) + PDV\left(1;\, i,\, t\right)B^{\ell}(t)\right) + b(t) \le PDV\left(\hat{s};\; r,\, t\right) \tag{2.1}$$

A *budgetary rule* at time t is a set of current and future values for, or a set of functions determining the current and future values for, the augmented primary surplus, the interest rate on money and either the short nominal interest rate on bonds or the nominal money stock, that is in effect at time *t* and is believed with complete certainty by households to be in effect at time t and for all future time; we denote it $\left\{\hat{s},\, i^{M},\, i \text{ or } M;\, t\right\}$. Assume the authorities have followed a Ricardian budgetary rule, $\left\{\hat{s}_R,\, i^{M}_R,\, i_R \text{ or } M_R;\, t < t_1\right\}$ until period t_1 (until the instant before time t_1) and that the private sector have, right up to period t_1, believed with complete certainty that this Ricardian rule would always continue to be implemented. In period t_1 (at time t_1) there is a 'black swan' event: unexpectedly the State changes its budgetary rule to a non-Ricardian rule, denoted $\left\{\hat{s}_N,\, i^{M}_N,\, i_N \text{ or } M_N;\, t \ge t_1\right\}$. This new non-Ricardian rule is again expected by the private sector, with complete confidence, to be implemented for all future time.

Under the old, Ricardian budgetary rule, it is true, by the definition of a Ricardian rule, that equation (2.1) is satisfied: $P(t)^{-1}\left(B(t) + PDV\left(1;\, i_R,\, t\right)B^{\ell}(t)\right) + b(t) \le PDV\left(\hat{s}_R;\, r,\, t\right)$ for $t < t_1$. The conventional or standard (non-FTPL) approach would check whether the assumption that the State remains solvent and there is no default risk under the new, non-Ricardian policy rule, by verifying whether $P(t_1)^{-1}\left(B(t_1) + PDV\left(1;\, i_N,\, t_1\right)B^{\ell}(t_1)\right) + b(t_1) \le PDV\left(\hat{s}_N;\, r,\, t_1\right)$ is satisfied in the new equilibrium. This can be done by treating the IBC of the State, holding with equality, as an equilibrium government bond pricing equation, determining the counterfactual market value of the bonds. If this counterfactual market value is equal to or greater than the contractual value of outstanding bond debt of the State, the maintained assumption of the model, that sovereign bonds are priced at their contractual values, is verified and the analysis can proceed. If it is falsified, a different model is required. A simple way of representing the counterfactual government bond pricing equilibrium condition is to write it as:

$$D(t_1)\left[P(t_1)^{-1}\left(B(t_1) + PDV\left(1;\, i_N,\, t_1\right)B^{\ell}(t_1)\right) + b(t_1) + b(t_1)\right] = PDV\left(\hat{s}_N;\; r,\, t_1\right) \tag{2.2}$$

where $D(t)$ is the 'bond revaluation factor' at time *t*, or the ratio of the (counterfactual) market value of the bonds to their contractual values. For simplicity, we assume all three bonds have the same revaluation factor. The market price of a bond obviously cannot be higher than its contractual value; government debt trades at its contractual value if the government is 'super-solvent', that is, if $P(t_1)^{-1}\left(B(t_1)+PDV\left(1;\, i_N,\, t_1\right)B^{\ell}(t_1)\right)+b(t_1)<PDV\left(\hat{s}_N;\, r\,,\, t_1\right)$. In this case, the government is wasting 'fiscal space'. Nor can the market value of a bond be negative, so, if $PDV\left(\hat{s}_N;\, r\,,\, t_1\right)<0$ and there is a positive net amount of government debt outstanding, the creditors of the sovereign get nothing – they cannot lose more than the entire contractual value of the government debt they hold. Therefore, if the net contractual value of the outstanding bonds is positive, we must have $0 \le D \le 1$.

Note for future reference that this counterfactual equilibrium bond pricing equation (2.2), works perfectly well if all government debt is index-linked or, in an open economy extension of the model, denominated in foreign currency. When all government bonds are index-linked, the counterfactual equilibrium government bond pricing equation simplifies to:

$$D(t_1)b(t_1)=PDV\left(\hat{s}_N;\, r\,,\, t_1\right) \qquad (2.3)$$

The FTPL introduces the intertemporal budget constraint of the State, holding with equality and with sovereign debt priced at its contractual value, as an additional equilibrium condition, but without adding the bond revaluation factor. So, it replaces equation (2.2) with

$$P(t_1)^{-1}\left(B(t_1)+PDV\left(1;\, i_N,\, t_1\right)B^{\ell}(t_1)\right)+b(t_1)=PDV\left(\hat{s}_N;\, r\,,\, t_1\right) \qquad (2.4)$$

The FTPL then assumes that the general price level, $P(t_1)$ can do the job of the bond revaluation factor. With all sovereign debt priced at its contractual value, the general price level jumps to the level required to ensure that the IBC of the State, holding with equality and with sovereign debt priced at its contractual value, is satisfied. Because a negative price level is not considered acceptable, if there is a non-zero stock of net nominal government bonds outstanding, the FTPL requires:

$$\text{sgn}\left\{B(t_1)+PDV\left(1;\, i_N,\, t_1\right)B^{\ell}(t_1)\right\}=\text{sgn}\left\{PDV\left(\hat{s}_N;\, r\,,\, t_1\right)-b(t_1)\right\} \qquad (2.5)$$

Consider the empirically most interesting case where there is positive net nominal public debt outstanding: $B(t_1)+PDV\left(1;\, i_N,\, t_1\right)B^{\ell}(t_1)>0$. In that case, the PDV of the augmented real primary surpluses has to be larger than or equal to the real value of the outstanding stock of index-linked public debt, if a negative price level is to be avoided.

The starting point in showing that the FTPL is a fallacy is quite independent of whether equation (2.4) solves for a positive value of the general price level. Equation (2.4), the IBC of the State holding with equality and with sovereign debt priced at its contractual value, *cannot* be imposed by the FTPL as an additional or independent equilibrium condition in a general equilibrium model that includes household consumption behavior derived from the optimizing behavior of forward-looking households. That is because this same IBC of the State (holding with equality and with sovereign debt priced at its contractual value), or its mirror image, the IBC of the optimizing household (holding with equality and with debt priced at its contractual value), has already been used to derive the optimal consumption rule. The only qualification of this "don't use the same equilibrium condition twice" requirement is to interpret the second application to be an equilibrium selection device in a model with multiple equilibria. We show below that this qualification cannot salvage the FTPL.

(2.B.) The FTPL is invalid economic theory because it uses the same equilibrium condition twice

It is intuitively obvious, and we show this rigorously in the formal model below, that, if the household utility function exhibits non-satiation (more is better) in consumption and/or real money balances, the IBC of the household will hold with equality: no resources that could be devoted to consumption or to accumulating additional real money balances are wasted.

The IBC of the household, holding with equality and with debt priced at its contractual value can, in equilibrium, be written as:

$$P(t)^{-1}\left(B(t)+PDV\left(1;\, i,\, t\right)B^{\ell}(t)\right)+b(t)=PDV\left(c+\tau+\sigma_1-y;\, r,\, t\right) \tag{2.6}$$

where y denotes real output (the real endowment) and c real household consumption. Equation (2.6) states that the real value of the net non-monetary financial assets held by the household equals the PDV of the augmented primary deficits of the household - its conventional primary deficit, $c+\tau-y$, plus the real value of its accumulation of money balances net of interest paid on money, σ_1.

In equilibrium (if an equilibrium exists), the economy-wide real resource constraint holds: real output (the real endowment) equals real household consumption plus real public spending on goods and services: $y=c+g$. Substituting the economy-wide real resource constraint into the household IBC, holding with equality and with debt valued at its contractual value turns the IBC of the household (equation (2.6)) into the IBC of the State, holding with equality and with the public debt priced at its contractual value – that is, it turns equation (2.6) into equation (2.1), holding with equality. It is a basic rule of sound general equilibrium economics that you cannot use the same equilibrium condition more than once. The FTPL therefore cannot impose the IBC of the State, holding with equality and with sovereign debt priced at its contractual value as an additional equilibrium condition when this 'additional' equilibrium condition is implied, in equilibrium, by another equilibrium condition, the IBC of the household, holding with equality and with debt priced at its contractual value, that has already been used in the derivation of the optimal household consumption rule. This fatal flaw invalidates the entire FTPL literature except for the New-Keynesian model developed in Sims (2011), which only uses the IBC of the State (or the household sector) once. The policy conclusions Sims draws from the Sims (2011) model are invalid for other reasons, as we show informally below in this Section and formally in Section 4.

(2.C.) The original FTPL, overdetermined systems, other inconsistencies and anomalies

Because the FTPL introduces an additional equilibrium condition (the IBC of the State, holding with equality and with sovereign bonds priced at their contractual values) without adding another endogenous variable (such as $D(t)$ in the standard approach of equation (2.2)), it should lead to an overdetermined system (more equations than unknows) in any model where the conventional approach yields a determinate equilibrium. And indeed, this is the case in many commonly used models, as we show in our formal model below in Section 3). There is, however, one class of models for which the standard approach results in indeterminacy of the all nominal variables – the general price level and the nominal money stock – although all real variables (including the stock of real money balances, the real interest rate, the rate of inflation, and the pecuniary opportunity cost of holding central bank money) are well-determined.

This is the class of models that has a freely flexible price level and a pegged risk-free (short) nominal interest rate on bonds.[4]

(2.C.a.) A flexible price level and a pegged nominal interest rate

When the short (instantaneous) nominal interest rate is pegged (set as an exogenous policy instrument or driven by a rule that does not make it a function of current and anticipated future nominal variables), the nominal money stock is endogenously determined. The monetary equilibrium condition in most standard models typically specifies the stock of real money balances demanded as an increasing function of some scale variable like real consumption, real output, real wealth or real transactions volumes, and a decreasing function of the difference between the short risk-free nominal interest rate on bonds and the nominal interest rate on money. In our formal models in Sections 3 and 4, real household consumption is the scale variable. The monetary equilibrium condition can be written as $M / P = \left(i - i^{M}\right)^{-1} \phi c,\ \phi > 0;\ i \geq i^{M}$.

In equilibrium, in the flexible price model, real household consumption equals the exogenous level of real output, $\bar{y}$, minus the level of real public spending on goods and services (also treated as exogenous for simplicity) $\bar{g}$. With both the nominal interest rate on bonds and the nominal interest rate on money pegged (and assuming we are away from the effective lower bound with the safe nominal interest rate on bonds higher than the own interest rate on money), the equilibrium stock of real money balances is uniquely determined:

$$M / P = \left(\bar{i} - \bar{i}^{M}\right)^{-1} \phi\left(\bar{y} - \bar{g}\right), \quad i > i^{M} \tag{2.7}$$

But neither the price level nor the nominal money stock are determined. This flexible price level, pegged nominal interest rate world is the only one where imposing the IBC of the State, holding with equality and with the sovereign debt priced at its contractual value, does not lead to a mathematically overdetermined system. We still have the fatal flow of a model that is misspecified from an *economic* perspective (the same equilibrium condition is imposed twice) but we don't have the problem of mathematical overdeterminacy.

The suggestion of Kocherlakota and Phelan (1999) that the FPTL be viewed as an equilibrium selection device to resolve the indeterminacy of the price level and the nominal money stock in the flexible price level model under a nominal interest rate rule is, in our view, the only conceivable rationalization for using the same equilibrium condition twice. However, in games or models with multiple equilibria, the selection mechanisms that are favored in the literature are those that select 'natural focal points'. Using the IBC of the State twice (holding with equality and with sovereign debt priced at its contractual value) does not, in our (admittedly subjective) view meet the 'natural focal point' criterion. More importantly, it does not cure the following five defects of the equilibrium selected in this manner:

(1) The price level can be negative unless condition (2.5), is satisfied.
(2) The theory ceases to function when all government debt is index-linked (or, in an open economy, denominated in foreign currency).
(3) The FTPL determines the price of money even in a world where there is no money except as an abstract numeraire, like phlogiston, the (imaginary) substance believed, in the pre-scientific world, to cause

[4] The own (nominal) interest rate on money will be treated as exogenous throughout and plays no role in our rejection of the FTPL as a false theory.

combustion in materials. The ability to price phlogiston, and to determine an equilibrium price without an associated quantity is a bridge too far, in our view.

(4) The logic of the FTPL can be applied to the IBC of the household sector or indeed to the IBC of an individual household, as long as it has positive nominal debt outstanding, follows a non-Ricardian rule for consumption and money accumulation, and the analogue of condition (2.5) is satisfied. The household theory of the price level (HTPL) or the Joneses' theory of the price level (JTPL) have equal standing (none, that is, in our view) with the FTPL.

(5) When the (counterfactual) equilibrium bond pricing equation is specified properly, say by introducing an addition endogenous variable like the bond revaluation factor D in equation (2.2) (thus introducing the market value of the bonds as a counterfactual separate variable from its contractual value), there is no FTPL. If there is positive net nominal sovereign debt outstanding, equation (2.2) determines the real market value of the outstanding public debt, $D(t_1)\left[\left(B(t_1)+PDV\left(1;\, i_N,\, t_1\right)B^{\ell}(t_1)\right)P(t_1)^{-1}+b(t_1)\right]$, but not $D(t_1)$ and $P(t_1)$ separately (nor the nominal money stock). When there is only index-linked debt, equation (2.2) determines $D(t_1)$ but not the price level or the nominal money stock, which are indeterminate.

Finally, if we were to accept the IBC of the State, holding with equality and with sovereign bonds priced at their contractual values, as the equilibrium selection device in the flexible price level model when the nominal interest rate is exogenous, it would surely make sense to also use it as the equilibrium selection device in the flexible price level model when the nominal money stock is exogenous. This model too has, as pointed out below in Section (2.C.b.), a continuum of price level equilibria for a given path of the current and future nominal money stock. The problem is that, in this case, the use of the FTPL equilibrium selection criterion would, almost surely, lead to the selection of a bubble equilibrium, with the real stock of money balances either exploding or imploding to zero.

(2.C.b.) A flexible general price level and a monetary rule

When the nominal money supply is exogenous or driven by a rule that does not depend on the general price level (current or future anticipated values) or anticipated future values of the nominal money stock, the equilibrium is overdetermined when we impose the IBC of the State, holding with equality and with the public debt priced at its contractual value, even when the price level is freely flexible We have two equilibrium conditions determining the price level: the monetary equilibrium condition, reproduced as equation (2.8) below and the IBC of the State, holding with equality and with sovereign priced at its contractual value, equation (2.1) holding with equality.

$$M\ /\ P=\left(i-\overline{i}^{\,M}\right)^{-1}\phi\left(\overline{y}-\overline{g}\right),\quad i>i^{M} \tag{2.8}$$

Some care is needed with this statement, because it is well-known that flexible price level models of the kind analyzed in the FTPL literature have infinitely many equilibria under an exogenous rule for the nominal money stock. Consider the simple case of a constant nominal money stock, a constant endowment, a constant level of real public spending on goods and services, a constant time preference rate (ensuring a constant equilibrium real interest rate) and a constant nominal interest rate on central bank money, that is below the (endogenous) short nominal interest rate on bonds.

