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Checking Gollier and Weitzman's solution of the "Weitzman–Gollier puzzle"

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Abstract
In "How should the distant future be discounted when discount rates are uncertain?" (2010) Gollier and Weitzman claimed having solved the Weitzman–Gollier puzzle, concluding from a risk-averse utility maximizing model that Weitzman discounting is qualitatively correct and that when uncertain annual interest rates are highly correlated, long term discount rates are declining functions of time. This paper quantifies a similar model and comes to the opposite conclusion. Weitzman discounting is wrong; there is no puzzle if the correct method is used. Risk-neutral discount rates are growing, rather than declining functions of time under the Weitzman assumptions. Risk-averse discount rates can be declining, but must not be used to discount risky project's cash flows; risk adjusted rates must be used instead. When long term market yields are a growing function of time, it makes no sense to invest in projects of similar risk but lesser yield, irrespective of one's degree of risk-aversion.

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Keywords Weitzman–Gollier puzzle; declining discount rates; discounting

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Testing Gollier and Weitzman's Solution of the "Weitzman-Gollier Puzzle"

1. Introduction

In their paper entitled "How Should the Distant Future Be Discounted When Discount Rates Are Uncertain?" Christian Gollier and Martin L. Weitzman (2010) claim to have solved the "Weizman-Gollier puzzle," which originated in the conflicting conclusions that the two authors had separately reached before, based on the same model.

In "Why the Far-Distant Future Should Be Discounted at Its Lowest Possible Rate" Weitzman (1998) postulated that certainty equivalent discount factors should be obtained by probability weighting the discount factors of possible scenarios in a stochastic model. He concluded that certainty equivalent discount rates (CERs) should be declining functions of time.

In "Maximizing the Expected Net FV as an Alternative Strategy to Gamma Discounting" Gollier (2004) derived the opposite conclusion, by postulating that certainty equivalent compound factors should be obtained by probability weighting the compound factors of the possible scenarios, and concluded that CERs should be increasing functions of time.

An extensive literature tried to resolve the puzzle, largely by discussing the problem in the context of risk-aversion, rather than in the risk-neutrality context of the papers that gave it rise. This is also the approach of Gollier and Weitzman (2010), which many view as having settled the issue, and in which they assert the correctness of Weitzman's conclusion: "The bottom-line message that we wish for readers to take away from this paper is the following. When future discount rates are uncertain but have a permanent component, then the "effective" discount rate must decline over time toward its lowest possible value." (p. 353)

The substantive import of this recommendation, namely that in cost benefit analyses of long lived projects, future benefits should be given much higher weight than what exponential discounting would give them, a recommendation that has already found its way into public policy. HM Treasury, *The Green Book, Appraisal and Evaluation in Central Government, Treasury Guidance* (2011:98), recommends using declining discount rates for long lived projects, explaining that "The main rationale for declining long-term discount rates results from uncertainty about the future. This uncertainty can be shown to cause declining discount rates over time." Weitzman (1998) is cited as evidence. The question is, therefore, of more than academic importance.

Gollier and Weitzman (2010) reach their conclusions by formally postulating a risk-averse utility maximizing model and showing that its solution is free of the puzzling conflict, yet it can be expressed in ways that resemble the original formulations of Weitzman (1998) and Gollier (2004), from which they conclude that Weitzman was qualitatively right. This is puzzling, because the formulation can also be made to look like Gollier's, which should lead to the opposite conclusion. As it is not obvious how a puzzle that exists in a risk-neutral context can be solved in a risk-averse one, and as the conceptual model proposed by the authors is eminently computable, this paper undertakes to test their claims through a numerical example of a model like the one they propose.

There are several advantages to using a numerical example to test the behavior of such a model: the calculations are easy to follow and to verify, the results are unambiguous, and instead of merely relying on qualitative resemblances, quantitative conclusions can be derived. The examples are illustrative, but extensive sensitivity analyses show that the results obtained are robust. Even more importantly, the key conclusions are fully independent of the examples chosen.

This paper summarizes the arguments of Gollier and Weitzman (2010) and verifies the following claims:

1. "The two rigorous formulations [of the certainty equivalent rate] give the same discount rate (as a function of time), thereby resolving the "Weitzman-Gollier puzzle"." (p. 352) Can the presented results, derived assuming risk-aversion, be transferred to the risk-neutrality case, and is the puzzle solved thereby?
2. "The bottom-line message that we wish for readers to take away from this paper is the following. When future discount rates are uncertain but have a permanent component, then the 'effective' discount rate must decline over time toward its lowest possible value." (p. 353) What is the scope of this sweeping statement?
    a. Does it apply to any degree of risk-aversion?
    b. Does it apply to safe yields only, or can declining CERs also be used to directly discount the cash flows of risky projects, as they seem to imply?

This paper proceeds as follows:

In Section 2 the Weitzman-Gollier puzzle is summarized.

In Section 3, the line of reasoning of Gollier and Weitzman (2010) is summarized.

In Section 4, a numerical example is built of their analysis, and the calculations performed within its framework are used to evaluate their claims.

In Section 5, an explanation is given for why the same discount rates are obtained through both discounting and compounding in Gollier and Weitzman (2010) but not in the formulations that gave rise to the puzzle. The correct risk-neutral CER is derived and the nature of Weitzman discounting is illustrated.

Section 6 is devoted to checking Gollier and Weitzman's (2010) claim that effective discount rates should be declining.

Section 7 presents conclusions.

The supporting calculations, performed with the data of a numerical example, are detailed in the Appendix, references to which, including relevant Section numbers, are provided in parentheses where needed.

2. Summary of the Weitzman-Gollier puzzle

Weitzman (1998:207) concluded that "Uncertainty about future discount rates provides a strong generic rationale for using certainty-equivalent social discount rates that decline over time." Assuming risk-neutrality, and stochastic but constant interest rates, Weitzman defined the certainty equivalent discount factor A as the probability $\{p_i\}$ weighted average of the discount factors corresponding to the possible interest rate scenarios $\{r_i\}$ of a safe future payoff of \$1 at time t, where p_i is the probability of r_i occurring.

$$A = \sum p_i e^{-r_i t} \tag{1}$$

from which the following certainty equivalent discount rate can be derived[2]:

$$R_W = -\left(1/t\right)\ln\left(\sum_i p_i e^{-r_i t}\right) \tag{2}$$

This – the same as expression (2) in Gollier and Weitzman (2010:351) – is a declining function of time and tends to the lowest possible r_i, because present values (PVs) of distant benefits computed at high rates of interest are much smaller than those computed at low rates of interest. At the limit only the lowest rate counts in determining the certainty equivalent. This is also observed for expression (4) of Gollier and Weitzman (2010:351).

The "Weizman-Gollier puzzle" arose with the publication of Gollier (2004), in which, under similar assumptions, Gollier derived a certainty equivalent rate (CER) from the expected compound factor F, obtained by probability weighting possible future values (FVs) at time t of an initial investment of \$1.

$$F = \sum p_i e^{r_i t} \tag{3}$$

from which the following certainty equivalent rate can be derived:

$$R_G = \left(1/t\right)\ln\left(\sum_i p_i e^{r_i t}\right) \tag{4}$$

This – the same as expression (7) in Gollier and Weitzman (2010:351) – is an increasing function of time and tends to the highest possible r_i, because FVs of a given investment computed at high rates of interest are much higher than those computed at low rates of interest. At the limit only the highest rate counts in determining the certainty equivalent. This is also observed for expression (8) of Gollier and Weitzman (2010:351)

In the Weitzman formulation, the FV is certain while the PV is stochastic, whereas in the Gollier formulation the PV is certain while the FV is stochastic. This observation led Gollier (2004:5) to state that "Taking the expected net future value is equivalent to assuming that all risks will be borne by the future generation." Even though risk is irrelevant in the context of risk-neutrality, assumed by both fundamental papers of the puzzle, most of the

[2] Subscripts W attribute to Weitzman, G to Gollier.

literature trying to reconcile the two approaches appeals to the notion of risk-aversion. This is what is done in Gollier and Weitzman (2010) as well.

3. Summary of the Gollier and Weitzman (2010) argument

"How might a person resolve this distressing paradox by choosing between two such seemingly symmetric formulations, with each one having diametrically opposed implications for distant-future discounting? The answer can only come from a 'careful rigorous analysis' " state Gollier and Weitzman (2010:352), who present a specific model from which they derive their conclusions. The model is unrealistic, but suited to the analysis. "Our purpose here is to focus sharply on clarifying this particularly thorny issue by using a crisp formulation that abstracts away from all other elements of CBA. […] We do not defend this model for its realism and immediate applicability to such long-term issues as CBA of climate change." (Gollier and Weitzman, 2010:351).

The model proposed in Gollier and Weitzman (2010:351) is as follows:

> "In the highly stylized model of this paper, time $t = 0, 1, 2, ...$, is measured in discrete periods of unit length. To state loosely the issue at hand, a decision must be taken now, just before time zero (call it time 0−), whether or not to invest a marginal cost δ that will yield a marginal benefit ε at future time t. Right now, at time 0−, it is unknown what will be the appropriate future rate of return on capital in the economy. There are n possible future states of the economy, indexed by $i = 1, 2, ..., n$. As of now (time 0^-), future state i is viewed as having marginal product of capital r_i with probability $p_i > 0$, where $\Sigma_i p_i = 1$. A decision must be made now (at time 0−, just before the "true" state of the world is revealed at time t = 0) about whether or not to invest δ now in order to gain payoff ε at future time t. To pose the problem sharply, it is assumed that immediately after the investment decision is made, at time 0, the true state of the world i is revealed and the marginal product of capital will thenceforth be r_i, from time $t = 0$ to time $t = \infty$."

Gollier and Weitzman (2010) postulate that if a decision maker optimizes his consumption path by reference to a linear budget constraint represented by interest rate r_i of state of the world i, then the optimal consumption trajectory for each scenario i must satisfy the following first order condition

$$V''_i\,(C_0) = V''_i\,(C_t)\, e^{(r_i\, t)} \tag{5}$$

where $V''_i\,(C_t)$ is the marginal utility of consumption of the period indicated by the subscript of C, in the state of the world identified by subscript i of V".

At time $t = 0^-$ a safe investment opportunity arises that expends marginal cost of δ in time period 0 to yield a safe benefit of ε in time period t. The investment project will increase the expected utility of the decision maker if and only if

$$\varepsilon \sum p_i\; V''_i\,(C_t) \geq \delta \sum p_i\; V''_i\,(C_0) \tag{6}$$

Using optimality condition (5) this can be rewritten in two ways. The one called the "Weitzman approach," defines an expected discount factor, and eliminates $V''_i(C_t)$ from (6) yielding:

$$\varepsilon \sum q^{W}{}_{i}\, e^{(-r_i t)} \geq \delta \tag{7}$$

where $q^{W}{}_{i} = p_i\, V''_i(C_0) / \sum p_i\, V''_i(C_0)$.