In the standard approach (without double use of the IBC of the State) such an economy has a barter equilibrium with $\frac{1}{P(t)} = 0$ for all time. It has one 'fundamental' equilibrium, which will have a constant price level (and a nominal interest rate equal to the real interest rate). And it has infinitely many sunspot or bubble equilibria, which can either be inflationary or deflationary (see Buiter and Sibert (2007). In equation (2.8), even with a constant nominal money stock, the price level can rise without bound, reducing the real money stock to zero and pushing the nominal interest rate towards infinity or it can fall without bound, raising the real money stock to zero and driving the nominal interest rate down to the ELB value of $\overline{i}^{M}$.

Note that this multiplicity of equilibria is different from the indeterminacy in the conventional approach under an interest rate rule. Under the interest rate rule, neither the price level nor the nominal money stock are determined. Under the monetary rule, the nominal money stock is (by construction) determined but there is a continuum of equilibria for the price level and the nominal interest rate. Our approach in the formal model is to select among this continuum of possible solutions for the current and future price level using the equilibrium selection criterion that stationary exogenous variables support stationary endogenous variables. We view this 'fundamental' solution as the 'natural focal point'. If we select the fundamental solution, adding the IBC of the State, holding with equality and with sovereign debt priced at its contractual value, as another equilibrium condition, the model is overdetermined.

Can we use the IBC of the State as an equilibrium selection device when there is this continuum of equilibria for the price level and the rate of inflation under a monetary rule? In principle, yes, because in the absence of a 'theory of equilibrium selection rules' anything can be an equilibrium selection rule. The FTPL equilibrium selection criterion appears highly unusual, however. Why would an equilibrium condition that has already been used to construct the equilibria of a model be used again to select among the multiple equilibria of the model? In the case where the nominal money stock is constant, unless the IBC of the State picks the stationary, 'fundamental' solution by happenstance, the FTPL solutions will be inflationary or deflationary bubbles with the nominal interest rate rising without bound or falling to the level of the interest rate on money. The FTPL would, under an exogenous nominal interest rate rule (assuming a constant nominal interest rate for simplicity) produce a possibly time-varying inflation rate driven by the evolution over time of the IBC of the State, with the real interest rate endogenously determined and, in our simple model, a constant stock of real money balances.

(2.C.c.) The original FTPL and fiscal dominance

The FTPL and non-Ricardian policies are frequently identified with fiscal dominance or active fiscal policy and passive monetary policy. This makes no sense. Ricardian budgetary rules can have either monetary or fiscal dominance, as the famous 'Unpleasant Monetarist Arithmetic' paper by Sargent and Wallace (1981) shows. Before the public debt reaches the (exogenously given) upper bound, monetary policy is active - the growth rate of the nominal money stock is exogenous. Government borrowing is passive. Once the debt ceiling is reached, monetary growth passively finances the public-sector deficit (public spending and taxes don't change).

The fiscal dimension of monetary policy (and specifically of central bank monetized balance sheet expansion) exists even if the central bank is operationally independent and even if there is 'monetary dominance' (or active monetary policy and passive fiscal policy) rather than the 'fiscal dominance' (or active fiscal policy and passive monetary policy), that characterizes the 'Unpleasant Monetarist Arithmetic' model after the ceiling on the government debt-to-GDP ratio is reached. The key insight is that, given the outstanding stocks of State assets and liabilities, if you want to ensure the State remains solvent (if you want a Ricardian budgetary rule), you cannot specify monetary policy (base money issuance) and fiscal policy (public spending and taxes) independently. Either there is a cooperative solution, or there is fiscal dominance and monetary issuance becomes endogenously determined (the residual), or there is monetary dominance and public spending and/or taxation have to adjust (becomes the residual) to maintain sovereign solvency.

(2.D.) The FTPL and sticky nominal prices

When the price level is predetermined (and updated, say, through an Old-Keynesian or New-Keynesian Phillips curve), it obviously cannot jump endogenously at $t = t_1$ to the level required to make the IBC of the State hold with equality, with the sovereign debt priced at its contractual value. If we impose the IBC of the State as an equilibrium condition *and* have also used the equilibrium mirror image of the IBC of the State – the IBC of the household, holding with equality and with household debt priced at its contractual value – to determine the optimal consumption rule, the system is overdetermined. There is no 'equilibrium selection mechanism' escape valve - however unconvincing one may consider such an escape valve to be in the flexible price level models.

(2.D.a.) Sims's new FTPL – the FTLEA

Sims (2011) does not fall into the overdeterminacy trap. In his analytical and numerical models, household consumption behavior is characterized by the Euler-equation for consumption (growth) – a first-order differential equation. The IBC of the household, holding with equality and with household and sovereign debt priced at its equilibrium value - the boundary condition which, together with the Euler equation, permits one to solve for optimal consumption behavior – is *replaced*, as an equilibrium condition, with the IBC of the State holding with equality and with sovereign debt priced at its contractual value. This, however, does not mean that all is well with the conclusions of the Sims (2011) model. Why would arbitrary non-Ricardian fiscal rules be consistent with government solvency if the price level cannot jump to the level necessary to satisfy equation (2.4) or, equivalently in equilibrium, equation (2.6)?

Sims argues that the default-risk-free real and nominal discount factors in equation (2.4), represented by $PDV\left(1;\, i_N,\, t_1\right)$ and $PDV\left(\hat{s};\, r_N,\, t_1\right)$ can do the job of ensuring sovereign solvency. According to Sims these discount factors (current and anticipated future default-risk-free nominal and real interest rates) will jump in the desired manner – to ensure sovereign solvency - when a non-Ricardian rule is unexpectedly introduced at t_1. These discount factors can indeed jump when a surprise hits the system, because household consumption, which is chosen by forward-looking optimizing households, can jump when the non-Ricardian rule is introduced unexpectedly at t_1. Because it is not the price level that jumps but consumption, and with it the demand-determined level of real economic activity, we call the Sims (2011) New-Keynesian model the fiscal theory of the level of economic activity (FTLEA).

Can a jump in consumption (and presumably, in a richer model, consumption and real capital expenditure) really do the job of setting the nominal and real discount factors at values that ensure government solvency? They could for certain non-Ricardian rules and for certain values of the exogenous variables, parameters and initial values of the predetermined state variables (we provide an example), but it is trivial to come up with examples of non-Ricardian rules that cannot do the job and will violate the IBC of the State with sovereign debt priced at its contractual value. Note that, in the flexible price level model too, nominal and real discount factors can jump, because there also, household consumption is non-predetermined, driven by optimizing, forward-looking households. Household consumption therefore can, in principle, jump in response to news. Of course, with real government spending and real output constant, equilibrium consumption will not jump in the classical model of the original FTPL. Because of that, real interest rates too will not change in equilibrium. If the nominal interest rate is pegged at the same level in both the Ricardian regime (pre-t_1) and the non-Ricardian regime, following the unexpected regime change at t_1, the nominal discount factors also would not change in the flexible price level model.

Because we have no way of determining a-priori whether an arbitrary, non-Ricardian budgetary rule is consistent with government solvency when the economic model is specified properly – that is, without double use of the IBC of the State - one always should do a counterfactual analysis, using equation (2.2), to determine whether the budgetary rule in question does indeed satisfy the IBC of the State in equilibrium, holding with equality and with sovereign debt priced at its contractual value, for a robust range of initial conditions and values of the exogenous variables and parameters. If it does, all is well. If the PDV of current and future real augmented primary surpluses exceeds the real contractual value of the outstanding sovereign debt, the sovereign is wasting fiscal space. If the PDV of current and future real augmented primary surpluses falls short of the real contractual value of the outstanding sovereign debt, there is at the very least default risk and possibly actual sovereign default and sovereign insolvency. The assumptions on which the model is based – default-risk-free bond pricing - are then falsified. The household cannot satisfy its IBC with equality and with its holdings of sovereign bonds valued at their contractual values, because the market value of that sovereign debt will be less than its contractual value. Depending on the procedures for dealing with sovereign default (including the seniority of old and new creditors of the government) the terms of access of the State to the bond markets will be different. The maintained assumption of no sovereign default risk and no sovereign default have been falsified. The model is not fit for purpose.

(3) The original FTPL: a more rigorous presentation

We first state the key results concerning the original FTPL for the case where the economy is never at the ELB. We choose the sequence of nominal interest rates or the sequence of nominal money stocks in such a way that the nominal interest rate on bonds exceeds the nominal interest on money in each period.

We employ a deterministic, continuous-time model. Time, *t*, begins at time zero and proceeds to infinity. There is a single, perishable consumption good and the model is inhabited by an infinite-lived government and a representative infinite-lived household.

(3.A.) The State

At each instant, the state collects real taxes τ and buys an amount g of the good. Variables depend on time, but this is suppressed in the notation where there is no ambiguity. The asset menu is the same as in Section 2. Notation used in Section 2 carries over to the rest of the paper.

The State's within-period budget constraint is thus

$$\frac{\dot{M}+\dot{B}+P^{\ell}\dot{B}^{\ell}}{P}+\dot{b}=g-\tau+\left(\frac{i^{M}M+iB+B^{\ell}}{P}\right)+rb, \tag{3.1}$$

Arbitrage implies that the expected return on real bonds and the expected real return on consols must equal the expected real return on (instantaneous) nominal bonds. Thus,

$$r = i - \pi \tag{3.2}$$

$$(1 + \dot{P}^{\ell}) / P^{\ell} = i, \tag{3.3}$$

where $\pi \equiv \dot{P} / P$ is the expected rate of inflation. With perfect foresight except at the point in time, $t = t_1 > 0$, when the authorities switch unexpectedly from a Ricardian to a non-Ricardian budgetary rule, actual and expected returns and inflation rates are the same. Solving (3.3) forward and imposing a no-bubble terminal condition yields that the price of a consol is equal to the present discounted value of its coupon payments.[5]

$$P^{\ell}(t) = \int_t^{\infty} \exp\left(-\int_t^{v} i(u)du \right) dv. \tag{3.4}$$

Substituting equations (2) and (3) into equation (1) yields

$$\dot{l} + \dot{m} = rl - (\pi - i^M)m + g - \tau, \tag{3.5}$$

where $l \equiv b + (B + P^{\ell} B^{\ell}) / P$ is the real value of non-monetary debt and $m \equiv M / P$ is the real money supply. Solving equation (5) forward yields the State's intertemporal budget identity

$$\begin{aligned} l(t) + m(t) = &\int_t^{\infty} \exp\left(-\int_t^{v} r(u)du\right)\left[\tau(v) - g(v) + \left(i(v) - i^M(v)\right)m(v)\right]dv \\ &+ \lim_{v\to\infty}\left\{\exp\left(-\int_t^{v} r(u)du\right)[l(v) + m(v)]\right\}, \end{aligned} \tag{3.6}$$

where $\sigma_2 \equiv (i - i^M)m$ can be viewed as another flow measure of real seigniorage.

The no-Ponzi game or solvency constraint requires that the present discounted value of the terminal value of the State's *non-monetary* liabilities is non-positive in the limit as the terminal date goes to infinity:

$$\lim_{v\to\infty}\left[\exp\left(-\int_t^{v} r(u)du\right)l(v)\right] \le 0. \tag{3.7}$$

Note that equation (3.7) does not put any restriction on what happens to the present discounted value of the terminal money supply. This is because central bank money is irredeemable: a holder of a central bank's money can never compel the central bank to exchange it for anything other than the same amount of the central bank's money. Although money is perceived as an asset by private holders, it is not in any meaningful sense a liability of the central bank. This asymmetry matters for monetary policy effectiveness at the effective lower bound (ELB) but is not relevant to our discussion of the FTPL.

Substituting equation (7) into equations (6) yields the IBC of the State:

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[5]The terminal condition in question is $\lim_{v\to\infty} P^{\ell}(v) \exp\left(-\int_t^{v} i(u)du \right) dv = 0.$

$$l(t)+m(t)\le\int_t^{\infty}\exp\left(-\int_t^v r(u)du\right)\left[\tau(v)-g(v)+\left(i(v)-i^M(v)\right)m(v)\right]dv$$
$$+\lim_{v\to\infty}\left[\exp\left(-\int_t^v r(u)du\right)m(v)\right]. \tag{3.8}$$

Through integration by parts, it can be seen that the two flow seigniorage measures, $\sigma_1\equiv\left(\dot{M}-i^M M\right)/P$ and $\sigma_2\equiv(i-i^M)m$ are related as follows:

$$\int_t^{\infty}\exp\left(-\int_t^v r(u)du\right)\sigma_1(v)dv=\int_t^{\infty}\exp\left(-\int_t^v r(u)du\right)\sigma_2(v)dv+\lim_{v\to\infty}\left[\exp\left(-\int_t^v r(u)du\right)m(v)\right]-m(t) \tag{3.9}$$

The IBC of the State can therefore also be written as follows:

$$l(t)\le\int_t^{\infty}\exp\left(-\int_t^v r(u)du\right)\left[\tau(v)-g(v)+\left(\mu(v)-i^M(v)\right)m(v)\right]dv, \tag{3.10}$$

where $\mu\equiv\dot{M}/M$ is the proportional growth rate of the nominal stock of central bank money. Let $\hat{s}_1$ be one measure of the real value of the augmented primary surplus of the State, that is,

$$\hat{s}_1\equiv s+\left(\mu-i^M\right)m, \tag{3.11}$$

where $s\equiv\tau-g$ is the real value of the primary surplus of the State. The other measure of the real value of the augmented primary surplus, $\hat{s}_2$, is defined as:

$$\hat{s}_2\equiv s+\left(i-i^M\right)m \tag{3.12}$$

The IBC of the State can therefore be written compactly as in equation (3.13):

$$l(t)\le\int_t^{\infty}\exp\left(-\int_t^v r(u)du\right)\hat{s}_1(v)dv \tag{3.13}$$

In equations (3.8) and (3.10) all three types of bonds are valued at their contractual values. Budgetary policies that always satisfy equation (3.8) or, equivalently, (3.10), are called *Ricardian*.

If instead of the solvency condition for the State given in equation (3.7), we were to adopt the solvency constraint of the State used in the FTPL literature - the condition that the PDV of the sum of the monetary *and* non-monetary liabilities of the State is non-positive asymptotically - i.e. that

$$\lim_{v\to\infty}\left[\exp\left(-\int_t^v r(u)du\right)\left(l(v)+m(v)\right)\right]\le 0. \tag{3.14}$$

the IBC of the State given in equation (3.8) changes to:

$$l(t)+m(t)\le\int_t^{\infty}\exp\left(-\int_t^v r(u)du\right)\hat{s}_2(v)dv. \tag{3.15}$$

The equivalent version of the ICB of the State given in equation (3.10) becomes:

$$l(t)\le\int_t^{\infty}\exp\left(-\int_t^v r(u)du\right)\hat{s}_1(v)dv-\lim_{v\to\infty}\exp\left(-\int_t^v r(u)du\right)m(v) \tag{3.16}$$

Equation (3.16) is not what is used in the FTPL literature, which instead uses our equation (3.13), holding with equality.

The right-hand sides of these weak inequalities will of course be the same when the PDV of the terminal money stock is zero. Because our rejection of the FTPL does not depend on it, we shall assume in what follows, unless stated explicitly otherwise, that

$$\lim_{v\to\infty} \exp\left(-\int_t^v r(u)du\right) m(v) = 0. \tag{3.17}$$

As noted earlier, this assumption only matters for monetary policy effectiveness at the ELB.

For non-Ricardian policies, equation (3.10) is replaced by the counterfactual government bond pricing equation or solvency test, equation (3.18), which says that the real *market value* of non-monetary government debt equals the PDV of the sum of current and anticipated future augmented primary surpluses, discounted using default-risk-free discount factors.

$$D(t)l(t) = \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\hat{s}_1(v)dv \tag{3.18}$$

The bond revaluation factor, D, is the ratio of the market value of the debt to its contractual value. If it were a true (non-counterfactual) bond pricing equation, D satisfies the following conditions:

$$\begin{aligned}
&0 \le D(t) \le 1 \\
&D(t) = 1 \text{ if } \; P(t)^{-1}\left(B(t) + P^{\ell}(t)B^{\ell}(t)\right) + b(t) \le \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\hat{s}_1(v)dv \\
&\qquad\quad \text{and} \quad P(t)^{-1}\left(B(t) + P^{\ell}(t)B^{\ell}(t)\right) + b(t) > 0 \\
&D(t) = 0 \text{ if } \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\hat{s}_1(v)dv \le 0 \; \text{ and } P(t)^{-1}\left(B(t) + P^{\ell}(t)B^{\ell}(t)\right) + b(t) > 0
\end{aligned} \tag{3.19}$$

If $D \ge 1$ the State is solvent and if $D > 1$ the State is 'super-solvent' and is wasting 'fiscal space'. If $D < 1$ the maintained assumption that all sovereign bonds are priced at their contractual values, is falsified. The model is invalid. Any value of $D(t)$ that solves (3.18) and is equal to or greater than 1 verifies the maintained assumption that the State is solvent. As far as the characterization of the State is concerned, the model is fit for purpose and the analysis can proceed. However, if the value of $D(t)$ that solves (3.18) is greater than 1, the household sector would be violating its IBC in equilibrium, as shown in Sections 3.B. and 3.C., and the model would have to be revised to allow for household insolvency, default risk and default. So only the outcome $D(t) = 1$ in the counterfactual analysis would allow the model solution to proceed without modifying some part of the model.

In the equilibrium bond pricing approach, the budgetary policies are whatever they are. Essentially arbitrary sequences or rules for public spending, taxes, monetary issuance and policy rates are permitted. Such budgetary policies that are not required to satisfy the intertemporal budget constraint in and out of equilibrium, that is for all possible values of the variables entering the IBC of the State with debt priced at its contractual value, are called *non-Ricardian* budgetary policies.[6]

[6] In Niepelt (2004), the author argues that even if one accepts the valuation equation approach, this implies an intertemporal budget constraint if one imposes rational expectations and if one goes back to a truly initial period where

Our analysis is restricted to the case where $l > 0$ and the government is a net debtor. If it were a net creditor, it would be necessary to verify the solvency of those who issued the bonds held by the State.

The FTPL considers non-Ricardian budgetary programs but does not add a bond revaluation factor to the equilibrium bond pricing equation. It effectively sets $D(t) \equiv 1$ in the counterfactual bond pricing equilibrium condition, (3.18). Since the FTPL adds an additional equation - (3.10) holding with equality – but does not add another unknown, any model of the economy that has a determinate equilibrium under Ricardian budgetary programs should be overdetermined under a non-Ricardian budgetary program, that is, mathematically inconsistent with more equations than unknowns.

With multiple bonds, the State has to choose not just the mix of monetary financing vs. bond financing of its budget deficits, but also the composition of its bond financing.

(3.B.) The household sector

The representative household receives an exogenous endowment $y > 0$ each instant, consumes $c \geq 0$ and pays a net real lump-sum tax and chooses money and bond holdings. The instantaneous budget identity for the representative household is:

$$\dot{l} + \dot{m} = y - \tau - c - (\pi - i^M)m + rl. \tag{3.20}$$

Solving equation (3.20) forwards to find the intertemporal budget constraint of the household yields:

$$\begin{aligned} l(t) + m(t) = &\int_t^\infty \exp\left(-\int_t^v r(u)du\right)\left[c(v) + \left(i(v) - i^M(v)\right)m(v) + \tau(v) - y(v)\right]dv \\ &+ \lim_{v\to\infty}\left\{\exp\left(-\int_t^v r(u)du\right)[l(v) + m(v)]\right\}. \end{aligned} \tag{3.21}$$

The household solvency constraint is the condition that the PDV of its terminal net financial debt must be non-negative

$$\lim_{v\to\infty}\left\{\exp\left(-\int_t^v r(u)du\right)[l(v) + m(v)]\right\} \geq 0. \tag{3.22}$$

By equations (3.21) and (3.22) the intertemporal budget constraint becomes

$$l(t) + m(t) \geq \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\left[c(v) + \left(i(v) - i^M(v)\right)m(v) + \tau(v) - y(v)\right]dv. \tag{3.23}$$

Using (3.9) the IBC of the representative household can equivalently be written as:

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debt is issued. Once this is accepted, the nominal anchor disappears and the possibility to run "arbitrary" (non-Ricardian) fiscal policies disappears as well—the price level cannot be relied upon to satisfy the IBC. If a bond revaluation factor less than 1 were to occur in the initial period when a government bond is issued, it would not be possible to price that bond at par. The State either sells it at the appropriate discount to its contractual value or the State cannot sell the bond. Niepelt's analysis and mine are substantially the same. We am indebted to Dirk Niepelt for pointing this out to us. Of course, as pointed out in Daniel (2007), if, in that initial period, all the necessary conditions for the FTPL to generate an equilibrium price level sequence are satisfied (flexible prices, exogenous nominal interest rate, non-zero stock of government bonds), the FTPL might be able to pick a price level sequence that yields a unique, non-explosive equilibrium. This amounts to 'relocating' the FTPL to the initial period. Even if there is no inconsistency (overdeterminacy) in this case, the five anomalies introduced below still invalidate the FTPL.

$$l(t) \geq \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\left[c(v) + \left(\mu(v) - i^M(v)\right)m(v) + \tau(v) - y(v)\right]dv - \lim_{v\to\infty} \exp\left(-\int_t^v r(u)du\right)m(v)$$

If equation (3.17) hold, this becomes:

$$l(t) \geq \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\left[c(v) + \left(\mu(v) - i^M(v)\right)m(v) + \tau(v) - y(v)\right]dv \tag{3.24}$$

Equation (3.23) and the equivalent equation (3.24) will hold with equality if there is non-satiation in either consumption or real money balances or both. The household utility function introduced below has that property.

It is ensured that, away from the ELB, households will hold central bank money that is pecuniary-rate-of-return-dominated by bonds by making real money balances an argument in the direct utility function. Alternatively, one could specify a transactions technology where money saves on real resources or some varieties of the cash-in-advance model. The representative household takes as given its initial money and bond holdings, $M(0) = M_0 \geq 0$, $B(0) = B_0$, $\mathrm{B}^\ell(0) = B_0^\ell$ and $b(0) = b_0$, and chooses bond holdings and non-negative consumption and money balances to maximize

$$\begin{gathered}\int_v^\infty \exp(-\delta(v-t))\left(\ln c(t) + \phi \ln m(v)\right)dv, \\ \delta > 0;\ \phi \geq 0\end{gathered} \tag{3.25}$$

subject to its IBC (3.23) or (3.24).

If $i > i^M$ (as we assume until further notice), then optimality requires, for $t \geq 0$

$$c(t)^{-1} = \eta(t) \tag{3.26}$$

$$\phi m(t)^{-1} = \left(i(t) - i^M(t)\right)\eta(t) \tag{3.27}$$

$$\dot{\eta}(t) = \left(\delta - r(t)\right)\eta(t) \tag{3.28}$$

$$\eta(t)\left[l(t) + m(t) - \int_t^\infty \exp\left(-\int_t^v r du\right)\left[c(v) + \left(i(v) - i^M(v)\right)m(v) + \tau(v) - y(v)\right]dv\right] = 0 \tag{3.29}$$

where $\eta(t)$ is the costate variable in the Hamiltonian dynamics or, equivalently, the Lagrange multiplier attached to the IBC of the household. Because the marginal utility of consumption and money balances are always positive (non-satiation is assumed), $\eta(t) > 0$. Equation (3.29) then implies that the IBC of the optimizing household holds with equality:

$$l(t) + m(t) = \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\left[c(v) + \left(i(v) - i^M(v)\right)m(v) + \tau(v) - y(v)\right]dv \tag{3.30}$$

or, equivalently:

$$l(t) = \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\left[c(v) + \left(\mu(v) - i^M(v)\right)m(v) + \tau(v) - y(v)\right]dv \tag{3.31}$$

Equations (3.30), (3.26), (3.27) and (3.28) allow us to obtain the following closed form household consumption function and money demand function:

$$c(t)=\delta(1+\phi)^{-1}\left[l(t)+m(t)+\int_t^{\infty}\exp\left(-\int_t^v r(u)du\right)\left(y(v)-\tau(v)\right)dv\right] \tag{3.32}$$

$$m(t)=\phi\left(i(t)-i^M(t)\right)^{-1}c(t) \tag{3.33}$$

(3.C.) The FTPL equilibrium in the flexible price level model

We first consider the original FTPL case of a freely flexible general price level. Until further notice, all equilibria considered are away from the ELB: $i>i^M$. For simplicity, it is assumed that the endowment, real government spending on the commodity and the nominal interest rate on money are constant: $y=\overline{y},\ g=\overline{g}<\overline{y}$ and $i^M=\overline{i}^M$. If the endowment is interpreted as labor time, there is continuous full employment in this version of the model: aggregate demand always equals the exogenous endowment. For $t\geq 0$

$$c(t)+\overline{g}=\overline{y}. \tag{3.34}$$

With constant consumption, the costate variable is constant and the following conditions will have to be satisfied in any equilibrium where the economy is never at the effective lower bound.

$$r(t)=\delta \tag{3.35}$$

$$i(t)=\delta+\pi(t) \tag{3.36}$$

$$l(t)+m(t)=\int_t^{\infty}\exp\left(-(v-t)\delta\right)\left[(i-\overline{i}^M)m+\tau-\overline{g}\right]dv. \qquad (3.37)[7]$$

For non-Ricardian budgetary rules, the conventional approach should add the intertemporal budget constraint of the government, holding with equality, as a counterfactual sovereign bond pricing equilibrium condition

$$D(t)l(t)=\int_t^{\infty}\exp\left(-(v-t)\delta\right)\left[\left(i(v)-\overline{i}^M\right)m(v)+\tau(v)-\overline{g}\right]dv-m(t)$$

or (3.38)

$$D(t)l(t)=\int_t^{\infty}\exp\left(-(v-t)\delta\right)\left[\left(\mu(v)-\overline{i}^M\right)m(v)+\tau(v)-\overline{g}\right]dv$$

When Ricardian budgetary programs are considered and the State is not wasting fiscal space, $D\equiv 1$ and equation (3.38) can be omitted from the rest of the analysis, which can proceed because the maintained assumption in the model — that sovereign bonds trade at their contractual values — has been verified. When non-Ricardian budgetary programs are considered, the model is solved including equation (3.38). If, in equilibrium, $D\geq 1,\ t\geq 0$, a maintained assumption in the model — that sovereign bonds trade at their contractual values — has been verified and the analysis can proceed. If $D<1$, that maintained assumption

[7] $\lim_{v\to\infty}\left[\exp\left(-\delta(v-t)\right)m(v)\right]=0$ is assumed to make for complete comparability with the original FTPL analysis and because for our purposes nothing depends on this assumption.

is false and the model is inconsistent. If $D > 1$ the maintained assumption that households satisfy their IBCs is falsified.

The FTPL adds the ICB of the State, holding with equality and pricing sovereign bonds at their contractual values, as an equilibrium condition. No additional endogenous variable, like the market value of the debt or $D(t)$. the ratio of its market value to its contractual value is added to the model. The FTPL therefore adds:

$$l(t) = \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\left[\tau(v) - \overline{g} + \left(\mu(v) - i^M(v)\right)m(v)\right]dv \tag{3.39}$$

Note the problem this creates: optimality conditions of the household consumption and asset allocation program imply that the IBC of the household holds with equality (equation (3.31)). Substituting the economy-wide equilibrium condition in the output market (or the equilibrium economy-wide real resource constraint (equation (3.34)) into the household IBC, holding with equality (equation (3.31), yields equation (3.39), the IBC of the State, holding with equality and with government bonds priced at their contractual values. The FTPL cannot impose equation (3.39) because it has already been used to derive the optimal consumption program (equations (3.26) to (3.29)), and the money demand function and consumption function they imply (equations (3.33) and (3.32)).

We are now ready to consider the inconsistencies and anomalies in the FTPL.

(3.C.a.) The nominal money stock is the policy instrument

If the central bank sets the nominal stock of money either through some exogenous open-loop rule or through some feedback rule that does not depend on current and anticipated future values of the general price level and the nominal money stock, then the instantaneous nominal interest rate on bonds, i, is endogenously determined.

It is supposed that the central bank sets a constant proportional money growth rate for the nominal money stock that keeps the short nominal interest rate above the ELB.

$$\dot{M} / M = \overline{\mu} > \overline{i}^M - \delta. \tag{3.40}$$

It is well-known that, with flexible prices, there are infinitely many equilibria under this policy rule. There is the barter equilibrium where money has no value: $1/P(t) = 0,\ t \geq 0$. Then there is a continuum of non-fundamental or bubble equilibria where, despite the fact that all the exogenous variables of the model are constant, the value of real balances goes to infinity or zero. (See Buiter and Sibert (2007)). We consider the unique fundamental or stationary equilibrium where real balances are constant. By (3.40), $\dot{m} = (\overline{\mu} - \pi)m$ and hence

$$\pi(t) = \overline{\mu},\ t \geq 0. \tag{3.41}$$

By (3.36) this determines the (constant) nominal interest rate. Thus, we can use (3.33) and (3.34)) to solve for the constant real balances:

$$i(t) = \delta + \overline{\mu} \tag{3.42}$$

$$M(t) / P(t) = \phi\left(\delta + \overline{\mu} - \overline{i}^M\right)^{-1}\left(\overline{y} - \overline{g}\right) \tag{3.43}$$

With the initial nominal money stock given, the monetary equilibrium equation (3.43) determines the general price level, because it can be written, for $t \geq 0$ as:

$$P(t)^{-1} M_0 \exp(\bar{\mu} t) = \left(\delta + \bar{\mu} - \bar{i}^M\right)^{-1} \left(\bar{y} - \bar{g}\right) \tag{3.44}$$

Consider a simple non-Ricardian rule where the real value of the augmented primary surplus is constant:

$$\hat{s}_1(t) = \bar{\hat{s}}_1 = \tau(t) - \bar{g} + \sigma_1(t) = \tau(t) - \bar{g} + \left(\bar{\mu} - \bar{i}^M\right) m(t) = \tau(t) - \bar{g} + \left(\frac{\bar{\mu} - \bar{i}^M}{\delta + \bar{\mu} - \bar{i}^M}\right) \phi\left(\bar{y} - \bar{g}\right) \tag{3.45}$$

This rule can be implemented by setting net real taxes, τ at the appropriate level, which is this simple example is constant.