According to Gollier and Weitzman (2010) this is equivalent to discounting ε at the following rate[3]:

$$R_d = -\left(1/t\right)\ln\left(\sum_i q_i^W e^{-r_i t}\right) \tag{8}$$

Alternatively, the "Gollier approach," defines an expected compound factor, and eliminates $V''_i(C_0)$ from (6) yielding:

$$\varepsilon \geq \delta \sum q^{G}{}_{i}\, e^{(r_i t)} \tag{9}$$

where $q^{G}{}_{i} = p_i\, V''_i(C_t) / \sum p_i\, V''_i(C_t)$

According to Gollier and Weitzman (2010) this is equivalent to discounting ε at the following rate:

$$R_c = \left(1/t\right)\ln\left(\sum_i q_i^G e^{r_i t}\right) \tag{10}$$

Since both (8) and (10) were derived from (6), it must be true that $R_d = R_c$. The authors go on to state that "This means that the adjustment of the valuation for risk resolves the "Weitzman-Gollier puzzle"," they call this the "risk adjusted discount rate" R_*, and go on to state that "qualitatively the properties of the efficient discount rate $R_*(t)$ resemble closely those of $R_W(t)$ recommended by Weitzman, with the only quantitative difference being the substitution of "Weitzman-adjusted probabilities" $\{q^{W}{}_{i}\}$ for the unadjusted probabilities $\{p_i\}$."

That the same could just as easily have been concluded about $R_G(t)$ recommended by Gollier went unstated.

4. A numerical example of the Gollier-Weitzman model

The utility function proposed by the authors is:

$$V(C) = \sum_t^{\infty} e^{-\rho t} U(C_t) \tag{11}$$

where $\rho>0$ is the pure rate of time preference and $U(C_t)$ is a utility function that the authors did not specify, but which will be taken in our proposed model to be of the constant-intertemporal-elasticity-of-substitution (CIES) type:

[3] The subscript d of R refers to the discounting approach and the subscript c to the compounding approach.

$$U(C) = \frac{C^{1-\sigma} - 1}{1-\sigma} \tag{12}$$

where consumption C > 0, and the elasticity of marginal utility with respect to consumption σ > 0 but not equal to 1. This is also the measure of the decision maker's constant proportional risk-aversion.

The above utility function will be maximized subject to the budget constraint postulated in Gollier and Weitzman (2010:352): inherited capital K_0 is the investor's only source of income. Given that Weitzman's (1998) original model has only two time periods (other than the mentioned t= 0^- used to decide, but not to act), and that it is used to analyze discounting the distant future, the budget constrained consumption path to be established in each scenario must be as follows:

$$C_t = e^{r_i t}(K_0 - C_0). \tag{13}$$

Where K_0 is the initial endowment and C are consumptions.

At the time 0^- K_0 is known, but the interest rates r_i are still unknown. Given that the arguments of Gollier and Weitzman (2010) are predicated on the decision maker being on his optimal consumption trajectory, it must be calculated for each scenario. The investor pondering whether to invest in the safe project of the Gollier Weitzman model at time 0^- must also prepare to bring himself into optimality once the interest rate to prevail in an instant and forever thence is revealed. He must solve for the optimal market investment x to be made by maximizing the following expression for each interest rate scenario:

$$V(C) = \frac{(K_0 - x)^{1-\sigma} - 1}{1-\sigma} + \frac{(e^{rt}x)^{1-\sigma} - 1}{e^{\rho t}(1-\sigma)} \tag{14}$$

subject to the constraints that $x \geq 0$ and $x \leq K_0$. Finding the utility maximizing investment x determines the consumption path as follows:

$$C_0 = K_0 - x \tag{15}$$

$$C_t = e^{rt}x \tag{16}$$

There will be one such consumption path for each scenario i.

A simple two-period, two-scenario numerical example with the following parameters will be used:

Table 1
Parameters of the numerical example

Scenario 1 interest rate, r_1	1%
Scenario 2 interest rate, r_2	5%
Probability of scenario 1, p_1	0.5
Probability of scenario 2, p_2	0.5
Endowment at time = 0, K_0	\$2,000
Constant proportional risk-aversion, σ	1.7

Pure rate of time preference, ρ	0%
Time t in years	200

All the data are arbitrary. The interest rate data choice will not affect the conclusions to be reached. The size of the endowment will affect the effective degree of risk-aversion. In sensitivity analysis, the effects of changing the chosen value to $200,000 are examined. The coefficient of risk-aversion was chosen to be 1.7 so that the results obtained illustrate the expectation that risk-averse CERs are declining functions of time, but sensitivity analysis shows that this is not always the case. The pure rate of time preference was set to zero to allow the rate of response of CERs to changes in the time horizon to be unaffected by the choice of a variable that is also multiplied by time, see expressions (11) and (14). The effect of assuming a positive rate of pure time preference is shown in sensitivity analysis.

The data of this simple example will be used to verify the claims made in Gollier and Weitzman (2010). The calculation of the optimal investment amount is shown in Section 1 of the Appendix, (referred to as A.1 henceforth). Our investor would invest $610.03 in the low interest rate scenario, and $32.04 in the high interest rate scenario. It will be on this basis that he will ponder whether to invest in a safe project costing δ and yielding ε.

At this point we can compute R_d. To this end expression (6) will be adapted to our simple example, as follows, keeping the notation of the previous Section:

$$\varepsilon\ (\ p_1\ V''_1 (C_t) + p_2\ V''_2 (C_t)\ ) \geq \delta\ (\ p_1\ V''_1 (C_0) + p_2\ V''_2 (C_0)\ ) \qquad (17)$$

Employing the “Weitzman approach,” which seeks to define a discount factor, this becomes:

$$\frac{p_1 V_1^{'}(C_t) + p_2 V_2^{'}(C_t)}{p_1 V_1^{'}(C_0) + p_2 V_2^{'}(C_0)} \geq \frac{\delta}{\varepsilon} \qquad (18)$$

If we convert the above relationship into a strict equality, to find the point of indifference, and replace δ/ε by D, we can interpret D as the expected value of the Weitzman approach discount factor, from which we can compute:

$$R_d = -\ (1/t)\ \ln(\ D\ ) \qquad (19)$$

To follow the “Gollier approach” of finding a compound factor instead, all we have to do is invert expression (18), make it a strict equality, and replace ε/δ by F, which can be interpreted as the expected value of the “Gollier approach” compound factor, from which we can compute:

$$R_c = (1/t)\ \ln(\ F\ ) \qquad (20)$$

It is clear from this that $D = 1/F$, which is as it should be[4], as expected discount factors are the inverses of expected compound factors, and therefore $R_d = R_c$. This means that CERs can be computed from either expected discount or compound factors. Notice that the use of adjusted probabilities defined by (7) and (9) is not needed for computational purposes.

[4] If $D = \delta/\varepsilon$ and $F = \varepsilon/\delta$, then $D = 1/F$.

Gollier and Weitzman (2010) only performed those to show that expressions that look like (2) and (3) can be derived from (6).

With the data of our example we obtain (A.2) that $R_d = R_c = 1.22\%$. This result was calculated directly from (17), without going through the transformations that Gollier and Weitzman (2010) used to obtain expressions morphologically similar to the original Weitzman (1998) and Gollier (2004) formulations, or using "risk adjusted probabilities." The same result is obtained, however, if one works through those transformations (A.3).

The first half of Gollier and Weitzman's (2010:352) assertion that "The two rigorous formulations give the *same* discount rate (as a function of time), thereby resolving the 'Weitzman-Gollier puzzle' " is correct, given that the two formulations R_d and R_c, (8) and (10), derived above in the context of risk-aversion, are equal. But the second part of the statement is not true, however, because this equality does not solve the puzzle that exists in the context of risk-neutrality. R_W and R_G, (2) and (3) respectively, are unequal and are unrelated to R_d and R_c, as the following table shows (A.4):

Table 2

Certainty equivalent rates for selected time horizons

Years till time *t*	**100**	**200**	**300**	**400**	**500**
Risk-neutral R_W	1.67%	1.35%	1.23%	1.17%	1.14%
Risk-averse R_d or R_c	1.41%	1.22%	1.17%	1.14%	1.12%
Risk-neutral R_G	4.33%	4.65%	4.77%	4.83%	4.86%

The pairwise morphological resemblance between the definitions of R_d and R_c, as defined in expressions (8) and (10), on the one hand, and those of R_W and R_G, as defined in expressions (2) and (4) on the other, respectively, does not make them all equal. The equality between R_d and R_c neither makes $R_W = R_G$ nor provides an explanation for $R_W \neq R_G$. Furthermore, there is no reason why R_d should equal R_W, as the former pertains to a risk-averse investor, whereas the latter applies to a risk-neutral one.

There is one instance in which Gollier and Weitzman (2010:353) argue that Weitzman's rule (transposed to the case of risk-averse investors) is "qualitatively and quantitatively correct." This happens to be a very special case, however: that of investors with $\sigma = 1$, which means that their utility function is logarithmic. In that case, the Gollier and the Weitzman CER calculation methods yield the same result, but display neither growing nor declining trends. This is due to the coincidence that the natural logarithm of the utility function is the inverse function of the exponentiation that continuous compounding involves. This case therefore neither proves that the Weitzman method is correct, nor supports declining or growing CERs. See A.6.

Having checked that the a utility maximizing model that follows the Gollier and Weitzman (2010) assumptions produces correct and consistent CERs for risk-averse investors, we can use it to calculate CERs for lower degrees of risk-aversion than the one tested so far, and, at the limit, to identify the correct risk-neutral CER, given that when the risk-aversion coefficient $\sigma = 0$, the utility function of expression (12) becomes $U(C) = C - 1$, which is an instance of the risk-neutral utility function implicitly assumed by Weitzman (1998). Thus the model can numerically verify the correctness of Weitzman discounting conclusively[5], in a way that goes well beyond the formulaic resemblance that Gollier and Weitzman (2010) considered sufficient to declare the puzzle solved.

The following table shows CERs calculated for selected coefficients of proportional risk-aversion (A.7).

Table 3
CERs for selected time horizons and degrees of risk-aversion

Years till time *t*	100	200	300	400	500
σ = 1.9	1.38%	1.22%	1.17%	1.14%	1.12%
σ = 1.7	1.41%	1.22%	1.17%	1.14%	1.12%
σ = 1.5	1.44%	1.22%	1.16%	1.13%	1.11%
σ = 1.3	1.50%	1.23%	1.16%	1.13%	1.11%
σ = 1.1	1.60%	1.28%	1.17%	1.13%	1.10%
σ = 0.9	1.79%	1.50%	1.41%	1.37%	1.36%
σ = 0.7	2.22%	2.13%	2.15%	2.17%	2.19%
σ = 0.5	2.88%	2.97%	2.99%	3.00%	3.00%
σ = 0.3	3.58%	3.76%	3.79%	3.80%	3.80%
σ = 0.1	4.11%	4.41%	4.51%	4.55%	4.59%
σ = 0.05	4.22%	4.54%	4.65%	4.71%	4.75%
σ = 0.01	4.31%	4.63%	4.75%	4.81%	4.84%
σ = 0	4.33%	4.65%	4.77%	4.83%	4.86%

Table 3 shows that while for higher values of σ CERs are declining functions of time (shaded cells of Table 3), this begins to gradually change as σ decreases. CERs become strictly increasing functions of time in the examined time range when $\sigma \leq 0.5$, including the risk-neutrality case. What is most significant, however, is that the last lines of Table 2 and Table 3 are identical. What this means is that R_G is the correct risk-neutral CER, not R_W. The proposed utility maximizing model does help to solve the Weitzman-Gollier puzzle after all: not by showing Weitzman discounting to be correct, as claimed by Gollier and Weitzman (2010), but by showing it to be wrong.