Under this non-Ricardian rule, the IBC of the State, holding with equality and with sovereign debt priced at its contractual values can be written as:

$$\frac{B(t) + \left(\frac{B^{\ell}(t)}{\delta + \bar{\mu}}\right)}{P(t)} = \frac{\bar{\hat{s}}_1}{\delta} - b(t) \tag{3.46}$$

Assume that all three bond stocks are predetermined – there is no instantaneous 'stock reshuffling' between the three bonds.
Equation (3.46) contains predetermined variables and parameters plus the general price level. It too determines a value for the general price level, alongside the monetary equilibrium condition.

Clearly, with this non-Ricardian budgetary rule, the price level determined by the monetary equilibrium condition (3.44) and the price level determined by the IBC of the State, holding with equality and with public debt priced at its contractual value, equation (3.46) will only be the same by happenstance. Under the FTPL the price level is determined twice and the model is therefore overdetermined and inconsistent.

The standard approach would be to replace (3.46) by

$$D(t)l(t) = D(t)\left(\frac{B(t) + \left(\frac{B^{\ell}(t)}{\delta + \bar{\mu}}\right)}{P(t)} + b(t)\right) = \frac{\bar{\hat{s}}_1}{\delta} \tag{3.47}$$

If equation (3.47) yields $D(t) \geq 1,\ t \geq 0$ 1, the analysis can proceed (as far as the sovereign is concerned). Otherwise, it is back to the drawing board, because the maintained hypothesis, that sovereign debt is priced at its contractual value, is false.

We summarize this discussion as follows:

Inconsistency 1. *In the flexible price level model, under an exogenous monetary rule and with the unique fundamental equilibrium selected from among the continuum of possible equilibrium solutions, the* *imposition of the IBC of the State, holding with equality and with sovereign bonds priced at their contractual values, results in an overdetermined system.*

(3.C.b.) The FTPL as an equilibrium selection rule

Under our monetary rule, there is a continuum of equilibria for the general price level (and for the entire future path of the price level). We chose among them the unique 'fundamental' solution which produces a constant inflation rate when the growth rate of the nominal money stock is constant.[8] It is the only solution, other than the barter solution, that does not produce explosive or implosive inflation or disinflation bubbles. We chose this unique solution because it seems to us to be a natural 'focal point': stationary inputs produce stationary outputs. Given our model selection criterion, the price level is overdetermined under the FTPL when we add the IBC of the State holding with equality and with sovereign debt priced at its contractual value.

We can get rid of the overdetermination problem while retaining the FTPL condition (3.46) if we drop our 'fundamental' equilibrium selection criterion (stationary inputs should produce stationary outputs if such equilibria exist) and *replace* it with the FTPL condition. Given the price level determined by the IBC of the state, equation (3.46), the nominal interest rate would be determined by the monetary equilibrium condition: $\frac{M_0 \exp(\bar{\mu} t)}{P(t)} = \frac{\phi}{i(t) - \bar{i}^M}\left(\bar{y} - \bar{g}\right)$.

Thus, if we don't impose our equilibrium selection criterion and instead interpret the FTPL's ICB of the State in equation (3.46) as our model selection criterion, the model is no longer overdetermined. The price level at time *t* is determined uniquely by equation (3.46) (as long as that yields a positive value for the price level). That is the good news for the FTPL. The bad news is that, unless the price level determined by the IBC of the State in equation (3.46) also satisfies the 'fundamental' monetary equilibrium condition under the monetary growth rule (3.44), which it will only do by happenstance, the price level will either rise explosively or fall without bound. The nominal interest rate can rise without bound (with the real money stock going to zero) or fall towards the ELB level, creating an infinite demand for real money balances.

We do not consider a selection criterion that generates almost always implosive or explosive solutions for the nominal interest rate in a model where all the fundamentals are constant to be an attractive one.

We recognize, of course, that the FTPL as equilibrium selection criterion was not proposed by Kocherlakota and Phelan (1999) for the case of a flexible price level and an exogenous money supply rule. Instead it was proposed for the case of a flexible price level and an exogenous nominal interest rate rule (considered in the next sub-section). The rationale for applying a fundamentally different equilibrium selection criterion in a flex-price-exogenous-money-endogenous-interest-rate model from that applied in a flex-price-exogenous-interest-rate-endogenous-money model is, however, not apparent to us. We summarize this as follows:

Anomaly 1: *Under a monetary rule, the use of the FTPL as an equilibrium selection rule almost always results in a bubble equilibrium being selected.*

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[8] The barter solution, $P^{-1}(t) = 0,\ t \geq 0$, could also be seen as a 'fundamental solution', but we will ignore it in what follows.

(3.C.c.) The nominal interest rate is the monetary policy instrument

We now assume that the instantaneous nominal interest rate on bonds is constant at a level that keeps the economy from the ELB:

$$i(t) = \bar{i} > \bar{i}^M. \tag{3.48}$$

We could allow for more elaborate exogenous (time-contingent or open-loop) rules. All results go through as long as the nominal interest rate is not made a function of current or anticipated future values of nominal variables such as the nominal money stock or the nominal price level. When the nominal interest rate is the policy instrument, the nominal stock of money is endogenous.

Equilibrium is now given by, for $t \geq 0$

$$c(t) + \bar{g} = \bar{y}. \tag{3.49}$$

$$r(t) = \delta \tag{3.50}$$

$$\pi(t) = \bar{i} - \delta \tag{3.51}$$

$$\frac{M(t)}{P(t)} = m(t) = \frac{\phi}{\bar{i} - \bar{i}^M}\left(\bar{y} - \bar{g}\right) \tag{3.52}$$

The FTPL adds:

$$\frac{B(t) + \left(\frac{B^{\ell}(t)}{\delta + \bar{\mu}}\right)}{P(t)} = \frac{\bar{\hat{s}}_1}{\delta} - b(t) \tag{3.53}$$

Under the standard approach, the flexible price level model with a pegged nominal interest rate produces nominal indeterminacy. Although all real variables, including the stock of real money balances are (uniquely) determined, neither the nominal money stock nor the general price level are determined.

The non-Ricardian budgetary rule under the FTPL now permits the general price level to be determined by the IBC of the state, equation (3.53), holding with equality and with the bonds priced at their contractual values. The nominal money stock is then determined from the monetary equilibrium condition (3.52). Note that the price level indeterminacy of the flexible price level with the monetary rule in the traditional approach is different from the indeterminacy under the interest rate rule. Under the interest rate rule, both the price level and the nominal money stock are undetermined.

For given initial values of the bond stocks, equation (3.53) does indeed uniquely determine the general price level. Is this the validation of the FTPL and indeed also of the model selection criterion interpretation of the imposition of the IBC of the State as an equilibrium condition - at least for the flexible price level money under an exogenous interest rate rule? The answer is a five-fold 'no'.

Anomaly 2: *The price level can be negative under the FTPL.*

There in nothing in equation (3.53), or its more general version

$\frac{B(t) + \left(\frac{B^{\ell}(t)}{\delta + \bar{\mu}}\right)}{P(t)} = \int_t^{\infty} \exp\left(-\int_t^v r(u)du\right)\hat{s}_1(v)dv - b(t)$, to ensure that

$$\operatorname{sgn}\left\{\frac{B(t)+\left(\frac{B^{\ell}(t)}{\delta+\bar{\mu}}\right)}{P(t)}\right\}=\operatorname{sgn}\left\{\int_t^{\infty}\exp\left(-\int_t^v r(u)du\right)\hat{s}_1(v)dv-b(t)\right\}.$$ Unless this condition is satisfied, the FTPL produces a negative price level.

Anomaly 3: *The FTPL vanishes, even under an interest rate rule, when there are no nominal bonds outstanding.*

If all debt is index-linked (or, in an open-economy extension of the model) foreign-currency-denominated, the general price level remains indeterminate. Without nominal bonds, equation (3.53) turns into $b(t)=\frac{\overline{\hat{s}_1}}{\delta}$. Imposing the FTPL, with $b(t)$ inherited from the past will therefore produce an inconsistency except by happenstance. Will the FTPL condition be dropped in this case? What is the economic rationale for making the equilibrium selection criterion depend on the asset menu in this way?

Anomaly 4: *When the equilibrium bond pricing equation is specified properly, nominal indeterminacy reappears under an interest rate rule.*

When we distinguish between the contractual and the market value of sovereign bonds, equation (3.53) becomes: $D(t)\left(\frac{B(t)+\left(\frac{B^{\ell}(t)}{\delta+\bar{\mu}}\right)}{P(t)}+b(t)\right)=D(t)l(t)=\frac{\overline{\hat{s}_1}}{\delta}$. The sovereign debt pricing equation sets the market value of the debt equal to the PDV of current and future primary surpluses. All it determines, however, is $D(t)l(t)$. In general, unless there is only index-linked debt outstanding, the general price level $P(t)$ and the bond revaluation factor $D(t)$ are not individually determined. Even if there is no index-linked debt, the model only determines D/P.

Anomaly 5: *The FTPL can price phlogiston – it can determine a price without an associated quantity***.**

Another anomaly of the FTPL is that it can determine the price of money even if money does not exist except as a numeraire. Suppose there were no money as an asset and store of value in the model. Formally, in our simple model, this means setting $\phi=0$, $i^M=0$ and $M(t)=0, t\geq 0$. Suppose there were only some imaginary concept called "money" that, for some reason, serves as the unit of account, numéraire or invoicing unit. A government bond is denominated in terms of this imaginary numeraire. The FTPL equilibrium is then given by equations (3.49), (3.50), (3.51) and (3.53). The monetary equilibrium condition vanishes, but since we lose an endogenous variable, M, we still have as many equations as unknowns. The price of money can still jump to satisfy equation (3.53).

Instead of something non-existing called 'money', we could use another abstract/imaginary numéraire – phlogiston, say. This is the substance formerly (in the pre-scientific age) believed to be embodied in all combustible materials. It this world, when the FTPL supports a positive general price level (see Anomaly 1), it manages to price non-existent phlogiston, just as it can price non-existent money. We consider this to be an undesirable, indeed unacceptable feature of the model.

To illustrate the deep conceptual bizarreness of the phlogiston economy, consider what a one-period maturity pure discount nominal bond actually is in such an economy.[9] It promises, in period *t*, to pay the purchaser, 'something' in period *t+1*. That something cannot be one unit of phlogiston, because phlogiston does not exist except as a unit of account. Instead it promises to pay the holder in period *t+1 something worth one unit of phlogiston* in that period. How do we know what a unit of phlogiston is worth in period *t+1* – in terms of things that actually exist other than as pure numéraires? We have this phlogiston-denominated bond equilibrium pricing condition in every period. It tells us that the real value of the phlogiston-denominated bond, priced at its contractual value in terms of phlogiston, has to be equal to the PDV of the current and future real augmented primary budget surpluses of the State.

So, in a world where money does not exist except as a pure numéraire, a nominal phlogiston-denominated bond is the ultimate *non-deliverable* forward contract.[10] We believe that it makes no sense to model a world where non-deliverable contracts exist without there also being a deliverable benchmark. Money has to exist either as a commodity (with or without intrinsic value) or as a (fiat) financial claim issued by some economic entity. There has to be a benchmark spot market for money and a deliverable forward contract for money if a non-deliverable forward contract for money is to make sense. In the preceding paragraph, the word 'money' can be replaced by 'phlogiston'.

The FTPL fails this test, insofar as it can price money (phlogiston) is a world where there are no deliverable spot or forward contracts for money (phlogiston). We recognize this is an anomaly rather than a logical inconsistency. We do, however, consider this anomaly to be as devastating as the logical inconsistencies inherent in the FTPL: it is inconceivable to us that one could work with a model of the economy that can determine the equilibrium price of something without an associated quantity of that something.

Anomaly 6: *The HTPL or Joneses theory of the price level is as plausible as the FTPL***.**

Another anomaly of the model is that we can apply its central idea to the household sector as a whole or even to an individual household. In this world, the government satisfies its intertemporal budget constraint with equality, say because it follows a Ricardian budgetary rule. The representative household chooses an arbitrary (non-Ricardian) path of consumption and money holdings and, as long as its consumption and money demand is not too large relative to its income (as long as the PDV of current and future augmented primary surpluses of the household is positive) and as long as the household has a positive stock of debt outstanding, the initial price level jumps to ensure its IBC is satisfied with equality and with household debt priced at its contractual value. This gives us the household sector theory of the price level or HTPL.

Indeed, in a world with many households, we can pick out one favored household, perhaps the Joneses. Every other household and the State follow Ricardian rules and satisfy their IBCs. The Joneses are non-Ricardian. As long as the Joneses have a positive stock of nominal debt outstanding and the PDV of

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[9]For this example, we briefly switch to a discrete time model.

[10] According to Investopedia *"A non-deliverable forward (NDF) is a cash-settled, short-term forward contract in a thinly traded or nonconvertible foreign currency against a freely traded currency, where the profit or loss at the settlement date is calculated by taking the difference between the agreed upon exchange rate and the spot rate at the time of settlement, for an agreed upon notional amount of funds. The gain or loss is then settled in the freely traded currency"*, http://www.investopedia.com/terms/n/ndf.asp. The key relevant point is that no payment in the thinly traded or nonconvertible currency is ever made. All payments are made in the freely traded currency. The amount of the freely traded currency /paid is given by the notional amount of the contract times the difference between the agreed upon forward rate and the spot rate at the time of settlement.

their current and future augmented primary surpluses is positive, the initial price level jumps to ensure that the Joneses remain solvent.

(3.D.) Equilibria at the ELB

We now consider equilibria where the economy is at the ELB. To make the point as dramatically as possible, we assume that the economy is permanently at the ELB.

Under the exogenous nominal interest rate rule this requires:

$$i(t) = \bar{i}^M \ \ , \ t \geq 0 \tag{3.54}$$

We again consider the non-Ricardian budgetary rule $\tau(t) = \bar{\hat{s}}_1 + \bar{g} - \sigma_1(t) = \bar{\hat{s}}_1 + \bar{g} - \left(\mu(t) - \bar{i}^M\right)m(t)$. The utility function (3.25) has global non-satiation in real money balances, so the demand for real money balances is infinite at the ELB (equation (3.55)). The only equilibrium conditions that are different at the ELB from what they are away from the ELB are the monetary equilibrium condition (3.55) and, of course, equation (3.54), which implies equation (3.56):

$$M(t)^{-1} P(t) = 0 \tag{3.55}$$

$$\pi(t) = \bar{i}^M - \delta \tag{3.56}$$

Monetary equilibrium requires an infinite stock of real money balances because of the non-satiation feature of the logarithmic utility function. This can be generated either by a zero price level and any finite nominal stock of money or by a positive price level and an infinite nominal money stock. In principle, at the ELB, the nominal money stock can be exogenous (policy-determined) or demand-determined and endogenous.