5. Explaining the Puzzle

5.1 The cause of the puzzle

The fact that $R_W \neq R_G$ is what constitutes the puzzle. Solving it requires finding the conditions that would make $R_W = R_G$. Using the notation of the previous Section, in which D is the expected discount factor and F is the expected compound factor, we can say that the puzzle will be solved if the following expression, obtained by setting the equivalents of expressions (2) and (4) to be equal, is true:

$$R_W = -\left(1/t\right)\ln\left(D\right) = R_G = \left(1/t\right)\ln\left(F\right) \tag{21}$$

[5] I am grateful to Claire Boeing-Reicher for the suggestion to use the proposed model to this end.

For this to be true, the following must hold:

$$-\ln(D) = \ln(F) \tag{22}$$

Which means that it must be true that

$$D = \frac{1}{F} \tag{23}$$

This condition is met in the numerical example of the analysis proposed by Gollier and Weitzman (2010). In both Sections A.2 and A.3 of the Appendix we can see that the expected compound factor F is 11.4783 and the expected discount factor is D is 0.08712: they are each other's reciprocals. Furthermore, risk-averse D complies with the definition of PV[6], as demonstrated in A.5: investing the PV of δ in the stochastic market has the same expected utility as the safe future amount ε.

There is no puzzle in Gollier and Weitzman's (2010) because the expected discount and compound factors are each other's reciprocals. However, Weitzman's original expected discount factor and Gollier's original expected compound factor, both of which pertain to risk-neutral investors, are not each other's reciprocals, as can be seen by multiplying (1) and (3):

$$\sum p_i e^{-r_i t} \sum p_i e^{r_i t} \neq 1 \tag{24}$$

This is what causes the puzzle.

The calculations performed have already shown that CERs derived by Weitzman's method are wrong. Formally extending the "rigorous" analysis proposed by the authors to the case of risk-neutrality can be used to derive the correct risk-neutral CER.

5.2 Deriving the correct risk-neutral CER

In the context of risk-neutrality the FV (*FV*) at time t of amount I invested at time 0 in each scenario i is $FV_i = I\, e^{(r_i t)}$. The expected FV of I is obtained by probability weighting all possible scenarios. As under risk-neutrality expected values are worth the same as certain amounts, the expected value operator *E()* will generally be omitted for simplicity:

$$FV = I \sum p_i e^{r_i t} \tag{25}$$

The expected compound factor F is the factor by which we need to multiply the invested amount I to get its expected future value FV, so it is defined by the following expression:

$$F \times I = FV \tag{26}$$

Therefore, from (26) and (25)

[6] "The present value of an asset is obtained by calculating how much money invested today would be needed, at the going interest rate, to generate the asset"s future stream of receipts." (Paul A. Samuelson and William D. Nordhaus, 1992:271.)

$$F = \frac{FV}{I} = \sum p_i e^{r_i t} \tag{27}$$

Similarly, we can define the expected discount factor *D* as the factor by which we need to multiply any FV (or expected FV) at time *t* to get its PV at time 0. In the case of the *FV* of *I*, *D* is defined by:

$$I = FV \times D \tag{28}$$

Therefore

$$D = \frac{I}{FV} = \frac{I}{I \sum p_i e^{r_i t}} = \frac{1}{\sum p_i e^{r_i t}} \tag{29}$$

Notice that the correct expected discount factor *D*, derived from the nature of the market to which it pertains, is markedly different from Weitzman's proposed measure *A*, expression (1)!

As the expected FV of *I* is

$$FV = I \sum p_i e^{r_i t} \tag{30}$$

the PV of the *FV* of *I* is

$$PV = FV \times D = \frac{I \sum p_i e^{r_i t}}{\sum p_i e^{r_i t}} = I \tag{31}$$

which is what compliance with the definition of PV requires. Notice that in the context of risk-neutrality it makes no difference whether we discount the expected FV of *I* by multiplying it by the discount factor *D*, or alternatively, multiply the scenario specific FVs by the same constant factor *D* and then probability weight the resulting terms. This is so because of the distributive property of multiplication. The key point is that the same *D*, a pure number derived from the known interest rate uncertainty, is used in both cases, and not scenario specific discount rates.

For $I = 1$, the certainty equivalent rate r^* is defined by $F = e^{(r^* t)}$, from which

$$r^* = \frac{1}{t} \ln\left(\sum p_i e^{r_i t}\right) \tag{32}$$

or it can be calculated from *D* by noting that $F = 1 / D$ and that $ln(F) = -ln(1/F) = -ln(D)$. Therefore

$$r^* = -\frac{1}{t} \ln\left(\frac{1}{\sum p_i e^{r_i t}}\right) = \frac{1}{t} \ln\left(\sum p_i e^{r_i t}\right) \tag{33}$$

Notice that r^* is the same regardless of whether it was derived from compound factor *F* or discount factor *D*. The "Weitzman-Gollier Paradox" is absent when the correct discount factor *D* is used. As has been noted by Gollier, r^* – equal to the certainty equivalent rate of Gollier (2004), expression (4) above – is a growing function of time, and tends to the highest possible interest rate, turning around the conclusion that should be drawn from the Weitzman (1998) model.

5.3 What is Weitzman discounting?

How can Weitzman's *A*, expression (1), be interpreted if it does not compute the correct expected PV? Notice that expression (1) corresponds exactly to expression (3) when the product $r_i \times t$ is negative. Weitzman's discount factor in expression (1) is the same as the compound factor of expression (3), but for the negative signs in the exponents. Having negative r_i would correspond to a capital market in which resources are stored for a fee, rather than being lent to someone willing to pay a positive interest rate. Having *t* negative would imply reversing the flow of time.

5.3.1 Weitzman discounting is time reversed negative compounding

Weitzman discounting is time reversed negative compounding: discounting with the negatives of the assumed interest rates and reversing the arrow of time by changing the sign of the resulting negative CERs, which is the equivalent of considering the FV obtained by such compounding to be a PV. Table 4 shows, with the data of our example, that Weitzman CERs are the absolute values of the CERs that result from compounding with the negatives of the interest rates assumed.

Table 4

Standard CERs with negative rates and Weitzman CERs with positive rates

Years	t	**100**	**200**	**300**	**400**	**500**
Compound factor ***F*** when r_1= –5% r_2= –1%	$\sum p_i e^{r_i t}$	0.1873	0.0677	0.02489	0.009158	0.003369
CERs when r_1= –5% r_2= –1%	$\frac{1}{t}\ln\left(\sum p_i e^{r_i t}\right)$	–1.67%	–1.35%	–1.23%	–1.17%	–1.14%
Weitzman CERs when r_1= 1% r_2= 5%	$-\frac{1}{t}\ln\left(\sum p_i e^{-r_i t}\right)$	1.67%	1.35%	1.23%	1.17%	1.14%

Changing the sign of the resulting negative CERs effectively reverses the arrow of time, because of the following relationship:

$$CER = \left(1/t\right)\ln(F) = -\left(1/t\right)\ln\left(\frac{1}{F}\right) \qquad (34)$$

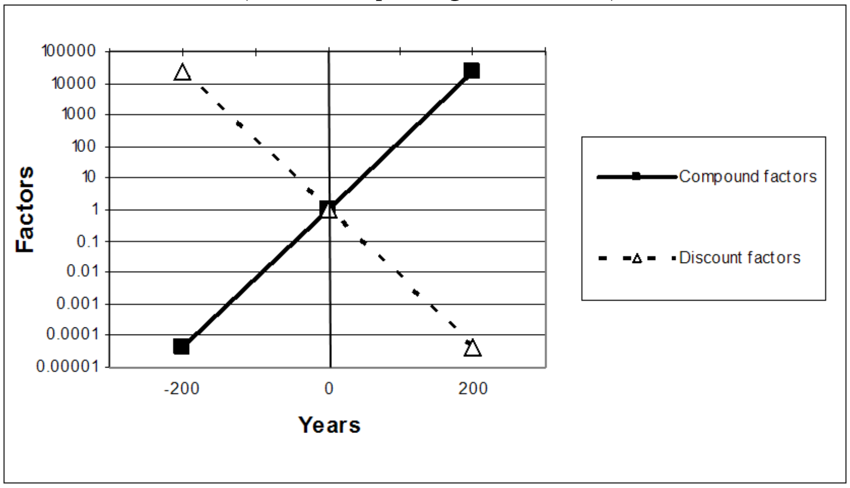
Making a discount factor *(1/F)* out of compound factor *F* changes the arrow of time by taking a FV to be a PV, and changes the sign of the negative CERs that correspond to the negative interest rates implicit.

But it isn't just signs that change, but also absolute values. The difference between discounting and negative compounding will be explained with the help of two Figures. Figure 1 shows the compound and discount factors curves applicable to an investment of $1 made at time 0, in continuous time, with a deterministic annual interest rate of 5%, between years –200 and 200. (We have negative compounding and discounting to the left of year 0. Time reversal occurs when the sign of the resulting CER is changed.) The equations being plotted are $e^{0.05t}$ for the compound factors curve, and $1/\ e^{0.05t}$ for the discount factors curve. The vertical scale in the figures is logarithmic, which is why both the compound factors and discount factors curves are seen to be linear. The fact that one is the inverse of the other is evidenced by their symmetry with respect to the horizontal line passing through the value of 1. Note that the negative range of the compound factor curve is symmetrical to the positive range of the discount factor curve around the vertical axis (year 0), which means that in the deterministic case discounting and time reversed negative compounding are equivalent.

Figure 1

Compound and discount factors

(5% interest p.a., logarithmic scale)



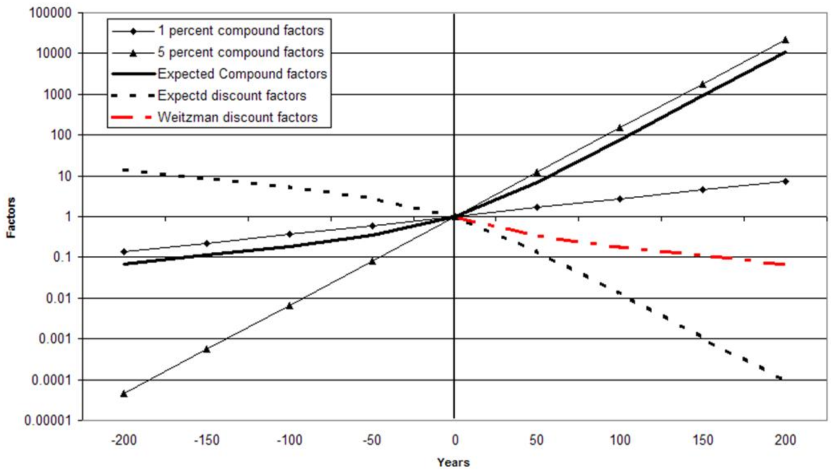
This is the reason why Weitzman's A, expression (1), was accepted by nearly everyone as being correct despite it not having been derived from anything. If time reversed negative compounding is correct in the deterministic case, why could not the stochastic expected discount factor be obtained, as Weitzman proposed, by probability weighing the discount factors of the alternative scenarios? The reason was already given analytically in the previous Section, but will be explained further with the help of another figure.

Figure 2 illustrates the stochastic case. It is assumed that interest rates can be either 1% or 5%, with equal probabilities. Figure 2 shows the compound factor curves corresponding to 1% and 5%, both of which are linear in logarithmic terms. Their expectation is no longer linear, however. Moving forward in time (positive range of years), compound factors corresponding to the high interest rate grow comparatively larger relative to those of the low interest rate, thereby pulling their expected value ever closer to the compound factors curve of the high rate. The same happens moving backwards into the past (negative range of years), in which case it is the compound factors corresponding to the low interest rate that grow relatively larger, and it is therefore towards the compound factors curve of the low interest rate that their expected values tend asymptotically. In other words, the higher compound factors pull the expected compound factors upwards over the entire time range, this effect being stronger as the absolute value of time increases.

The immediate consequence of this is that the expected compound factors curve is no longer linear logarithmically. This is also true of the expected discount factors curve, which is the inverse of the expected compound factors curve. Because of this lack of linearity, the negative range of the expected compound factor curve is not symmetrical to the expected discount factor curve with respect to the vertical axis, and cannot be used, therefore, to calculate PVs correctly. As Figure 2 shows, the negative range of the compound factors curve is significantly higher than the positive range of the discount factors curve for all absolute values of time.

Figure 2

Compound factors at 1% and 5%, their expected value and the corresponding discount factors, logarithmic



This is the reason why the probability weighted average of the conditional discount factors of alternative interest rate scenarios (which is what the negative range of the expected compound factors curve is, and which Weitzman used to calculate expected PVs) does not yield the correct expected PV of amounts compounded to the future. To facilitate comparison with the correct discount factors, the former are mapped to the positive range of years and labeled Weitzman discount factors in Figure 2. They significantly overstate the PV of future sums.

5.3.2 *What is the nature of the capital market that can be derived from Weitzman discounting?*

Neither Weitzman (1998) nor Gollier and Weitzman (2010) derive *A*, expression (1), from a market description or from anything else; it is just presented in both papers as an obvious truth. Given that discount and compound factors are each other's reciprocals, we can derive the compound factors that correspond to Weitzman discount factors *A* and see what kind of a capital market is implicit when *A* is taken to be the expected discount factor[7].

It follows from the definition of *A* that *PV*, the expected PV of a future sum *P*, is given by:

$$PV = A \times P \tag{35}$$

If we define *B* to be the expected compound factor such that it converts a PV into an expected FV, it must satisfy the following condition:

[7] I am grateful to Derek Lemoine for the suggestion that presenting this would help understand the nature of Weitzman discounting.

$$PV \times B = P \tag{36}$$

Therefore, using (35):

$$B = P / PV = P / (A \times P) = 1 / A \tag{37}$$

Combining (1) and (37) it must be true that

$$B = 1 \Big/ \sum p_i e^{-r_i t} \tag{38}$$

Because *B* is the reciprocal of *A*, the two together will comply with the definition of PV. Discounting amount *P* with discount factor *A* will yield an expected PV such that when multiplied by compound factor *B* it will yield a FV equal to *P*. This means that investing the PV at the interest rates implicit in *A* will yield amount P. As will be seen below, however, the rates implicit in *A* are not those of the market.

For $PV = 1$, the certainty equivalent rate r_w is defined by $B = e^{(r_w \times t)}$, from which

$$r_W = \frac{1}{t}\ln\left(1 \Big/ \sum p_i e^{-r_i t}\right) = -\frac{1}{t}\ln\left(\sum p_i e^{-r_i t}\right) \tag{39}$$

or it can be calculated from *A* by noting that when $P = 1$, $A = e^{(-r_w t)}$. Therefore

$$r_W = -\frac{1}{t}\ln\left(\sum p_i e^{-r_i t}\right) \tag{40}$$

Again, there is no paradox. CERs derived from both discount and compound factors are identical. As has been noted by Weitzman (1998), r^w is a declining function of time, and tends to the lowest possible discount rate.

The difference between r^* – expression (29) or (30) – and r_w – expression (39) or (40) – does not derive from one having been defined from a compound factor and the other from a discount factor, but rather from what each assumes about future returns in the market for which CERs are sought. This difference will be illustrated though a simple numerical example: two equally likely scenarios, with possible interest rates of $r_1 = 1\%$ and $r_2 = 5\%$.

We first establish as a benchmark the behavior of the market that Weitzman explicitly assumed, as described at the beginning of Section 3, for the case of our specific numeric example. It results in the following expected FVs of investing $1:

Table 5
Behavior of the market assumed by Weitzman (1998)

Years	*t*	**100**	**200**	**300**	**400**	**500**
FV low	$e^{(0.01\,t)}$	2.718282	7.389056	20.08554	54.59815	148.4132
FV high	$e^{(0.05\,t)}$	148.4132	22026.47	3269017	4.85E+08	7.2E+10
E(FV)	(FV low +FV high)/2	75.56572	11016.93	1634519	2.43E+08	3.6E+10

The values of the compound factors F, derived as in the previous Section, and the corresponding CERs for the same years, are the following:

Table 6
Standard compound factors and CERs

Years	t	100	200	300	400	500
F	$\sum p_i e^{r_i t}$	75.56572	11016.93	1634519	2.43E+08	3.6E+10
CER	$\frac{1}{t}\ln\left(\sum p_i e^{r_i t}\right)$	4.33%	4.65%	4.77%	4.83%	4.86%

The first line of Table 6 is identical to the last line of Table 5. Unsurprisingly, the values of F correspond to the assumed market behavior, given that F was derived from (3).

In contrast, the values of compound factor B, derived not from the market description, but from Weitzman's discount factor A, yields different results:

Table 7
Weitzman compound factors and CERs

Years	t	100	200	300	400	500
B	$1/\sum p_i e^{-r_i t}$	5.3387	14.77316	40.1708	109.1963	296.8263
CER	$-\frac{1}{t}\ln\left(\sum p_i e^{-r_i t}\right)$	1.67%	1.35%	1.23%	1.17%	1.14%

Clearly Weitzman compound factor values B do not correspond to the assumed market behavior, but the CERs are the same as the Weitzman CERs already computed in Table 4. What stochastic market interest rate assumptions could compound factors B correspond to? An infinite number of assumptions could yield the values of B in Table 7, but for illustrative purposes let's find out what the interest rates implicitly assumed by Weitzman's expression A are if we enforce the constraint of our numerical example, namely that the high annual interest rate exceed the low annual interest rate by 4 percentage points. Then Weitzman's A implies the following market interest rates:

Table 8
A set of market rates compatible with Weitzman discounting

Years	t	100	200	300	400	500
Low rate	Found by iteration to get the value of B	–1.65%	–2.31%	–2.54%	–2.65%	–2.72%

High rate	Low rate + 4%	2.35%	1.69%	1.46%	1.35%	1.28%
FV low	$e^{(\text{Low rate} \times t)}$	0.192049	0.009908	0.000494	2.46E-05	1.22E-06
FV high	$e^{(\text{High rate} \times t)}$	10.48551	29.5364	80.34116	218.3926	593.6526
E(FV) = *B* above	(FV low +FV high)/2	5.33878	14.77316	40.17083	109.1963	296.8263
CER	ln(*B*)/t	1.67%	1.35%	1.23%	1.17%	1.14%

These results agree with common sense. Assuming the perfect year to year correlation of annual interest rates implicit in the two-period long-term Weitzman model, in which a single rate remains unchanged for each scenario, the acceleration effect enjoyed by higher rates will make them outweigh the contributions of lower rates and result in increasing CERs. In such circumstances the only way to get declining risk-neutral CERs while preserving the imposed difference between high and low rates is to assume the possibility of negative rates, which Weitzman (1998:204) explicitly excluded, or failing that, to assume declining annual rates, contrary to the assumption of unchanging annual rates, or both. When interest rates are non-negative and perfectly correlated, is impossible to have declining risk-neutral CERs. In other words, Weitzman's discounting is incompatible with Weitzman's market description.

5.4 Comparing the performance of the correct and Weitzman's discounting methods

Weitzman proposed discount factor *A* for the calculation of risk-neutral CERs. The purpose of any market CER is to establish a hurdle rate that the CERs of uncertain projects have to exceed in order to increase investors' welfare over and above what investing their costs in that market would bring. Weitzman's measure *A* fails at this task, for the risk-neutral market CER it computes does not correctly measure the opportunity cost of not investing in the market characterized by the interest rates it assumes.

First, *A* fails to comply with the definition of PV. With the data of our example, it computes the PV of $1 in year 200 as $0.06769 (A.9), which when invested in the market has an expected yield of $745.74 in year 200 (A.10). In contrast discount factor *D* does comply, as the PV of $1 in year 200 is $ 9.0769E-05, which when invested will have an expected yield of $1 in year 200 (A.8).

Second, making a small investment yielding $R_d = R_c$ (4.65% in our example, A.8) will leave the welfare of the risk-neutral investor unchanged, because the opportunity cost of making the small investment is equal to its yield. Someone investing in a small project yielding R_W (1.35% in our example, A.9), however, will incur a loss, as the opportunity cost of the funds exceeds the project's yield. Therefore, using Weitzman's discount factor *A* to judge investment decisions will lead to welfare loss. Someone paying $0.06769 for a gain of $1 in year 200, when he could instead invest just $ 9.0769E-05 in the market for the same gain, will lose the difference between these amounts. This is equivalent to wasting 99.9% of the investment with the data of our example (A.11).

This just illustrates an assertion that will hold true in all cases under the assumptions of the Weitzman model: an investor paying a Weitzman PV for any future yield will always suffer an opportunity loss, because the Weitzman PV, computed at declining discount rates, will always be higher than the PV computed at the growing discount rates of the market.

In fact, an investor using Weitzman discounting will become a money pump. For example, a risk-neutral investor willing to pay \$0.06769 for a gain of \$1 in year 200 will find many takers in the market, as it would be possible to hedge the bet for just \$9.0769E-05 with another risk-neutral counterparty.