The infinite demand for real money balances (equation (3.55)) at the ELB is implausible both a-priori and empirically. Japan, the Eurozone, Sweden and Denmark have been at the EBL for a significant amount of time, and there has been no evidence of an unbounded demand for central bank money in any of these countries. To make sure that the results don't depend on this feature, we (briefly) consider the alternative household utility function below, which exhibits satiation in real money balances when $\frac{M}{P} = \frac{k_2}{k_1} \geq 0$.

$$\begin{aligned}
&u(t) = \int_t^{\infty} e^{-\delta(v-t)} \left[\ln c(v) + \xi\left(\frac{M(v)}{P(v)}\right)\right] dv \\
&c(v), M(v) \geq 0;\ \delta,\ \varphi \geq 0 \\
&\xi\left(\frac{M}{P}\right) = \varphi\left(-\frac{k_1}{2}\left(\frac{M}{P}\right)^2 + k_2 \frac{M}{P}\right) \quad \text{if } 0 \leq \frac{M}{P} \leq \frac{k_2}{k_1} \\
&= \frac{(k_2)^2}{2k_1} \quad \text{if } \frac{M}{P} > \frac{k_2}{k_1} \\
&c, M \ \geq 0;\ k_1,\ k_2,\ \delta, \varphi > 0
\end{aligned} \tag{3.57}$$

The only thing that changes as a result of this alternative utility function is the demand for real money balances, which becomes:

$$\frac{M(t)}{P(t)} = \frac{k_2}{k_1} - \left(\frac{i - \bar{i}^M}{k_1 \varphi c(t)}\right) \quad \text{if } i(t) > \bar{i}^M$$
$$\geq \frac{k_2}{k_1} \quad \text{if } i(t) = \bar{i}^M \tag{3.58}$$

The monetary equilibrium condition at the ELB becomes, instead of equation (3.55):

$$\frac{M(t)}{P(t)} \geq \frac{k_2}{k_1} \tag{3.59}$$

Note that satiation in real money balances at a finite level of real money balances only refers to the non-pecuniary, *direct* utility derived from money balances. Even at the ELB, money remains a valuable store of value and larger real money balances make a household better off because wealth is higher. If there is no satiation in consumption (a property of both utility functions), higher holdings of real money balances will boost household demand for consumption and the household IBC will continue to hold with equality.

There is a unique exogenous money stock rule that supports the economy being permanently at the ELB only if there is satiation in real money balances at a finite stock of real money balances at the ELB *and* the utility of holding real money balances declines for real money holdings larger than the satiation level (a case we don't consider because we view it as a-priori implausible). In that case:

$$\frac{\dot{M}(t)}{M(t)} = \bar{\mu} = \bar{i}^M - \delta, \; t \geq 0$$
$$M(0) = M_0 > 0$$

However, if there is an infinite demand for real money balances when the pecuniary opportunity cost of holding money is zero - as there is with the logarithmic utility function of equation (3.25) - then, if the price level is positive, an infinite stock of nominal money balances will always be demanded. Even if there is satiation in real money balances at a finite stock of real money balances, but the utility of money remains constant at the satiation level when the stock of real money balances rises above the minimum level at which satiation occurs (the utility function given in equation (3.57)), the monetary equilibrium condition does not in general yield a unique price level when the nominal money stock is exogenous and the price level is freely flexible.

The direct analogue of the FTPL in an economy at the ELB with the FTPL in an economy away from the ELB is where, away from the ELB, the nominal money stock is endogenously determined – the exogenous interest rate rule. If the misspecified equilibrium bond pricing equation,

$\frac{B(t) + B^{\ell}(t)P^{\ell}(t)}{P(t)} + b(t) = \int_t^{\infty} \exp\left(-\int_t^v r(u)du\right)\hat{s}_1(v)dv$ implies a positive price level, we have the FTPL again.

If there is non-satiation in real money balances, an exogenous and finite nominal money stock is only consistent with monetary equilibrium and a flexible price level if the price level is zero (equation (3.55)). That would be inconsistent with the price level implied by the misspecified bond pricing equilibrium equation (3.53).

What happens to the ELB to Inconsistency 1 and the six Anomalies? Inconsistency 1 - a non-Ricardian budgetary rule implies an overdetermined model when (a) an exogenous monetary rule setting a (finite) nominal money stock is followed, (b) we select the 'fundamental' equilibrium and (c) we impose the IBC of the State, holding with equality and with sovereign bonds priced at their contractual values – does not carry over without qualifications. As we saw earlier, when there is no satiation in real money balances, the infinite demand for real money balances at the ELB can only be satisfied at a zero general price level, making the FTPL overdetermined on a monetary rule if the nominal money stock is finite. However, if there is satiation in real money balances at a finite stock of real money balances (equation (3.59) holds), the system is not necessarily overdetermined under an exogenous money supply rule even at the ELB, because, there is no unique 'fundamental' solution: as long as the exogenous nominal money stock and the (positive) price level determined by the misspecified bond pricing equilibrium condition, (3.53), satisfy equation (3.59), the monetary equilibrium condition will not uniquely determine the price level: the household is indifferent between holding real money balances in an amount k_2 / k_1 and holding any amount of real money balances greater than k_2 / k_1, which can be supported with the same nominal money stock and different price levels.[11]

Anomaly 1, that under a monetary rule the use of the FTPL as an equilibrium selection rule almost always results in a bubble equilibrium being selected also does not occur at the ELB. Consider the case where the nominal money stock is constant and the nominal interest rate on money is less than the pure rate of time preference. It follows from equation (3.56) that the equilibrium price level will be falling. Assume that money demand is characterized by satiation at a finite level of money balances and that the FTPL picks an initial price level, at $t = t_1$, that satisfies $M(t_1) / P(t_1) \geq k_2 / k_1$. If follows that the monetary equilibrium condition will also be satisfied for all future time, with the real stock of money balances rising at a proportional rate $\bar{i}^M - \delta$. Of course, the phlogiston anomaly cannot occur at the ELB, because there can be no ELB if money does not exist as a store of value. The other four anomalies occur at the ELB also.

4. The FTPL and the FTLEA in sticky price level models

Because the entire thrust of the FTPL is to make the general price level do the work of the bond revaluation factor, *D*, it would seem pretty self-evident that in models with a predetermined or sticky general price level (any Old-Keynesian or New-Keynesian model), the FTPL would find itself facing the familiar problem of an overdetermined system, with the general price level determined twice – once by the IBC of the State and once by history. That presumption is indeed correct if, as in the original flexible price level FTPL, the IBC of the State, holding with equality and with sovereign bonds priced at their contractual values, is used twice: once to derive the optimal consumption and money demand sequences and once more to do its FTPL duty. We summarize this as Inconsistency 2.

Inconsistency 2: *If the general price level is predetermined (sticky), the imposition of the IBC of the State, holding with equality and with sovereign bonds priced at their equilibrium values, leads to an overdetermined equilibrium, if the IBC of the household has been used to derive the optimal consumption and money demand sequences.*

Sims, however, does not make this mistake. He constructs a conventional sticky price level New-Keynesian model (Sims (2011)) where the IBC of the State is used (correctly) once only, and an Old-Keynesian sticky price level model (Sims (2016a)) where the IBC of the State is (correctly) not used at all. Instead of using the counterfactual market equilibrium pricing version of the IBC of the State to verify whether his non-Ricardian budgetary rules are consistent with sovereign solvency, Sims studies the

[11] If, at the price level determined by equation (3.53), the real money stock is smaller than k_2 / k_1, we cannot be at the ELB.

dynamics of his models, concluding that if the behavior of the real public debt is non-explosive, for constant values of the exogenous variables, the IBC of the State, holding with equality and with the sovereign bonds priced at their contractual values, will be satisfied. That too is appropriate methodology. In other words, in his Old-Keynesian and New-Keynesian models Sims uses perfectly sound conventional analytical tools and models.

Where Sims goes wrong is in overstating the case that his non-Ricardian budgetary rules will be consistent with sovereign solvency. Thus, while Sims's New-Keynesian and Old-Keynesian models are not overdetermined, the use of non-Ricardian policy rules cannot be guaranteed to lead to public debt sequences that satisfy the IBC of the State, holding with equality and with sovereign debt priced at its contractual value.

In what follows, we shall use simplified versions of the New-Keynesian and Old-Keynesian models of Sims. This permits us to use analytical methods rather than the numerical solution methods used by Sims. No issue of any importance is missed through these simplifications, however.

Both the general price level and the rate of inflation are predetermined. The inflation rate is updated through an accelerationist Phillips curve;

$$\begin{aligned} \dot{\pi}(t) &= \alpha\left(y(t)-\bar{y}\right) \\ \alpha &> 0 \end{aligned} \tag{4.1}$$

Actual output, $y(t)$, can differ from the exogenous and constant level of potential output, $\bar{y}$. Actual output is demand-determined:

$$y(t)=c(t)+\bar{g} \tag{4.2}$$

We keep the rest of the model the same as before, except for the interest rate rule and the budgetary rules - the rules governing public spending, taxation and money issuance, and assume that the nominal interest rate is the policy instrument, with the nominal money stock endogenous. We restrict the analysis, for sake of brevity, to the case where the economy is not at the ELB. Sims's models are actually phlogiston models – money does not exist as an asset but only as the numeraire - so there is no ELB in his models. We also revert to the logarithmic utility function for real money balances.

The optimizing, forward-looking household whose optimal consumption and money demand are characterized by equations (3.26) through (3.29), with closed form solution for optimal consumption and money demand given in equations (3.32) and (3.33) respectively, follows a Ricardian consumption, money demand and bond holding plan and its IBC holds with equality, implying that, in equilibrium (should an equilibrium exist), the IBC of the State also holds with equality.

The model can be summarized as follows, for $t \geq 0$:

$$\dot{\pi}(t)=\alpha\left(y(t)-\bar{y}\right) \tag{4.3}$$

$$\frac{\dot{P}(t)}{P(t)}=\pi(t) \tag{4.4}$$

$$\dot{l}(t)=y(t)-\tau(t)-c(t)-\left(\mu(t)-\bar{i}^{M}\right)m(t)+r(t)l(t) \tag{4.5}$$

$$\dot{c}(t)=\left(r(t)-\delta\right)c(t) \tag{4.6}$$

$$\dot{B}^{\ell}(t)=\bar{\dot{B}}^{\ell}(t) \tag{4.7}$$

$$\dot{b}(t)=\bar{\dot{b}}(t) \tag{4.8}$$

$$y(t) = c(t) + \bar{g} \tag{4.9}$$

$$m(t) = \frac{\phi}{i(t) - i^M(t)} c(t) \tag{4.10}$$

$$l(t) \equiv \frac{B(t) + B^\ell(t)\int_t^\infty \exp\left(-\int_t^v i(u)du\right)dv}{P(t)} + b(t) \tag{4.11}$$

The initial conditions for the predetermined state variables are:

$$\begin{aligned} \pi(0) &= \pi_0 \\ P(0) &= P_0 \\ B(0) &= B_0 \\ B^\ell(0) &= B_0^\ell \\ b(0) &= b_0 \end{aligned} \tag{4.12}$$

The boundary condition for private consumption is:

$$l(t) = \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\left[\tau(v) + c(v) - y(v) + \left(\mu(v) - i^M(v)\right)m(v)\right]dv \tag{4.13}$$

Note that we assume that the dynamics of the short nominal bond stock and of the nominal consol stock are exogenously given, in equations (4.7) and (4.8), with the dynamics of the index-linked bond endogenously or residually determined.

For the moment, consider the nominal interest rate, $i(t)$, the real primary surplus, $s(t) = \tau(t) - \bar{g}$ and the real value of seigniorage $\sigma_1(t) = \left(\mu(t) - \bar{i}^M\right)m(t)$ to be exogenously given for all $t \geq 0$.

The model has six first-order differential equations, five predetermined state variables, π, P, B^ℓ, l and b and one non-predetermined (or forward-looking) state variable, c. The boundary conditions for the five predetermined variables are the five initial conditions given in equation (4.12). The boundary condition for consumption is equation (4.16) the IBC of the household, holding with equality and with bonds priced at their contractual values. There are three endogenous variables that are not state variables but whose values can be expressed as functions of the state variables and the exogenous variables: y, the demand-determined level of real output (equation (4.9), m, the real money stock, whose value is determined by the monetary equilibrium condition (4.10), and B, the stock of short nominal bonds, obtained from equation (4.11). So, we have the same number of equations and unknowns, the same number of state variables and first-order differential equations and the right number (and type) of boundary conditions. Does that mean all is well with the FTLEA? Meeting the 'counting tests' is just a necessary condition for the system to have one or more solutions. It means that the system is not overdetermined, but the equations describing it still may not have a solution – may be inconsistent. Finally, even if the equations have one or more solutions, these solutions may not make economic sense (the JTPL and the ability to price phlogiston are two examples from the flexible price model).

Note that in equilibrium, the IBC of the household and the output market equilibrium condition ($y = c + \bar{g}$, referred to as the equilibrium real resource constraint by Sims) imply the IBC of the State. The flow budget constraint of the household and the output market equilibrium condition imply the flow budget constraint of the State. It follows that we can replace equation (4.5) by

$$\dot{l}(t) = \overline{g} - \tau(t) - \left(\mu(t) - \overline{i}^{M}\right)m(t) + r(t)l(t) \tag{4.14}$$

and equation (4.16) by

$$l(t) = \int_t^{\infty} \exp\left(-\int_t^v r(u)du\right)\left[\tau(v) - \overline{g} + \left(\mu(v) - i^M\right)m(v)\right]dv \tag{4.15}$$

Obviously, the current price level, which is predetermined, cannot do the FTPL job of ensuring that equation (4.15) holds. So what else can do the job of ensuring that, even when the government follows a non-Ricardian budgetary rule, the IBC of the State will be satisfied with equality and with the sovereign bonds priced at their contractual values? Sims (2011) argues that the nominal and real discount factors can ensure that equation (4.15) holds despite $P(t)$ being given by history. Using (4.11) equation (4.15) can be written as:

$$\frac{B(t) + B^{\ell}(t)\int_t^{\infty} \exp\left(-\int_t^v i(u)du\right)dv}{P(t)} + b(t) = \int_t^{\infty} \exp\left(-\int_t^v r(u)du\right)\left[\tau(v) - \overline{g} + \left(\mu(v) - i^M\right)m(v)\right]dv \tag{4.16}$$

he real discount factors, $\exp\left(-\int_t^v r(u)du\right) = \exp\left(-\int_t^v \left(i(u) - \pi(u)\right)du\right)$, $v \geq t$, that discount current and future (non-Ricardian) real augmented primary surpluses plus, if there is long-dated nominal debt like nominal consols, the nominal discount factors, $\exp\left(-\int_t^v i(u)du\right)$, $v \geq t$, $v \geq t$, are, according to Sims, capable of ensuring that the IBC of the State holds in equilibrium for any non-Ricardian budgetary rule. The discount factors can take on the required values because real consumption, c, is a non-predetermined state variable when the household is optimizing, forward-looking and able to borrow and lend in efficient capital markets. Current and future values of c will take on the values required to make the current and future discount factors take on the values to ensure that equation (4.16) holds for all $t \geq 0$. This theory we refer to as the fiscal theory of the level of economic activity or FTLEA.