Anyone using Weitzman discounting for decision making will violate the principle of transitivity of preferences and will not be able to maximize his utility. This is why any utility maximizing model will show Weitzman discounting to be incorrect in the case of risk-neutrality.

5.5 Conclusion

The rigorous analysis proposed by Gollier and Weitzman (2010) shows that CERs can be computed either from compound or discount factors, because these are each other's reciprocals. This is true irrespective of the degree of risk-aversion assumed. The Weitzman method of CER calculation under risk-neutrality is incorrect precisely because it fails to comply with this proviso.

Contrary to the widely-held belief, the Weizman-Gollier puzzle is not caused by one CER being derived from a discount factor and another one from a compound factor. It is caused by the fact that the interest rates implicit in Weitzman's discount factor *A* are not the ones explicitly assumed for the market.

This is a simple matter of financial arithmetic. The unproven notion of calculating expected PVs by probability weighting deterministic discount factors that seems so appealing *a priori* turns out to be specious. The expected value of the conditional discount factors is not the correct risk-neutral expected discount factor. The inverse of the expected compound factor is, and consequently there is no puzzle.

6. Which discount rates are declining and how are they to be used?

That the bottom line message of Gollier and Weitzman (2010) is not true for risk-neutral investors was already shown in Section 5.2 above. Correctly calculated risk-neutral CERs are increasing functions of time and tend to the highest possible rate under the assumptions of the Weitzman model.

Risk-averse market CERs can be declining, however, for certain degrees of risk-aversion (although not for all, as shown in Table 3 above). To explore the question of how risk-averse discount rates can be used, and what happens to the relative weight of future benefits and costs, it is useful to distinguish three concepts:

1. Market CER. This is the certainty equivalent rate corresponding to a utility preserving safe investment. As extensively analyzed in Section 4, it is computed from the probability distribution of the utilities of market yields. This is how Gollier and Weitzman (2010) defined the discount rate.

2. Project CER. This is computed the same way as the market CER, but from the probability distribution of the utilities of project yields.
3. Project IRR. This is the internal rate of return computed from the expected values of project net flows (benefits less costs). It differs from the project CER in that the calculation is based on the monetary values, not on the utility of such values.

For an investor to prefer investing in a project rather than in the market, the project CER must exceed the market CER, due to the requirement of transitivity of preferences. Both CERs are defined by comparable safe or certainty equivalent yields, and the higher yield, hence the higher CER, must be preferred to the lower one.

Because the utility function of risk-neutral investors is a linear transformation of monetary receipts, project CERs and IRRs are identical in their case. Market CERs are also computed based on monetary yields for risk-neutral investors. In their case, therefore, the project IRR is directly comparable to the expected market monetary return. This is the basis of the common CBA test of seeing if the project's IRR exceeds the discount rate, which can equivalently be ascertained by checking if the project's NPV is positive when discounted by the discount rate.

For risk-averse investors, however, this shortcut is not open. The discount rate (market CER) can be compared to the project CER, computed by reference to the same utility function, but not to the project IRR, which is not based on the utility function. Consequently, the discount rate cannot be used to discount the monetary flows of risky projects. (It can be used to discount risk free yields, however, since the market CER is the IRR of a risk-free monetary yield).

As discounting risky project cash-flows with a risk-averse discount rate leads to very large errors (see A.12), how can we examine the impact of declining risk-averse discount rates on the relative weight that risk-averse investors should give to future costs and benefits? To do this we introduce the concept of required monetary return, which is the IRR that a project must yield for the project's CER to equal the market CER, as computed for a risk-averse investor. As shown in A.14, this is also a rate at which the project cash flow can be discounted. This rate, which can also be called the risk adjusted discount rate (RAR), will yield the same PV (when applied to the project cash flow) as will the risk-averse project CER (when applied to the certainty equivalents of the project's yields). This rate is project specific, however, so it cannot act as a general hurdle rate. The only rule available to risk-averse investors for risky projects is the comparison of project and market CERs.

Numerical examples will be used to explore the question of relative weights trough time. Calculations will be made of the required (or risk adjusted) monetary returns (RARs) of small investment projects that have the same expected annual return as the market, but (a) are riskier, or (b) are equally risky, but their risk is negatively correlated with the market's risk. These will be computed for selected time horizons, using the same data as before. For each selected time horizon, the following is done:

1. The market's expected monetary return is computed. This is the market CER of risk-neutral investors. This is the risk-neutral discount rate.
2. The optimal consumption path is computed for each scenario and the expected utility of time-period *t* is calculated. Time zero utility can be ignored, because

the choice of investing a small amount in a project rather than in the market does not affect consumption at time zero.

3. The risk-averse market CER is computed by finding the rate of return of an investment of $0.0001 with a safe yield such that the investor is indifferent between investing in it or at the stochastic market rate. This is the risk-averse discount rate.
4. A small project is defined with an investment cost of $0.0001, an expected annual return of x, and high and low rates defined in such a way that the coefficient of variation of rate x is 1.5 times the coefficient of variation computed from the low and high scenario market rates (1% and 5%, respectively). It is assumed that the market risk and project risk are perfectly correlated. For each time horizon, the value of x that leaves the total utility of the risk-averse decision maker unchanged is found, and from that the required monetary rate of the small project so defined is computed. This is the risk adjusted discount rate (RAR) of the project. Notice that by definition its CER is the same as the risk-averse market CER or the discount rate.
5. Another small project costing $0.0001 is defined, the coefficient of variation of its expected annual return x is the same as that of the market rates, but this time the risk of this project is inversely correlated with the risk of the market. The project's RAR is computed.

With these assumptions, the following results were obtained (see A.12 for details of the calculations):

Table 9
Certainty equivalent rates for selected time horizons

Years till time *t*	**100**	**200**	**300**	**400**	**500**
Risk-neutral CER	4.33%	4.65%	4.77%	4.83%	4.86%
Risk-averse CER (σ = 1.7)	1.41%	1.22%	1.17%	1.14%	1.12%
Risky project's RAR	5.16%	5.16%	5.08%	5.04%	5.03%
Negative correlation project's RAR.	0.75%	0.87%	0.94%	0.97%	0.97%

The implications of these results on the question of relative weights that should be given to future project benefits are the following:

- The discount rate to be used by risk-neutral investors (including by the public sector, by the common CBA assumption of its risk-neutrality) is the market CER, which is an increasing function of time (with Weitzman model assumptions).
- Risk-averse CERs decline when $\sigma = 1.7$, but this is not the case for all degrees of risk-aversion, as Table 3 shows. However, risk-averse CERs can only be used to discount safe amounts (such as certainty equivalent values of project net flows or divine IOUs). Discounting risky cash-flows with this rate is a grievous error.
- Investment projects with risk equal to that of the market must have (growing) expected monetary returns equal to those of the market for them to have sufficiently high (possibly declining) risk-averse CERs to be acceptable to risk-

averse investors. This is true for any degree of risk-aversion. In other words, these projects must have IRRs equal to growing risk-neutral CERs, even for risk-averse investors!

- Projects that are riskier than the market must have IRRs even higher than those of the market (risk-averse RARs are then higher than risk-neutral CERs), but as Table 9 shows these could be declining functions of time.
- If the risks are the same as those of the market, but negatively correlated with it, then RARs are lower than risk-averse market CERs, because such projects are better than safe projects. As the Table 9 shows, these RARs can be positive functions of time.

It is instructive to examine the behavior of risky project RARs as a function of the degree of risk-aversion. This is shown in the following table.

Table 10
Risk adjusted discount rates of the risky project for selected time horizons and degrees of risk-aversion

Years till time *t*	**100**	**200**	**300**	**400**	**500**
σ = 1.9	5.19%	5.16%	5.07%	5.04%	5.02%
σ = 1.7	5.15%	5.15%	5.08%	5.04%	5.02%
σ = 1.5	5.10%	5.15%	5.08%	5.05%	5.03%
σ = 1.3	5.04%	5.13%	5.09%	5.06%	5.04%
σ = 1.1	4.94%	5.07%	5.07%	5.06%	5.05%
σ = 0.9	4.79%	4.88%	4.89%	4.89%	4.90%
σ = 0.7	4.59%	4.71%	4.78%	4.83%	4.86%
σ = 0.5	4.43%	4.66%	4.77%	4.83%	4.86%
σ = 0.3	4.36%	4.66%	4.77%	4.83%	4.86%
σ = 0.1	4.33%	4.65%	4.77%	4.83%	4.86%
σ = 0.05	4.33%	4.65%	4.77%	4.83%	4.86%
σ = 0.01	4.33%	4.65%	4.77%	4.83%	4.86%
σ = 0	4.33%	4.65%	4.77%	4.83%	4.86%

Comparing Table 10 to Table 3 we can see that generally higher degrees of risk-aversion result in lower CERs but higher RARs. These results make economic sense. Risk-aversion implies a willingness to trade yield for safety. Hence the higher the degree of risk-aversion, the lesser the value of a project of a given risk, or the greater the required return of a risky project. Conversely, the lower the degree of risk-aversion, the lower the RAR. When risk-neutrality is reached, risk has no cost, and the RAR is equal to the risk-neutral market CER.

These results are in stark contrast with the assertion of Gollier and Weitzman (2010) that declining risk-averse discount rates will give higher weight to future benefits. And we have seen already that discounting costs and benefits of risky projects with risk-averse CERs leads to large errors!

Weitzman's original model was framed in the context of risk-neutrality, and therefore his recommendation to use declining discount rates is equivalent to the prescription of discounting the expected monetary flows of projects at declining rates. As the results of Table 9 illustrate – but expressions (32) and (33) demonstrate – this recommendation is not justified for risk-neutral investors, and neither is it justified for projects that are as risky or riskier than the market for risk-averse investors. Risk-averse CERs of market rates for high enough degrees of risk-aversion do decline, because risk-averse investors are willing to trade yield for safety, but by the same reasoning risk-averse CERs of project yields also decline. Therefore, for a project to be preferred to the market by a risk-averse investor, its risk-averse CER would have to be higher than that of the market, and for that to occur, its expected monetary return (IRR) should be higher than the market's as well, provided its risks are not lower than those of the market, or are not negatively correlated with them.

The above results are illustrative examples, which sensitivity analyses show to be robust[8] (A.15). Independently of the illustrative results, however, the following generalizations are universally valid:

1. Risk-neutral CERs are a growing function of time if the Weitzman (1998) assumption of perfectly auto-correlated interest rates holds, as shown in Section 5.
2. Risk-averse CERs can be declining functions of time, but must not be used to discount future benefits of projects; risk adjusted discount rates should be used instead.
3. The process of converting monetary risk to expected utility affects market and project yields alike, and for that reason projects with risks like those of the market (and correlated with them) must yield at least as much as the market for them to be acceptable to risk-averse investors; their required yields will therefore be growing with time, rather than declining. Required monetary yields may be lower, however, for independent, lower, or negatively correlated risks.

Gollier and Weitzman's (2010:353) "bottom-line message that […] when future discount rates are uncertain but have a permanent component, then the 'effective' discount rate must decline over time toward its lowest possible value" is not correct as stated, because risk-neutral CERs that can be used to discount benefits and costs are increasing, not declining functions of time under their assumptions, and risk-averse CERs, which can be declining, cannot be used to discount project benefits and costs directly, unless they are risk free. Their unqualified bottom line conclusion that future benefits of projects should effectively be discounted by declining rates is wrong.

7. Conclusions

In their paper entitled "How Should the Distant Future Be Discounted When Discount Rates Are Uncertain?" Christian Gollier and Martin L. Weitzman (2010) claim to have solved the "Weitzman-Gollier puzzle." They conducted their analysis on a model that replicates the assumptions of the original Weitzman (1998) paper, but in which the behavior of a risk-averse investor is analyzed, rather than that of the risk-neutral investor of Weitzman's model and the puzzle. This paper used a numerical example to verify the validity

[8] All conclusions remain valid even if the assumption of clairvoyance made in Gollier and Weitzman (2010) is dropped. In such case the investor will choose an optimal time 0 consumption and will be uncertain about consumption at time t, which will be scenario dependent. He will not be on the optimal consumption path in either scenario, and the CER calculation formulas dependent on such optimality cannot be used, but optimal expected consumption and CERs can still be calculated using numerical methods.

of their assertions by carrying out the calculations suggested in their paper, and arrived at the following conclusions:

1. Regarding the claim of having solved the "Weizman-Gollier puzzle:"
   a. CERs derived from either discounting or compounding are the same in the model proposed in Gollier and Weitzman (2010), just as claimed.
   b. CERs derived from either discounting or compounding in the Gollier-Weitzman (2010) model are the same because the corresponding expected compound and discount factors are each other's reciprocals.
   c. Gollier and Weitzman (2010) declare Weitzman discounting to be qualitatively correct by showing that it is possible to find expressions involving risk adjusted probabilities that are morphologically similar to the Weitzman (1998) and Gollier (2004) expressions for risk-neutral CERs. However, their equality in the risk-averse case does not translate into the equality of the analogous expressions of the risk-neutral case, and therefore their claim to have thereby solved the puzzle is not sustained.
   d. The extension of the Gollier and Weitzman (2010) type model to the case of risk-neutrality shows that the correct risk-neutral CERs are those proposed by Gollier (2004) and that Weitzman's (1998) CERs are quantitatively wrong.
   e. The "Weitzman-Gollier Puzzle" is due to the fact that Weitzman's (1998) expected discount factor is not the inverse of Gollier's (2004) expected compound factor. Using the reciprocal of the latter as a discount factor eliminates the puzzle.
   f. Weitzman's risk-neutral expected discount factor is inconsistent with the market interest rates assumed, does not comply with the definition of PV, leads to the violation of the requirement of transitivity of preferences, and will lead to significant welfare loss if relied upon for investment decisions.

2. The claim that "when future discount rates are uncertain but have a permanent component, then the "effective" discount rate must decline over time toward its lowest possible value" is not sustained.
   a. When stochastic interest rates are non-negative and perfectly serially correlated, risk-neutral discount rates are not declining, but increasing functions of time, and tend not to the lowest, but to the highest possible rate.
   b. Risk-averse CERs of market rates can decline depending on the degree of risk-aversion and the probability distribution of the investor's endowment, but so will risk-averse CERs of project yields.
   c. Risk-averse CERs can only be used to discount risk-free cash flows. Using them to discount risky benefits is a serious error. The feasibility of risky projects can be tested by comparing their CERs with the risk-averse market CER or by discounting their cash flows by RARs.
   d. For a project to be preferred to the market by a risk-averse investor, its risk-averse CER would have to be higher than the risk-averse CER of the market. For that to occur, its expected monetary return (IRR) should be higher than the market's risk-neutral CER, provided its risks is not lower than those of the market, or are not negatively correlated with it.
   e. Risk-averse RARs of projects with the same risks as those of the market, but which are negatively correlated with them, will be lower than the risk-averse market's CERs because such projects are preferable to safe yields.

In summary, the attempt to verify the claims made in Gollier and Weitzman (2010) by replicating the “careful rigorous analysis” through the quantification of their suggested model yields the conclusion that Gollier and Weitzman (2010) have not provided the solution of the “Weitzman-Gollier Puzzle.” The true solution of the puzzle lies in the recognition that Weitzman’s (1998) expected discount factor is incorrect. Their “bottom-line message that [] when future discount rates are uncertain but have a permanent component, then the “effective” discount rate must decline over time toward its lowest possible value” has not been sustained. If market interest rates are sufficiently auto-correlated to make market CERs a growing function of time, it makes no sense to invest in projects of similar risk but lesser yield, irrespective of one’s degree of risk-aversion. This is what corresponds to common sense.

The discussion presented in this paper was based on Weitzman’s assumption of perfect auto-correlation of stochastic interest rates. If there is no correlation, or it isn’t high enough, then the term structure of interest rates will be flat (Gollier 2009:1), meaning that risk-neutral certainty equivalent discount rates will be constant. Because the empirical evidence for the requisite auto-correlation[9] is not sufficiently robust, the conclusion that risk-neutral discount rates should be growing, as derived from the Weitzman model, cannot be asserted, but that they should not be declining can.

[9] For the numerical example of this paper interest rates should be autocorrelated with a coefficient higher than 0.9 for risk-neutral CERs of any time horizon to exceed the expected value of annual interest rates, and at least 0.997 to show a growing trend (A.16).

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Appendix

Details of Calculations Performed[10]

A.1. Optimal investment or borrowing in each scenario

Maximize

$$V(C)=\frac{(K_0-x)^{1-\sigma}-1}{1-\sigma}+\frac{(e^{rt}x)^{1-\sigma}-1}{e^{\rho t}(1-\sigma)} \tag{41}$$

Differentiating[11] this expression with respect to *x*, and setting the result equal to 0, gives the first order condition of the optimization:

$$\frac{e^{r(1-\sigma)t-\rho t}}{x^{\sigma}}+\frac{1}{(K_0-x)^{\sigma}}=0 \tag{42}$$

The optimal market action to be taken can be obtained by solving for *x* in the above expression[12]:

$$x=\frac{e^{\frac{rt}{\sigma}}K_0}{e^{\frac{\rho t}{\sigma}+rt}+e^{\frac{rt}{\sigma}}} \tag{43}$$

Table A.1
Optimal investment in each scenario

Concept	Calculation	Result
Optimal market investment in scenario1	$x=\dfrac{e^{\frac{.01\times 200}{1.7}}2{,}000}{e^{\frac{0\times 200}{1.7}+.01\times 200}+e^{\frac{.01\times 200}{1.7}}}$	610.03
Optimal market investment in scenario 2	$x=\dfrac{e^{\frac{.05\times 200}{1.7}}2{,}000}{e^{\frac{0\times 200}{1.7}+.05\times 200}+e^{\frac{.05\times 200}{1.7}}}$	32.04

Thus, with the data of our simple example, the investor invests \$610.03 in the low interest rate scenario and \$32.04 in the high interest rate scenario. This defines the following scenario dependent optimal consumption paths:

[10] There are slight discrepancies in some of the tables of this Appendix due to rounding.
[11] Result courtesy of http://www.derivative-calculator.net
[12] Solution courtesy of http://www.wolframalpha.com

Table A.2
Optimal borrowing or involvement in each scenario

Consumption	Scenario 1	Scenario 2
Time 0	2,000 – 610.03 = 1,389.97	2,000 – 32.04 =1967.96
Time t	610.03 × $e^{(0.01 \times 200)}$ = 4,507.55	32.04 × $e^{(0.05 \times 200)}$ = 705,727.96

A.2. Calculation of the certainty equivalent discount factor

Differentiating[13] (11) with respect to C_0 and C_t we obtain the marginal utilities:

$$V_i^{'}(C_0) = \frac{1}{C_0^{\sigma}} \tag{44}$$

$$V_i^{'}(C_t) = \frac{1}{e^{\rho t} C_t^{\sigma}} \tag{45}$$

With the data of the quantitative example, these assume the following values:

Table A.3
Marginal utilities of consumption

Marginal utilities	Scenario 1	Scenario 2
Time 0	$\frac{1}{1{,}389.97^{1.7}} = 4.538\text{E-}06$	$\frac{1}{1{,}967.96^{1.7}} = 2.513\text{E-}06$
Time t	$\frac{1}{e^{0\times 200} \times 4{,}507.55^{1.7}}$ = 6.142E-07	$\frac{1}{e^{0\times 200} \times 705{,}727.96^{1.7}}$ = 1.141E-10

Using the computed marginal utilities in expression (15) allows us to calculate the expected discount factor D as follows:

$$D = \frac{0.5 \times 6.142\text{E-}07 + 0.5 \times 1.141\text{E-}10}{0.5 \times 4.538\text{E-}06 + 0.5 \times 2.513\text{E-}06} = 0.08712 \tag{46}$$

and from that R_d can be computed using (16):

$$R_d = -(1/200)\ln(0.08712) = 1.22\% \tag{47}$$

Inverting the expected discount factor, we obtain the expected compound factor:

$$F = 1 / 0.08712 = 11.4784 \tag{48}$$

[13] Results courtesy of http://www.derivative-calculator.net

from which R_c can be computed using (17) as follows:

$$R_c = (1/200)\ \ln(11.4784) = 1.22\% \qquad (49)$$

A.3. Test of the Gollier and Weitzman (2010) calculation method

Gollier and Weitzman's (2010) calculation method can be tested as well. From (7) the expected discount factor D following the "Weitzman" approach is:

$$D = \Sigma\, q^W{}_i\, e^{-r_i t} \qquad (50)$$

For the "Weitzman approach" the "risk adjusted" probabilities are defined as:

$$q^W{}_i = p_i\, V''_i\,(C_0) \,/\, \Sigma\, p_i\, V''_i\,(C_0) \qquad (51)$$

which using our data become:

Table A.4
Weitzman approach adjusted probabilities

	Scenario 1	Scenario 2
$\boldsymbol{q^W}$	$\dfrac{0.5\times 4.538\text{E-}06}{0.5\times 4.538\text{E-}06 + 0.5\times 2.513\text{E-}06}$ = 0.6436	$\dfrac{0.5\times 2.513\text{E-}06}{0.5\times 4.538\text{E-}06 + 0.5\times 2.513\text{E-}06}$ = 0.3564

The expected discount factor therefore can be calculated as follows:

$$D = 0.6436\times e^{(-0.01\times 200)} + 0.3564\times e^{(-0.05\times 200)} = 0.08712 \qquad (52)$$

which is the same result as was obtained in A.2, and from which, using (16), we get:

$$R_d = -\,(1/200)\ \ln(0.08712) = 1.22\% \qquad (53)$$

Alternatively, from (9), the expected compound factor F following the "Gollier" approach is:

$$F = \Sigma\, q^G{}_i\, e^{r_i t} \qquad (54)$$

For the "Gollier approach" the "risk adjusted" probabilities are defined as:

$$q^G{}_i = p_i\, V''_i\,(C_t) \,/\, \Sigma\, p_i\, V''_i\,(C_t) \qquad (55)$$

which using our data become:

Table A.5
Gollier approach adjusted probabilities

	Scenario 1	Scenario 2
$\boldsymbol{q^G}$	$\dfrac{0.5 \times 6.142\text{E-}07}{0.5 \times 6.142\text{E-}07 + 0.5 \times 1.141\text{E-}10}$ = 0.9998	$\dfrac{1.141\text{E-}10}{0.5 \times 6.142\text{E-}07 + 0.5 \times 1.141\text{E-}10}$ = 0.0001857

The expected compound factor therefore can be calculated as follows:

$$F = 0.9998 \times e^{(0.01 \times 200)} + 0.0001857 \times e^{(0.05 \times 200)} = 11.48 \qquad (56)$$

which is the same result as was obtained before, and from which, using (17), we get again

$$R_c = (1/200)\ \ln(2540) = 1.22\% \qquad (57)$$

A.4. Certainty equivalent rates for selected time horizons

Table A.6
Certainty equivalent rates for selected time horizons

Years till time *t*	**100**	**200**	**300**	**400**	**500**
Risk-neutral R_W	1.67%	1.35%	1.23%	1.17%	1,14%
Risk-averse R_d or R_c	1.41%	1.22%	1.17%	1.14%	1.12%
Risk-neutral R_G	4.33%	4.65%	4.77%	4.83%	4.86%

The values of R_W and R_G (risk-neutral case) were obtained by evaluating expressions (2) and (4) of the main text for the values of t shown in the column headings, with all other required data being the same as given in main text Table 1 for our numerical example. The values of R_d or R_c (risk-averse case) were obtained by repeating the calculations shown in Sections A.1 and A.2 of this Appendix for the relevant time horizons.

A.5. Test of compliance with the definition of present value (risk-averse)

The calculated expected discount factor will calculate a correct expected present value of a future safe sum if investing the expected present value in the capital market will yield a stochastic return such that its expected utility is the same as that of the future safe sum to which the discount factor was applied. In this Section, we will test if D = 0.08712 complies with this requirement. As only time 200 utilities are compared, discounting by the rate of time preference is omitted as it would affect both of the examined cases equally.

Table A.7
Sample compliance calculation

Concept	**Calculation**	**Result**
Safe yield at time 200 (an arbitrary small sum).	ε	0.001
Utility of consumption at time 200 in scenario 1, including safe yield of 0.001.	$\dfrac{(4507.55+0.001)^{1-1.7}-1}{1-1.7}$	1.42461645301
Utility of consumption at time 200 in scenario 2, including safe yield of 0.001.	$\dfrac{(705{,}727.96+0.001)^{1-1.7}-1}{1-1.7}$	1.42845638673
Expected utility with safe yield of 0.001.	(1.424616453+1.428456387)/2	**1.42653641987**
Expected present value of safe yield at time 200.	δ = D × ε = 0.08712 × 0.001	8.71207E-05
Yield at time 200 of investing δ in scenario 1	8.71207E-05 × $e^{(200 \times 0.01)}$	0.000643741
Yield at time 200 of investing δ in scenario 2	8.71207E-05 × $e^{(200 \times 0.05)}$	1.91896409
Utility of consumption at time 200 in scenario 1, including yield of investing δ.	$\dfrac{(4{,}507.55+0.000643741)^{1-1.7}-1}{1-1.7}$	1.42461645279
Utility of consumption at time 200 in scenario 2, including yield of investing δ.	$\dfrac{(705{,}727.96+1.91896409)^{1-1.7}-1}{1-1.7}$	1.42845638695
Expected utility having invested present value δ.	0.5×1.424616453+ 0.5×1.428456387	**1.42653641987**

The above table shows that the expected utility of the safe yield is the same as the expected utility of investing in the market the expected present value of that yield.

A.6. Standard and Weitzman CERs when the utility function is logarithmic

The verification of Gollier and Weitzman (2010) assertion that when the utility function is logarithmic the Weitzman Gollier approaches are quantitatively equivalent was tested only under the simplifying assumption that the there is no correlation between the yield of a small investment opportunity and the investor's income from other sources. In other words, the full consumption path optimization was not conducted.

The calculation of the CERs proceeds in the standard way: computing the expected utility of investing \$1 in the market, and then computing the certainty equivalent future amount by evaluating the expected utility with the inverse utility function, which in this case is exponentiation. For the Gollier approach CER this yields:

$$F = e^{\sum p_i \ln(e^{r_i t})} = e^{t\sum p_i r_i} = e^{t\bar{r}} \tag{58}$$

where $\bar{r}$ is the expected value of the possible annual interest rates. Consequently,

$$CER = \frac{1}{t}\ln(e^{t\bar{r}}) = \bar{r} \tag{59}$$

which means that the CER is constant as a function of time, and equal to the expected annual rate of interest.

The same happens with the Weitzman CER approach of probability weighting the scenario specific discount factors:

$$A = e^{\sum p_i \ln(e^{-r_i t})} = e^{-t\sum p_i r_i} = e^{-t\bar{r}} \tag{60}$$

$$CER = -\frac{1}{t}\ln(e^{-t\bar{r}}) = \bar{r} \tag{61}$$

So indeed the two methods yield quantitatively identical results, but only if the utility is logarithmic, which does not make the Weitzman method correct for other cases, nor does it support declining or growing discount rates.

A.7. Certainty equivalent rates for selected time horizons and degrees of risk-aversion

Most of the results shown in Table 3 of the text can be computed as in Section A.2 above, but not all. For very low values of σ, consumption in period 0 is 0, and then the expression for marginal utility becomes undefined. This is not a problem, however, for when consumption is zero it generates no utility, which therefore does not need to be computed. As the welfare consequences of diverting funds from the market to a specific project play themselves out entirely at time t, it is enough to consider utility at time t to compute risk-averse CERs. The question is: what yield ε at time t makes it worth diverting amount δ from investment in the market at time 0? This is obtained by (1) computing the benchmark expected utility at time t of making the optimal investment in the market, as utility maximization requires, and (2) finding by iteration yield ε at time t such that, considering the opportunity cost of not investing δ in the market, the same expected utility is reached as the computed benchmark.

For example, when t = 200 and σ = 1.7, the optimal investment at time 0 is \$610.03 in Scenario 1 and \$32.04 in Scenario 2. The calculation proceeds as shown in Table A.8. As only time 200 utilities are computed, discounting by the pure rate of time preference is omitted.

Table A.8
Alternative risk-averse CER calculation

Concept	Calculation	Result
Utility of consumption at time 200 in scenario 1 without safe investment.	$\frac{(610.03e^{0.01\times 200})^{1-1.7}-1}{1-1.7}$	1.4246164526
Utility of consumption at time 200 in scenario 2, without safe investment.	$\frac{(32.04e^{0.05\times 200})^{1-1.7}-1}{1-1.7}$	1.4284563867
Expected utility without safe investment.	(1.4246164526+ 1.4284563867) / 2	**1.4265364197**
Yield of safe investment costing \$0.0001	ε (obtained by iteration to make the two expected utilities equal)	\$0.0011478
Utility of consumption at time 200 in scenario 1 with investment of \$0.0001.	$\frac{((610.03-0.0001)e^{0.01\times 200}+0.0011478)^{1-1.7}-1}{1-1.7}$	1.4246164528
Utility of consumption at time 200 in scenario 2, with investment of \$0.0001	$\frac{((32.04-0.0001)e^{0.05\times 200}+0.0011478)^{1-1.7}-1}{1-1.7}$	1.4284563865
Expected utility with investment of \$0.0001.	(1.4246164528 + 1.4284563865)/2	**1.4265364197**
CER	= ln (0.0011478/0.0001) / 200	**1.22%**

A.8. Standard risk-neutral compound and discount factors

The risk-neutral compound factor for the numerical example can be computed as follows, on the assumption that \$1 is invested:

$$F = 0.5 \times e^{(0.01 \times 200)} + 0.5 \times e^{(0.05 \times 200)} = 11{,}016.93 \tag{62}$$

Consequently the discount factor is:

$$D = 1 / 11{,}016.93 = 9.0769\text{E-}05 \tag{63}$$

from which, using (16), the CER computes to:

$$R_d = -(1/t)\ \ln(\ D\ ) = -(1/200)\ \ln(9.0769\text{E-}05) = 4.65\% \tag{64}$$

A.9. Weitzman risk-neutral discount factor

Weitzman's risk-neutral discount factor for the numerical example can be computed as follows, on the assumption that a safe \$1 in year 200 is to be discounted:

$$A = 0.5 \times e^{(-0.01 \times 200)} + 0.5 \times e^{(-0.05 \times 200)} = 0.06769 \tag{65}$$

from which the CER computes to:

$$R_W = -(1/t)\ \ln(\ A\ ) = -(1/200)\ \ln(0.06769) = 1.35\% \tag{66}$$

A.10. Test of compliance with the definition of present value (risk-neutral)

The standard discount factor D is compliant (values from A.8 and A.9 above):

$$\mathrm{D} \times \mathrm{F} = 1 \tag{67}$$

$$9.0769\text{E-}05 \times 11{,}016.93 = 1 \tag{68}$$

Weitzman's discount factor A is not compliant:

$$\mathrm{A} \times \mathrm{F} \neq 1 \tag{69}$$

$$0.06769 \times 11{,}016.93 = 745.74 \tag{70}$$

A.11. Loss from following the Weitzman present value rule

As the Weitzman discounting rule results in present values much higher than those of the conventional method, anyone investing in a project that has a positive present value according to the Weitzman rule would suffer an opportunity loss, for the same future benefits could be had much more cheaply by investing in the market instead. The loss can be quantified using the difference between Weitzman's and the correct discount factors. For each \$1 of future benefits, this is as follows with the data of our example:

Absolute loss =\$0.06769 – \$9.0769E-05 = \$0.06760 (71)

Fractional loss = \$0.06760 / \$0.06769 = 99.9% (72)

The fact that investing Weitzman's present value A in a project yielding a safe \$1 leads to welfare loss can be shown for the formal Gollier-Weitzman (2010) model by assuming that the decision maker is risk-neutral. In that case the utility function to be used in (11) should be linear. In its simplest form $U(C) = C$, and expression (11) becomes:

$$V(C) = \sum_{t}^{\infty} e^{-\rho t} C_t \tag{73}$$

The risk-neutral investor would invest all he could in both the scenarios, because both the low and high interest rates exceed his pure rate of time preference, making the postponement of consumption the welfare maximizing action in both scenarios. Therefore, the welfare consequence of investing A = \$ 0.06769 (A.9) in the safe project yielding \$1 is the safe yield obtained less the opportunity cost at t = 200, which is (see A.10) 1 – 745.74 = –744.74. Discounting this at the pure rate of time preference of 0% yields a utility loss of 744.74[14]. Had A been replaced by the correct discount factor, the welfare loss would have been zero, as the opportunity loss of investing in any project with an expected return equal to that of the market is zero.