Simply stating this proposition ought to be enough to convince the reader that it cannot be true for arbitrary non-Ricardian policies. It is trivial to come up with non-Ricardian budgetary rules that will cause equation (4.16), to be violated in equilibrium. Consider the non-Ricardian rule in equation (4.17):

$$\begin{aligned} &i(t) = \delta + \pi(t), \ \delta > 0, \ t \geq 0 \\ &s(v) + \sigma_1(v) = \overline{z} + \frac{B^{\ell}(t)}{P(t)}\exp\left(-\int_t^v \pi(u)du\right), \ v \geq t \geq 0 \end{aligned} \tag{4.17}$$

Under this budgetary rule equation (4.16) becomes:

$$\frac{B(t)}{P(t)} + b(t) = \frac{\overline{z}}{\delta} \tag{4.18}$$

With every variable in equation (4.18) exogenous or predetermined, it will only hold by happenstance. Or consider the non-Ricardian budgetary rule in equation (4.19):

$$\begin{aligned} &i(t) = \text{ anything at all}, \ t \geq 0 \\ &s(v) + \sigma_1(v) < 0, \ v \geq t \end{aligned} \tag{4.19}$$

Assume that $B(t) \geq 0, B^{\ell}(t) \geq 0$ and $b(t) \geq 0$ with at least one of these three inequalities holding strictly. A permanent augmented primary deficit obviously cannot generate the resources required to service a positive net stock of sovereign debt.

The model just analyzed is a simplified version of the New Keynesian model analyzed in Sims (2011). Sims (2011) is actually a 'phlogiston model' – there is no money in it except as a numeraire and the unit in which the contractual payments due on the nominal bonds are denominated. Setting $M(t) = \dot{M}(t) = \sigma_1(t) = \sigma_2(t) = \phi = 0,\ t \geq 0$ and omitting the monetary equilibrium condition (4.10), does not alter the conclusion that non-Ricardian budgetary rules, including the two just analyzed, can cause the IBC of the State to be violated. The proposition that the nominal and real discount factors in the IBC of the State (equation (4.16)) can cause it to hold even for arbitrary rules for $s(t) + \sigma_1(t)$, the augmented primary surplus, and for the policy rate, $i(t)$, is incorrect.

There no doubt exist non-Ricardian rules that satisfy the IBC of the State in equilibrium - even equation (4.18) could be satisfied, given the historically determined values of the debt stocks and the price level and the exogenously given pure rate of time preference, if the authorities picked the unique value of $\bar{z}$ for which equation (4.18) holds. But neither the price level (in the original FTPL) nor the real and nominal discount factors (in the FTLEA) can be counted on to always make non-Ricardian budgetary rules consistent with sovereign solvency in equilibrium.

(4.A.) A Ricardian and a 'realistic' non-Ricardian budgetary rule in the New-Keynesian model

It is easy to come up with a Ricardian set of rules for the nominal interest rate and the augmented primary surplus that ensures that the model of equations (4.3) to (4.13) is well-behaved. The Ricardian budgetary rule is given in equations (4.20) and (4.21).

$$\begin{aligned} & i(t) = \delta + \pi^* + \beta_1\left(\pi(t) - \pi^*\right) + \beta_2\left(y(t) - \bar{y}\right) \\ & \beta_1 > 1,\ \beta_2 > 0 \end{aligned} \tag{4.20}$$

$$\begin{aligned} & s(t) + \sigma_1(t) = r(t)l(t) + \zeta\left(l(t) - l^*\right) \\ & \zeta > 0 \end{aligned} \tag{4.21}$$

Equation (4.21) can be implemented by adjusting lump-sum real net tax revenue appropriately. The interest rate rule is a Taylor rule, where the nominal policy rate increases with the excess of the inflation rate over the target rate of inflation, π^*. and the output gap. Setting $\beta_1 > 1$ means that the real policy rate rises when the rate of inflation rises; this tends to be stabilizing in a variety of models. The real augmented primary surplus equals the real interest rate bill plus some fraction of the gap between the actual real stock of debt and its target value, l^* which we treat as exogenous and constant. With the Taylor rule and the convergent real public debt rule, the economic system can be reduced to a system of three first-order linear differential equations.

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \\ \dot{l} \end{bmatrix} = \begin{bmatrix} \beta_2 & \beta_1 - 1 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & -\zeta \end{bmatrix} \begin{bmatrix} c \\ \pi \\ l \end{bmatrix} + \begin{bmatrix} (1-\beta_1)\pi^* - \beta_2\bar{y} \\ -\alpha\bar{y} \\ \zeta l^* \end{bmatrix} \tag{4.22}$$

Under this Ricardian budgetary rule, the real stock of government debt converges asymptotically to its target value l^*, unaffected by the dynamics of c and π, whose dynamics are in turn unaffected by the dynamics of l. In the unique steady state, $c = \bar{y},\ \pi = \pi^*, l = l^*$ and $r = \delta$.

The two eigenvalues, λ_1 and λ_2 governing the dynamics of consumption and inflation satisfy:

$$\begin{aligned} \lambda_1 + \lambda_2 &= \beta_2 > 0 \\ \lambda_1 \lambda_2 &= \alpha(1-\beta_1) < 0 \end{aligned} \tag{4.23}$$

So, both roots are real, one root is positive and one negative. With one predetermined state variable (inflation) and one non-predetermined state variable (consumption), there is a unique solution (saddle path) that converges to the unique steady state. The third eigenvalue, which only drives the real debt stock, is $\lambda_3 = -\zeta < 0$ if $\zeta > 0$). The real stock of government debt converges exponentially to its target value l^*.

This is completely standard, non-FTPL and non-FTLEA economics. The budgetary rule is Ricardian and guarantees that the IBC of the State is always satisfied, in and out of equilibrium. It is hard to imagine any fiscal rule robustly satisfying the IBC of the State with equality and with sovereign debt priced at its contractual value, unless there is, sooner or later, directly or indirectly (in more general models this could be through the two other state variables, c and π) feedback by the real augmented primary surplus, $s + \sigma_1$, from the level of the real stock of outstanding non-monetary public debt, l. The real interest bill, rl, produces a 'snowball effect' in the debt dynamics if the real interest rate is positive (in models with positive growth of potential output, the snowball effect is represented by the term $(r-\gamma)\frac{l}{y}$ where $\gamma = \dot{y}/y$). Our model has a positive real interest rate equal to the pure rate of time preference in steady state. These explosive dynamics have to be neutralized by the augmented primary surplus.

It is therefore surprising that in the Sims (2011) New Keynesian model, the numerical simulations support stable, convergent behavior even though there is no feedback by the augmented primary surplus from the debt stock.[12] In terms of the New-Keynesian model analyzed in this paper, the Sims non-Ricardian budgetary rule can be approximated reasonably well by our Taylor rule (equation (4.20)) and the following fiscal rule:

$$\begin{aligned} s + \sigma_1 &= \omega c \\ \omega &> 0 \end{aligned} \tag{4.24}$$

Noting that $s = \tau - \overline{g}$, equation (4.24) is rather like an automatic fiscal stabilizer, where real tax receipts net of transfers vary positively with the level of real economic activity.

The dynamics of consumption and inflation are the same as before – linear with the eigenvalues λ_1 and λ_2 given in equation (4.23). The debt accumulation equation under the fiscal rule in (4.24) is non-linear, and we take a linear approximation at the unique steady state where $c = \overline{c} = \overline{y}$, $\pi = \overline{\pi} = \pi^*$, $l = \overline{l} = \frac{\omega \overline{y}}{\delta}$ and $r = \overline{r} = \delta$.

The dynamics of c and π are again independent of the behavior of the dynamics of l – and vice versa.

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \\ \dot{l} \end{bmatrix} \approx \begin{bmatrix} \beta_2 & \beta_1 - 1 & 0 \\ \alpha & 0 & 0 \\ \beta_2 \overline{l} - \omega & (\beta_1 - 1)\overline{l} & \delta \end{bmatrix} \begin{bmatrix} c - \overline{c} \\ \pi - \overline{\pi} \\ l - \overline{l} \end{bmatrix} \tag{4.25}$$

[12] Since Sims's (2011) New-Keynesian model does not include money as an asset, the augmented primary surplus is the same as the regular primary surplus. This does not change the analysis.

With two predetermined state variables, π and l, and one non-predetermined state variable, c, we need two stable and one unstable characteristic root for local stability. One characteristic root, the one driving the real debt dynamics is $\lambda_3 = \delta > 0$. This is really the end of the story, but for completeness, note that the two other eigenvalues (the ones driving c and π) are the same as under the Ricardian fiscal rule given in equation (4.21), and satisfy equation (4.23), with one stable and one unstable root if the normal Taylor rule assumptions hold with $\beta_1 > 1$ and $\beta_2 > 0$.

Of course, the model of the economy analyzed here is a simplified version of the Sims (2011) New-Keynesian model and the budgetary rule just considered is only 'in the spirit of' Sims's non-Ricardian budgetary rule – not identical to it. It is therefore possible that the true Sims (2011) non-Ricardian budgetary rule was a lucky choice that avoided explosive behavior of the public debt despite the absence of any stabilizing feedback loop from the debt stock to the augmented primary surplus. Even with the stabilizing Taylor rule, we would not recommend (4.24) to any policy maker, however: there has to be sufficient responsiveness of the (augmented) real primary surplus to the real debt stock to guarantee the debt dynamics will not become explosive. Responsiveness of taxes to the level of economic activity ($\omega > 0$) is, in general, not sufficient. That responsiveness can be more flexible than what is embodied in the Ricardian rule in equation (4.21). For instance, the stabilizing feedback of the real augmented primary surplus from the real debt stock might not kick in until the real debt stock exceeds some critical level. An example would be

$$\begin{aligned} s + \sigma_1 &= \omega c, \ \omega > 0 && \text{if } l \le l^* \\ &= rl + \zeta\left(l - l^*\right), \ \zeta > 0 && \text{if } l > l^* \end{aligned}$$

(4.B.) Financial Repression in the New-Keynesian model

Sims states in Sims (2016a, pages 4 and 5): *"… the existence of stable, unique equilibria under policies that peg the interest rate and leave fiscal effort unresponsive to the level of real debt does not rely on instant, far-sighted adjustments by rational agents. All that is required is a strong wealth effect on consumption and sufficiently rapid response of inflation to demand."*

In this Subsection, we show that even instant, far-sighted adjustments by rational agents are not sufficient to ensure *the existence of stable, unique equilibria under policies that peg the interest rate and leave fiscal effort unresponsive to the level of real debt.* We will therefore replace the Taylor rule given in (4.20) with a nominal interest rate pegging rule:

$$i(t) = \bar{i} \tag{4.26}$$

Note that this is just a special case of the Taylor rule in equation (4.20), with $\beta_1 = \beta_2 = 0$ and $\bar{i} = \delta + \pi^*$.

The optimizing, forward-looking New-Keynesian consumer does not have a financial wealth effect on consumption in equilibrium: the (forward-looking) consumption function in our New-Keynesian model looks like the is a financial wealth effect: $c(t) = \frac{\delta}{1+\phi}\left(l(t) + m(t) + \int_t^\infty e^{-\int_t^v r(u)du}\left(y(v) - \tau(v)\right)dv \right)$. The apparent financial wealth effect – the term $\frac{\delta}{1+\phi}\left(l(t)+m(t)\right)$, vanishes when we substitute the State's IBC, holding with equality, $l(t) = \int_t^\infty \exp\left(-\int_t^v r(u)du\right)\left[\tau(v) - \bar{g} + \left(\mu(v) - i^M\right)m(v)\right]dv$, into the consumption function, which becomes $c(t)=\delta\left(\int_t^{\infty} e^{-\int_t^v r(u)du}\left(y(v)-\overline{g}\right)dv\right)$ and exhibits debt neutrality or Ricardian equivalence.[13]

Because this model is a special case of the model summarized in equation (4.25), the debt dynamics will once again be unstable, because the fiscal effort is unresponsive to the level of the public debt. Again, the dynamics of c and π are decoupled from the dynamics of the real debt stock. The eigenvalues driving c and π are the complex conjugate solutions to:

$$\lambda_1+\lambda_2=0 \tag{4.27}$$

and

$$\lambda_1\lambda_2=\alpha \tag{4.28}$$

These roots are purely imaginary numbers, $\lambda_1=\iota\sqrt{\alpha}$ and $\lambda_1=-\iota\sqrt{\alpha}$, where $\iota^2=-1$. So, consumption and inflation circle around the steady state without ever getting there. The real debt dynamics are, once again, driven by the unstable characteristic root $\lambda_3=\delta>0$. Public debt behavior is, again, explosive. This, of course, invalidates the benign 'sovereign IBC' equilibrium assumption on which the (non-explosive) consumption and inflation dynamics are based.

(4.C.) Financial repression in an Old-Keynesian model: *NOT* the FTPL.

In Sims (2016a) an Old-Keynesian model is analyzed. The key difference with the Sims (2011) New-Keynesian model is that consumption is driven by an ad-hoc differential equation and is viewed as a predetermined variable. That means that the boundary condition for consumption is an initial condition: $c(0)=c_0$. If we still impose equation (4.16), the IBC of the State, holding with equality and with sovereign bonds priced at their contractual values, as an equilibrium condition, we have an overdetermined system. Sims has stated in private correspondence that equation (4.16) is not part of the Sims (2016a) model and that, consequently, the Sims (2016a) model is not an FTPL/FTLEA model. Indeed, the IBC of the household is not used to characterize consumption behavior either. With both the household and the State pursuing non-Ricardian programs, there is no presumption that either the IBC of the household or the IBC of the State will be satisfied in equilibrium, let alone satisfied with equality.

We first analyze a stripped-down version of the Sims (2016a) model. It differs from the actual Sims (2016a) model in two ways. In the Sims (2016a) model, all exogenous variables and policy instruments follow stable, univariate, first-order dynamic systems. We assume instead that all exogenous variables and policy instruments are constant. These constant values can be viewed as the steady-state values of Sims's univariate dynamic processes for these variables. Second, Sims adds additive stochastic shocks to many of the dynamic processes of the model. We leave these out. Nothing essential is lost by these simplifications. The stripped-own model turns out to have a number of weaknesses that make it unfit

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[13] There are two qualifications to the absence of a financial wealth effect in the forward-looking, optimizing household consumption function. First, financial ownership claims to 'outside' assets (assets for which there is no corresponding liability, like physical capital, land and real estate will show up as wealth in the consumption function, even after consolidating the household and State IBCs. Second, if there is money in the model and if there is irredeemability of central bank money, the term $\lim_{v\to\infty}\exp\left(-\int_t^v r(u)d\right)m(v)$ will appear in the consumption function even after consolidation, if the irredeemability property of central bank money is recognized.

as a guide to policy. We therefore developed an alternative Old-Keynesian model that does not share these weaknesses.

The (stripped-down) Sims (2016a) model can be summarized as follows:

$$\begin{aligned}
&\tau = \omega_0 \\
&i = \bar{i} \\
&\pi = \gamma_p\left(v - \ln\theta\right) + \gamma_p\left(\frac{1-\theta}{\theta}\right)\ln c \\
&\frac{d}{dt}\ln c = \gamma_c\left(l - \frac{\bar{\tau}}{\bar{i}}\right) \\
&\dot{v} = -\gamma_p\left(v - \ln\theta\right) + \left(\frac{\gamma_w - \gamma_p(1-\theta)}{\theta}\right)\ln c \\
&\dot{l} = (i-\pi)l - \tau \\
&\gamma_p, \gamma_w, \gamma_c > 0;\ 0 < \theta < 1
\end{aligned} \tag{4.29}$$

where v is the logarithm of the real wage. Sims assumes that $\bar{g} = 0$, that $m = 0$, $\sigma_1 = 0$ and that $\ln\bar{y} = 0$. It follows that $s = s + \sigma_1 = \tau$

Note that there is no responsiveness of the 'fiscal effort' or the augmented real primary surplus, τ, or in a more general model $s + \sigma_1$, to the level of the public debt or to the level of economic activity, but that the growth rate of consumption responds positively to household financial wealth.

The steady state of the system is given by

$$\begin{aligned}
&\ln c = \ln\bar{c} = 0 \\
&\pi = \bar{\pi} = 0 \\
&l = \bar{l} = \frac{\omega_0}{\bar{i}} \\
&v = \bar{v} = \ln\theta \\
&r = \bar{r} = \bar{i}
\end{aligned} \tag{4.30}$$

The linear approximation at the steady state of this system is:

$$\begin{bmatrix} \frac{d}{dt}\ln c \\ \dot{v} \\ \dot{l} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & \gamma_c \\ \frac{\gamma_w - \gamma_p\left(1-\theta\right)}{\theta} & -\gamma_p & 0 \\ -\frac{\omega_0}{\bar{i}}\gamma_p\left(\frac{1-\theta}{\theta}\right) & -\frac{\omega_0}{\bar{i}}\gamma_p & \bar{i} \end{bmatrix} \begin{bmatrix} \ln c - \ln\bar{c} \\ v - \bar{v} \\ l - \bar{l} \end{bmatrix} \tag{4.31}$$

All three state variables are predetermined, so all three eigenvalues must have negative real parts for the system to be stable. The three eigenvalues must satisfy the following conditions:

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 = \bar{i} - \gamma_p < 0 \\
&\lambda_1\lambda_2\lambda_3 = -\gamma_c\gamma_p\gamma_w\frac{\omega_0}{\theta\bar{i}} < 0
\end{aligned} \tag{4.32}$$

Sims assigns a positive steady-state nominal interest rate: $\bar{i} = 0.03$ and very large responsiveness of inflation to aggregate demand: $\gamma_p = 4.00$. The first condition in equation (4.32) is therefore satisfied.

The second condition will then be satisfied if and only if $\omega_0 > 0$: the (steady state) value of the (exogenous) (augmented) real primary surplus is positive. The characteristic equation is

$$\lambda^3 + (\gamma_p - \bar{i})\lambda^2 - \gamma_p\left(\bar{i} - \frac{\gamma_c\omega_0(1-\theta)}{\bar{i}\theta}\right)\lambda + \frac{\gamma_c\gamma_w\gamma_p\omega_0}{\bar{i}\theta} = 0$$

With the numerical values assigned by Sims, the characteristic equation is $\lambda^3 + 3.97\lambda^2 + 0.309\lambda + 0.343 = 0$, which indeed has three roots with negative real parts: $\lambda_1 = -3.9134;\ \lambda_2 = -0.02828 + 0.2947\imath;\ \lambda_2 = -0.02828 - 0.2947\imath$.[14]

Two things must be emphasized about this result. First, it has nothing at all to do with the FTPL/FTLEA. This is a classic fiscal stimulus plus financial repression (a constant nominal interest rate despite permanently higher inflation) policy. The assumption that the IBC of the State holds with equality is not used. The budgetary rule resembles what has been recommended by some economists to the Japanese authorities: peg the policy rate near zero for a long time (in Japan, peg the yield curve near zero for maturities up to 10 years) and provide a (large/long-lasting) deficit-financed fiscal stimulus.

Second, this result is not robust, two ways. First, the model is unstable with minor changes in the parameter values: $\omega_0 < 0$ suffices to produce instability. A shallow slope of the price Phillips curve ($\gamma_p < \bar{i}$) also produces instability. Second, and more important, there are two problems with this model that make it quite inadequate – even at a purely qualitative level - as a guide to the interaction of (nominal) public debt, fiscal policy, nominal interest rate rules and inflation.

First, the price Phillips curve is always upward-sloping (increasing in real aggregate demand) rather than vertical at least in steady state. This means that the inflationary impulse of a cut in taxes, through consumption, will persist as long as the real public debt burden does not start to decline (see the third and fourth equations in (4.29)). With the nominal interest rate pegged, the real interest rate can, for certain parameter values, be reduced enough by enough to run a budget surplus. This permanently upward-sloping price Phillips curve represents a form of permanent inflation illusion that makes the model not fit for purpose.

Second, the reason steady-state inflation is zero in the Sims (2016a) model is that the dynamics of real consumption are driven by the *nominal* interest rate. Again, we don't consider such permanent inflation illusion to be a desirable property of consumption behavior, even in an Old-Keynesian model. In steady state, real taxes have to equal the real interest rate bill to keep the real public debt constant. In the Old-Keynesian model, real taxes have to equal the nominal interest rate bill to keep real consumption constant. This certainly pins down the steady-state inflation rate – at zero - despite the long-run upward-sloping price Phillips curve, but it really makes no economic sense. If the consumption equation were instead written as $\frac{d}{dt}\ln c = \gamma_c\left(l - \frac{\bar{\tau}}{\bar{i} - \bar{\pi}}\right)$, that is, the growth rate of consumption depends on the (steady state) real interest rate, there would be infinitely many steady state solutions for $c, v, l,\ \pi$ and r. With the nominal interest rate in the consumption function pinning down steady-state inflation at zero and with a long-run upward-sloping Phillips curve, inflation can wipe out the public debt burden during the transition and return safely to zero in the long run. If only.

[14] The numerical values are: $\bar{i} = 0.03; \omega_0 = 0.03; \theta = 0.70; \gamma_w = 0.30; \gamma_c = 2.00; \gamma_p = 4.00.$

To get a more robust Old-Keynesian world than the one of Sims (2016a), we will revert to the New-Keynesian model, but replace the optimizing household/consumer with the 'financial wealth effect augmented' Keynesian consumption function in equation (4.33):

$$\begin{aligned} & c = -\eta_1 \tau + \eta_2 l - \eta_3 r \\ & \eta_1 > 0, \eta_2 > 0, \eta_3 > 0 \end{aligned} \tag{4.33}$$ [15]

Consumption depends negatively on real taxes, negatively on the real interest rate and positively on real financial wealth. We will again addition adopt the (for our purposes unimportant) simplifying assumptions of Sims (2016a), that $\bar{g} = \sigma_1(t) = m(t) = 0$, so $s + \frac{\sigma_1}{P} = \tau$. The Phillips curve is again an accelerationist one.

$$\begin{aligned} & \dot{\pi} = \alpha(y - \bar{y}) \\ & \alpha > 0 \end{aligned} \tag{4.34}$$

The tax function is:

$$\begin{aligned} & \tau = \omega_0 + \omega_1 l \\ & \omega_0 < \bar{y}, \omega_1 > 0 \end{aligned} \tag{4.35}$$

The model is completed with:

$$c = y$$

$$r = i - \pi$$

$$\dot{l} \equiv rl - \tau$$

The steady-state equilibrium is given by:

$$\begin{aligned} & \pi = -\left(\frac{\eta_2 - \eta_1 \omega_1}{\eta_3} \right) l + \bar{i} + \frac{\bar{y} + \eta_1 \omega_0}{\eta_3} \\ & \left(\bar{i} - \pi - \omega_1 \right) l = \omega_0 \end{aligned} \tag{4.36}$$

Even in this extremely basic model, there are in general two steady-state equilibria. The only way to avoid two steady-state equilibria is to assume that the real interest rate does not affect aggregate demand. In that case, the only way the real interest rate enters the model is through the public debt dynamics. We consider this highly implausible especially because aggregate demand in the real world includes capital expenditure.

[15] Because Sims works with a model without money, financial wealth equals the value of the net stock of government bonds held by the households. Outside the phlogiston economy, real financial wealth would be equal to $l + m$.

$$l=\bar{l}=\frac{\omega_1+\frac{\bar{y}+\eta_1\omega_0}{\eta_3}\pm\sqrt{\left(\omega_1+\frac{\bar{y}+\eta_1\omega_0}{\eta_3}\right)^2+4\left(\frac{\eta_2-\eta_1\omega_1}{\eta_3}\right)\omega_0}}{2\left(\frac{\eta_2-\eta_1\omega_1}{\eta_3}\right)}$$

$$\pi=\bar{\pi}=-\left(\frac{\eta_2-\eta_1\omega_1}{\eta_3}\right)\bar{l}+\bar{i}+\frac{\bar{y}+\eta_1\omega_0}{\eta_3}$$

$$=\frac{-\left(\omega_1+\frac{\bar{y}+\eta_1\omega_0}{\eta_3}\right)\pm\sqrt{\left(\omega_1+\frac{\bar{y}+\eta_1\omega_0}{\eta_3}\right)^2+4\left(\frac{\eta_2-\eta_1\omega_1}{\eta_3}\right)\omega_0}}{2}+\bar{i}+\frac{\bar{y}+\eta_1\omega_0}{\eta_3} \tag{4.37}$$

$$r=\bar{r}=\left(\frac{\eta_2-\eta_1\omega_1}{\eta_3}\right)\bar{l}-\left(\frac{\bar{y}+\eta_1\omega_0}{\eta_3}\right)$$

$$=\frac{\left(\omega_1+\frac{\bar{y}+\eta_1\omega_0}{\eta_3}\right)\pm\sqrt{\left(\omega_1+\frac{\bar{y}+\eta_1\omega_0}{\eta_3}\right)^2+4\left(\frac{\eta_2-\eta_1\omega_1}{\eta_3}\right)\omega_0}}{2}-\left(\frac{\bar{y}+\eta_1\omega_0}{\eta_3}\right)$$

Note that the steady-state real interest rate is independent of the pegged value of the nominal interest rate, unlike the Sims (2016a) model where the two are equal. We will assume, in the spirit of Sims (2016a), that a higher real stock of public debt boosts consumption demand, even after the effect of a higher public debt on taxes is allowed for, so $\eta_2-\eta_1\omega_1>0$. We also assume that $\bar{y}+\eta_1\omega_0>0$; a sufficient condition for this is $\omega_o>0$. This means that the real debt stock will be positive in one steady state and negative in the other steady state. In the steady state with the negative level of real public debt, the inflation rate is positive if the pegged nominal interest rate is not too negative. In the steady state with the positive level of real public debt, the inflation rate can be either positive or negative given the a-priori restrictions we have imposed on the parameters. The real interest rate is negative in the steady state with the negative real debt stock.

Eliminating equilibrium consumption, the real interest rate and real tax revenues, the model can be reduced to two first-order differential equations in π and l:

$$\dot{\pi}=\alpha\eta_3\pi+\alpha\left(\eta_2-\eta_1\omega_1\right)l-\alpha\left(\bar{y}+\eta_1\omega_0+\eta_3\bar{i}\right)$$
$$\dot{l}=\left(\bar{i}-\pi-\omega_1\right)l-\omega_0$$

Linearizing this at the two steady states, we obtain the following dynamic system:

$$\begin{bmatrix}\dot{\pi}\\ \dot{l}\end{bmatrix}\approx\begin{bmatrix}\alpha\eta_3 & \alpha(\eta_2-\eta_1\omega_1)\\ \bar{l} & \bar{i}-\omega_1-\bar{\pi}\end{bmatrix}\begin{bmatrix}\pi-\bar{\pi}\\ l-\bar{l}\end{bmatrix} \tag{4.38}$$

The characteristic roots are solved for from

$$\begin{aligned}\lambda_1+\lambda_2&=\alpha\eta_3+\bar{i}-\omega_1-\bar{\pi}\\ \lambda_1\lambda_2&=\alpha\eta_3\left(\bar{i}-\omega_1-\bar{\pi}\right)-\alpha\left(\eta_2-\eta_1\omega_1\right)\bar{l}\end{aligned} \tag{4.39}$$

Consider the case where the tax function is independent of the real value of the public debt, $\omega_1 = 0$. The characteristic roots then simplify to:

$$\begin{aligned} \lambda_1 + \lambda_2 &= \alpha\eta_3 + \bar{i} - \bar{\pi} \\ \lambda_1\lambda_2 &= \alpha\eta_3\left(\bar{i} - \bar{\pi}\right) - \alpha\eta_2\bar{l} \end{aligned} \tag{4.40}$$

Because both π and l are predetermined state variables, we need both roots to have negative real parts for the system to be stable. From the second equation in (4.39), this requires $\eta_3\bar{r} > \eta_2\bar{l}$. The simplified steady-state equations for the case where $\omega_1 = 0$ are:

$$l = \bar{l} = \frac{\frac{\bar{y} + \eta_1\omega_0}{\eta_3} \pm \sqrt{\left(\frac{\bar{y} + \eta_1\omega_0}{\eta_3}\right)^2 + 4\left(\frac{\eta_2}{\eta_3}\right)\omega_0}}{2\left(\frac{\eta_2}{\eta_3}\right)}$$

$$\pi = \bar{\pi} = -\left(\frac{\eta_2}{\eta_3}\right)\bar{l} + \bar{i} + \frac{\bar{y}}{\eta_3} = \frac{-\left(\frac{\bar{y} + \eta_1\omega_0}{\eta_3}\right) \pm \sqrt{\left(\frac{\bar{y} + \eta_1\omega_0}{\eta_3}\right)^2 + 4\left(\frac{\eta_2}{\eta_3}\right)\omega_0}}{2} + \bar{i} + \frac{\bar{y} + \eta_1\omega_0}{\eta_3} \tag{4.41}$$

$$r = \bar{r} = \left(\frac{\eta_2}{\eta_3}\right)\bar{l} - \left(\frac{\bar{y} + \eta_1\omega_0}{\eta_3}\right) = \frac{\left(\frac{\bar{y} + \eta_1\omega_0}{\eta_3}\right) \pm \sqrt{\left(\frac{\bar{y} + \eta_1\omega_0}{\eta_3}\right)^2 + 4\left(\frac{\eta_2}{\eta_3}\right)\omega_0}}{2} - \left(\frac{\bar{y} + \eta_1\omega_0}{\eta_3}\right)$$

If the exogenous component of the tax function is positive, $\omega_0 > 0$, one of the steady-state equilibria will have a negative stock of real debt and an associated negative real interest rate. This condition, similar to the stability condition in the Sims (2016a) model (given in equation (4.32)) is, however, only necessary but not sufficient for local stability. Consider the following numerical values of the parameters: $\bar{y} = 1;\ \omega_1 = 0.03;\ \eta_1 = 2.00;\ \eta_2 = 2.00;\ \eta_3 = 0.50;\ \alpha = 2.00$. The steady-state equilibrium with the positive stock of real public debt (which also has a positive real interest rate) is locally unstable ($\lambda_1\lambda_2 = -2.06$). The steady-state equilibrium with the negative stock of real public debt and the associated negative real interest rate is also locally unstable. To make either of both of the steady-state equilibria locally stable, we require fiscal effort responsiveness to the public debt. It can be checked numerically that a large positive value for ω_1 is not sufficient for local stability. To guarantee stability of the debt accumulation process, the fiscal effort has to be able to "change sign", that is, move from an augmented primary deficit to an augmented primary surplus, if the real value of the interest bill changes sign, either because the real interest rate changes sign or because the real debt stock changes sign. Only a flexible Ricardian fiscal effort rule like the one given in equation (4.21), $\tau(t) = r(t)l(t) + \zeta\left(l(t) - l^*\right),\ \zeta > 0$, will guarantee stable public debt dynamics.

The purpose of this section was not to develop a robust Old-Keynesian model for its own sake. We leave that as an exercise for the reader. The purpose is twofold. First, to make clear that the permanent financial repression model of Sims (2016a) has nothing to do with the FTPL/FTLEA and, second, to show that the fiscal policy effectiveness proposition of Sims (2016a) and his message about debt sustainability is not robust. We summarize Sims's message as *don't worry about the fiscal effort responding to the public debt burden; a sufficiently strong (positive) response of private demand to the real value of the public debt and of inflation to private demand (and to a fiscal stimulus) will take care of debt sustainability*. We do not consider the recommendation to engage in a (permanent) fiscal stimulus, to keep the policy rate pegged for a long time (technically forever) and not to have the augmented primary surplus respond intelligently to the public debt burden, to be sensible.

(5) Conclusion

The fiscal theory of the price level rests on a fundamental fallacy: the confusion of the intertemporal budget constraint of the State with a misspecified equilibrium nominal bond pricing equation and the double use of this IBC. This fundamental fallacy generates a number of internal inconsistencies and anomalies that should have led to the rejection of the FTPL as a logically coherent theory. This has not happened. This paper aims to rectify that error.

The issue is not an empirical one. Neither does it concern the realism of the assumptions that are made to obtain the FTPL. It is about the flawed internal logic of the FTPL.

Interpreting the FTPL as an equilibrium selection mechanism in models with multiple equilibria does not improve matters. The FTPL remains internally inconsistent and riven with unacceptable anomalies also when the economy is at the ELB. The attempt by Sims (2011) to extend the FTPL to models with nominal price rigidity is a failure. Current and future anticipated real and nominal interest rates cannot be relied upon to ensure solvency of the sovereign when non-Ricardian budgetary rules are implemented.

The fiscal theory of the price level died for the first time more than 15 years ago. Its attempted resurrection failed. It is time to bury it again – for the last time.

References

Abai, Katsuhiki and Kiichi Murashima (2017), "On the Fiscal Theory of the Price Level", *Citi Research, Economics, Japan Economics Flash*, 9 February, https://www.citivelocity.com/t/eppublic/12kGu.
Bassetto, Marco, "A Game-Theoretic View of the Fiscal Theory of the Price Level," *Econometrica,* LXX (2002), 2167–2196

Bergin, Paul R. (2000), "Fiscal Solvency and Price Level Determination in a Monetary Union," *Journal of Monetary Economics,* XLV, 37–53.

Buiter, Willem H. (1998), "The Young Person's Guide to Neutrality, Price Level Indeterminacy, Interest Rate Pegs, and Fiscal Theories of the Price Level," National Bureau of Economic Research Working Paper No. 6396, 1998 and CEPR Discussion Paper No. 1799, Mar. 1998.

Buiter, Willem H. (1999) "The Fallacy of the Fiscal Theory of the Price Level," Centre for Economic Policy Research Discussion Paper No. 2205, 1999.

Buiter, Willem H. (2001), "The fallacy of the fiscal theory of the price level, again," Bank of England Working Paper Series No. 141, July 2001. For an abstract, click here.

Buiter, Willem H. (2002), "The Fiscal Theory of the Price Level: A Critique," *Economic Journal,* CXII, 459–480.

Buiter, Willem H. (2003), "Helicopter money: Irredeemable fiat money and the liquidity trap," NBER Working Paper No. W10163, Dec. 2003. For an abstract, click here.

Buiter, Willem H. (2005), "New developments in monetary economics: Two ghosts, two eccentricities, a fallacy, a mirage and a mythos, "*Economic Journal* 115(502), Mar. 2005, C1-C31, presented as the Royal Economic Society 2004 Hahn Lecture. For an abstract, click here.

Buiter, Willem H. (2007), "Seigniorage," *Economics,* the Open-Access, Open-Assessment E-Journal, 2007-10.

Buiter, Willem H. (2014), "The Simple Analytics of Helicopter Money: Why It Works - Always," in *Economics*, The Open-Access, Open-Assessment E-Journal, 8 (2014-28): 1-51.

Buiter, Willem H. (2017a), "Bad and Good 'Fiscal Theories of the Price Level'", Willem H. Buiter, Citi Research, Multi-Asset, Global Economics View, 17 March 2017.

Buiter, Willem H. (2017b), "The Fallacy of the Fiscal Theory of the Price Level - Once More", Unpublished paper, 2 April 2017. Also available as Columbia University, SIPA, CDEP and CGEG Working Paper No. 37, March 2017, here.; also available as CEPR Discussion Paper No. 11941, 27 March 2017.

Buiter, Willem H. (2017c), "The Good and the Bad Fiscal Theory of the Price Level", paper prepared for the C. D. Deshmukh Memorial lecture at the Reserve Bank of India on 11 April 2017, revised 9 May 2017.

Buiter, Willem H. and Anne C. Sibert (2007), "Deflationary bubbles," *Macroeconomic Dynamics* 11(4), Sept., 431-454. For working paper version, click here.

Christiano, Lawrence J. and Terry J. Fitzgerald (2000), "Understanding The Fiscal Theory Of The Price Level," *FRB Cleveland - Economic Review*, v36 (2,Qtr-2), 2-37.

Cochrane, John H. (1999), "A Frictionless View of U. S. Inflation," *NBER Macroeconomics Annual 1998,* Ben S. Bernanke and Julio. J. Rotemberg, eds. (Cambridge, MA: MIT Press: 1999).

Cochrane, John H. (2001), "Long-Term Debt and Optimal Policy in the Fiscal Theory of the Price Level," *Econometrica,* LXIX (2001), 69–116.

Cochrane John H. (2005), "Money as stock," *Journal of Monetary Economics*, vol. 52(3), pages 501-528.

Cochrane, John H. (2016a), "Do higher interest rates raise or lower inflation", Unpublished paper, February, https://faculty.chicagobooth.edu/john.cochrane/research/papers/fisher.pdf.

Cochrane, John H. (2016b), "The Fiscal Theory of the Price Level and its Implications for Current Policy in the United States and Europe", paper presented at the "Next Steps for the Fiscal Theory of the Price Level" conference at the Becker Friedman Institute for Research on Economics at the University of

Chicago, 1 April, https://faculty.chicagobooth.edu/john.cochrane/research/papers/cochrane_fiscal_theory_panel_bfi.pdf.

Cochrane, John. H. (2016c), “Next Steps for the FTPL”, *The Grumpy Economist*, John Cochrane’s Blog, http://johnhcochrane.blogspot.com/2016/04/next-steps-for-ftpl.html.

Cushing, Matthew J. (1999), “The Indeterminacy of Prices under Interest Rate Pegging: The Non-Ricardian Case,” *Journal of Monetary Economics,* XLIV, 131–148.

Daniel, Betty C. (2001), “The Fiscal Theory of the Price Level in an Open Economy,” *Journal of Monetary Economics,* XLVIII (2001), 293–308.

Daniel, Betty C. (2007), “The Fiscal Theory of the Price Level and Initial Government Debt”, *Review of Economic Dynamics*, Vol. 10, Issue 2, April, pp. 193-206. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.589.7022&rep=rep1&type=pdf .

Dupor, Bill (2000), “Exchange Rates and the Fiscal Theory of the Price Level,” *Journal of Monetary Economics,* XLV, 613– 630.

Friedman, Milton (1969),” The Optimum Quantity of Money”, in Milton Friedman, *The Optimum Quantity of Money and Other Essays*, Chapter 1. Adline Publishing Company, Chicago.

Jacobson, Margaret, Eric Leeper and Bruce Preston (2016), “Fiscal Inflation in 1933”, Lecture slides, https://bfi.uchicago.edu/sites/default/files/file_uploads/BFI_Jacobson_Leeper_Preston.pdf.

Kocherlakota, Narayana, and Christopher Phelan (1999), “Explaining the Fiscal Theory of the Price Level,” *Federal Reserve Bank of Minneapolis Quarterly Review,* XXIII, 14–23.

Lara Resende, André (2017), “Juros e conservadorismo intellectual”, *Valor Econômico*, 13 January, http://www.valor.com.br/cultura/4834784/juros-e-conservadorismo-intelectual.

Leeper, Eric E. M. (1991): “Equilibria under ‘Active’ and ‘Passive’ Monetary and Fiscal Policies,” *Journal of Monetary Economics*, 27, 129–47.

Leeper, Eric E. M. (2015), Real Theory of the Price Level, Briefing, 21 September 2015, https://bfi.uchicago.edu/sites/default/files/file_uploads/Leeper_Background_3%20copy.pdf.

Loyo, Eduardo (1999), “Tight Money Paradox on the Loose: A Fiscalist Hyperinflation,” mimeo, Kennedy School of Government, Harvard University.

McCallum, Bennett T. (2001), “Indeterminacy, Bubbles, and the Fiscal Theory of Price Level Determination,” *Journal of Monetary Economics,* XLVII, 19–30.

Niepelt, D. (2004). ‘The fiscal myth of the price level’, *Quarterly Journal of Economics*, vol. 119(1) (February), pp. 277–300.

Sargent, Thomas J. (1987), *Dynamic macroeconomic theory.* Harvard University Press, Cambridge.

Sargent, Thomas J., and Neil Wallace (1981), "Some Unpleasant Monetarist Arithmetic," *Federal Reserve Bank of Minneapolis Quarterly Review,* V(3), 1–17.

Schmitt-Grohe´, Stephanie, and Martin Uribe (2000), "Price Level Determinacy and Monetary Policy under a Balanced-Budget Requirement," *Journal of Monetary Economics,* XLV, 211–246.

Sims, Christopher A. (1994), "A Simple Model for Study of the Determination of the Price Level and the Interaction of Monetary and Fiscal Policy," *Economic Theory,* IV (1994), 381–399.

Sims, Christopher A. (1999a) "Domestic Currency Denominated Government Debt as Equity in the Primary Surplus," mimeo, Princeton University, 1999a. Paper prepared for presentation at the August 1999 Latin American Meetings of the Econometric Society at Cancun, Mexico, http://sims.princeton.edu/yftp/Cancun/DebtEquity.pdf.

Sims, Christopher A. (1999b), "The Precarious Fiscal Foundations of EMU," *De Economist,* December, Volume 147, Issue 4, pp 415–436

Sims, Christopher A. (2001), "Fiscal Consequences for Mexico of Adopting the Dollar," *Journal of Money, Credit and Banking,* XXXIII (2001), 597– 625.

Sims, Christopher A. (2011): "Stepping on a Rake: The Role of Fiscal Policy in the Inflation of the 1970's," *European Economic Review* 55 (2011) 48â˘A ¸S56, 55, 48–56.

Sims, Christopher A. (2013): "Paper Money," *American Economic Review*, 103(2), 563–84.

Sims, Christopher A. (2016a): "Active fiscal, passive money equilibrium in a purely backwardlooking model," Discussion paper, Princeton University, http://yftp/OldKeynesianFTPL/BackwardAFPM.pdf.

Sims, Christopher A. (2016b), "Making government paper a bad investment", slides presented at the "Next Steps for the Fiscal Theory of the Price Level" conference at the Becker Friedman Institute for Research on Economics at the University of Chicago, 1 April. https://bfi.uchicago.edu/sites/default/files/file_uploads/6_MplsFedExpctns.pdf.

Sims, Christopher A. (2016c), "Fiscal policy, monetary policy and central bank independence", paper presented at the Kansas Citi Fed Jackson Hole Conference, August 26, 2016.

Uhlig, Harald (2016), "The FTPL: some skeptical remarks", slides presented at the "Next Steps for the Fiscal Theory of the Price Level" conference at the Becker Friedman Institute for Research on Economics at the University of Chicago, 1 April. https://bfi.uchicago.edu/sites/default/files/file_uploads/10_FTPL_Remarks_Uhlig.pdf.

Woodford, Michael (1994), "Monetary Policy and Price Level Determinacy in a Cash-In-Advance Economy," *Economic Theory,* IV (1994), 345–380.

Woodford, Michael (1995), “Price Level Determinacy without Control of a Monetary Aggregate,” *Carnegie-Rochester Conference Series on Public Policy,* XLIII (1995), 1– 46.

Woodford, Michael (1996), “Control of the Public Debt: A Requirement for Price Stability?" NBER working paper no. 5684, July 1996. Shorter version published in G.A. Calvo and M. King, eds., *The Debt Burden and its Consequences for Monetary Policy*, London: Macmillan, 1997.

Woodford, Michael (1998a), “Doing Without Money: Controlling Inflation in a Post-Monetary World", *Review of Economic Dynamics*, 1:173-219 (1998a).

Woodford, Michael (1998b) Comment on John Cochrane, `A Frictionless View of U.S. Inflation'," NBER Macroeconomics Annual, 13: 390-418 (1998b).

Woodford, Michael (1998c), “Public Debt and the Price Level,” conference draft, National Bureau of Economic Research Summer Institute. http://www.columbia.edu/~mw2230/BOE.pdf.

Woodford, Michael (1999), “Price-Level Determination under Interest-Rate Rules," unpublished, Princeton University, April 1999a. Available at http://www.princeton.edu/ .

Woodford, Michael (2001), “Fiscal Requirements for Price Stability,” *Journal of Money, Credit and Banking,* XXXIII, p.669–728.