A.12. The error of discounting monetary benefits with risk-averse CERs

Discounting monetary cash flows with discount rates derived from a risk-averse utility functions seriously overestimates the PV of future benefits. This will be illustrated with the data of Table A.7. The expected present value of safe future sum of \$0.001 is \$8.71207E-05. A project that has the same risk as the market, and an investment cost equal to this present value, would yield \$0.000643741 in the low interest scenario and \$1.42461645279 in the high interest scenario. The expected value of this project's benefit in year 200 is \$0.71263, obtained by probability weighting the yields. Discounting this expected value by the risk-averse CER of 1.22% yields a PV of \$0.062113446. This is an overestimate of 71,196%, given that the correct risk-averse PV is \$8.71207E-05, as shown it Table A.7.

A.13. CERs and required monetary returns

The analysis proceeds as described in the Section 6 of the main text. The risk-neutral CERs are calculated as in A.8. The following table shows the calculation of the monetary CER required by a risk-averse investor of a project that is riskier than the market. The small project has an investment cost of \$0.00001, the coefficient of variation of its annual rate is 1.5 times that of the market, the correlation between market and project risks is 1, and t = 200. The calculation of the risk-averse CER itself is identical to this, but in that case the small project's annual return probability distribution is the market's.

[14] Discounting at a positive rate of time preference would just result in this value being multiplied by a scaling factor. The conclusion of there being a welfare loss would remain the same.

Table A.9

Calculation of required monetary returns of a riskier investment

Concept	Calculation	Result
Consumption at time 200 in scenario1, do nothing.	See Table A.2	4,507.55
Consumption at time 2000 in scenario2, do nothing.	See Table A.2	705,727.96
Expected utility at time *t*, do nothing	$0.5\times((4{,}507.55^{(1-1.7)}-1)/(1-1.7))$ $+ 0.5\times((705{,}727.96^{(1-1.7)}-1)/(1-1.7))$	**1.42653642**
Expected annual monetary yield *x* of the small project costing $0.0001.	This is the solution of the iterative search for annual yield *x*. The search stops when expected total utilities are the same in the row above and in the last row of this table.	2.4996%
Coefficient of variation multiple.	Assumption. This is what makes investing in the project riskier than investing in the market.	1.5
Coefficient of variation of the project's rates.	The coefficient of variation of the market rates is $((0.01–0.03)^2+(0.05–0.03)^2)^{0.5}/0.03=0.9428$, which multiplied by 1.5 gives the coefficient of variation of the project.	1.4142
Deviation *d* from annual mean of project high and low rates, assumed to be the same as the market's.	$d = 0.03 \times v / 2^{0.5}$ $= 0.03 \times 1.4142 / 2^{0.5}$	3.00%
Project yield low.	0.024996 – 0.03	-0.5004%
Project yield high.	0.024996 + 0.03	5.4996%
Project RAR	$(1/200) \times \mathrm{LN}\ (0.5 \times e^{-0.005004\times 200}$ $+ 0.5 \times e^{0.054996 \times 200})$	**5.15%**
Correlation with market.	Assumption. The risks of the project are correlated with those of the market.	1
Consumption at time 200 in scenario 1.	Investment plus project yields: $(610.03–.0001) \times e^{(200 \times 0.01)}$ $+ 0.0001 \times e^{(-0.0050039 \times 200)}$	4507.5452
Consumption at time 200 in scenario 2.	Do nothing consumption plus project yield: $705{,}727.96 + 0.0001 \times e^{(200 \times 0.054996)}$	705,731.7441
Expected total utility	$0.5 \times ((4{,}507.5452^{(1-1.7)} – 1) / (1 – 1.7))$ $+0.5 \times ((705{,}731.7441^{(1-1.7)} – 1) / (1 – 1.7))$	**1.42653642**

The results of repeating these calculations for the selected values of t and other values of the calculation's parameters are shown in the main text.

A.14. Showing that the required monetary rate is the right discount rate for future benefits

Discounting a project's future cash flows by its RAR gives the correct risk-averse certainty equivalent present value. This will be illustrated with the data of Table A.9. The project's initial investment is \$0.0001, its expected annual yield is 2.4996%, but the coefficient of variation of its low and high rates is 1.5 times that of the market, so that its low and high annual yields are –0.50039% and 5.49961% respectively. Therefore in year 200 the project yields $0.0001 \times e^{(-0.0050039 \times 200)}$ = \$0.00003676 in the low scenario, and $0.0001 \times e^{0.0550195 \times 200}$ = \$5.9826 in the high interest scenario. The expected value of this project's benefit in year 200 is \$2.9913, obtained by probability weighting the yields. Discounting this expected future value by the RAR of 5.15% yields a PV of \$0.0001, which shows that 5.15% is the correct discount rate for this project's cash flow. The same present value is obtained if we first compute the certainty equivalent of these alternative yields (\$0.0011478), and discount it at the CER of 1.22%. This illustrates the fact that CERs should only be used to discount safe amounts (the certainty equivalents of project cash flows), while RARs should be used to discount those cash flows directly.

A.15. Sensitivity analysis of CERs

The following tables recalculate the values of Table 3 in the main text for alternative parameter values of the model. For reference Table 3 is replicated below as Table A.10.

Table A.10
CERs for selected time horizons and degrees of risk-aversion
Copy of Table A.3 from the main text

Years till time t	100	200	300	400	500
σ = 1.9	1.38%	1.22%	1.17%	1.14%	1.12%
σ = 1.7	1.41%	1.22%	1.17%	1.14%	1.12%
σ = 1.5	1.44%	1.22%	1.16%	1.13%	1.11%
σ = 1.3	1.50%	1.23%	1.16%	1.13%	1.11%
σ = 1.1	1.60%	1.28%	1.17%	1.13%	1.10%
σ = 0.9	1.79%	1.50%	1.41%	1.37%	1.36%
σ = 0.7	2.22%	2.13%	2.15%	2.17%	2.19%
σ = 0.5	2.88%	2.97%	2.99%	3.00%	3.00%
σ = 0.3	3.58%	3.76%	3.79%	3.80%	3.80%
σ = 0.1	4.11%	4.41%	4.51%	4.55%	4.59%
σ = 0.05	4.22%	4.54%	4.65%	4.71%	4.75%
σ = 0.01	4.31%	4.63%	4.75%	4.81%	4.84%
σ = 0	4.33%	4.65%	4.77%	4.83%	4.86%

Table A.11

CERs for selected time horizons and degrees of risk-aversion

Change pure rate of time preference from 0% to 0.5%

Years till time t	100	200	300	400	500
σ = 1.9	1.42%	1.26%	1.20%	1.16%	1.14%
σ = 1.7	1.45%	1.27%	1.20%	1.16%	1.13%
σ = 1.5	1.49%	1.27%	1.20%	1.16%	1.13%
σ = 1.3	1.54%	1.28%	1.20%	1.16%	1.13%
σ = 1.1	1.61%	1.31%	1.21%	1.16%	1.13%
σ = 0.9	1.76%	1.43%	1.31%	1.24%	1.20%
σ = 0.7	2.10%	1.94%	1.94%	1.96%	1.97%
σ = 0.5	2.74%	2.83%	2.89%	2.91%	2.93%
σ = 0.3	3.51%	3.73%	3.78%	3.79%	3.80%
σ = 0.1	4.11%	4.42%	4.51%	4.56%	4.59%
σ = 0.05	4.22%	4.54%	4.65%	4.71%	4.75%
σ = 0.01	4.31%	4.63%	4.75%	4.81%	4.84%
σ = 0	4.33%	4.65%	4.77%	4.83%	4.86%

Table A.12

Certainty equivalent rates for selected time horizons and degrees of risk-aversion

Change initial endowment from $2,000 to $200,000

Years till time *t*	100	200	300	400	500
σ = 1.9	1.26%	1.15%	1.12%	1.09%	1.08%
σ = 1.7	1.40%	1.22%	1.16%	1.13%	1.12%
σ = 1.5	1.44%	1.22%	1.16%	1.13%	1.11%
σ = 1.3	1.50%	1.24%	1.16%	1.13%	1.11%
σ = 1.1	1.60%	1.28%	1.17%	1.13%	1.10%
σ = 0.9	1.79%	1.50%	1.41%	1.38%	1.37%
σ = 0.7	2.21%	2.13%	2.15%	2.17%	2.19%
σ = 0.5	2.88%	2.97%	2.99%	3.00%	3.00%
σ = 0.3	3.58%	3.76%	3.79%	3.80%	3.80%
σ = 0.1	4.11%	4.42%	4.51%	4.55%	4.58%
σ = 0.05	4.22%	4.54%	4.65%	4.71%	4.74%
σ = 0.01	4.31%	4.63%	4.75%	4.81%	4.84%
σ = 0	4.33%	4.65%	4.77%	4.83%	4.86%

Comparing the above tables, it can be ascertained that while CERs change slightly in response to the parameter changes, their pattern remains unchanged.

A.16. Required degree of autocorrelation

Gollier (2009:1) states that “Weitzman’s claim is qualitatively correct if shocks on the [interest rates] are persistent. On the contrary, in the absence of any serial correlation in the [interest rates], the term structure of discount rates should be flat." Both Weitzman (1998) and Gollier and Weitzman (2010) implicitly assume perfect autocorrelation, as this is what is implicit in a long-term two-period model in which a single interest rate prevails throughout the entire time horizon in each scenario.

It is interesting to pose the question of what would happen if the degree of autocorrelation changed between zero, which should result in a flat term structure of risk-neutral CERs, and the perfect correlation assumed by the cited authors, which result in CERs being a positive function of time. This can easily be done while keeping the two-period nature of the models as far as consumption and investment decisions are concerned, but computing the required expected compound factors from annually simulated interest rates over entire period, assuming varying degrees of autocorrelation.

This was done by performing a Monte Carlo simulation of 10,000 trials for each random variable. Correlated pairs of unit random numbers were generated, and used to generate autocorrelated interest rates. Because the assumed interest rate probability distribution is discrete (probability of high and low rates each year given by the correlated random number drawn), the actually observed autocorrelation of the interest rates differed from the autocorrelation of the generated random numbers. The target correlation was therefore changed by trial and error to obtain the desired measured autocorrelation of interest rates.

On this basis, the following pattern of computed CERs was obtained, as a function of time and of selected degrees of autocorrelation:

Table A.12
CERs for selected time horizons and degrees of interest rate autocorrelation

Correlation Coefficients		Years				
		100	**200**	**300**	**400**	**500**
Theoretical	**0.000**	3.00%	3.00%	3.00%	3.00%	3.00%
Simulated	**-0.001**	2.95%	2.94%	2.95%	2.95%	2.95%
Simulated	**0.900**	2.97%	2.95%	2.95%	2.95%	2.95%
Simulated	**0.993**	4.02%	4.19%	4.05%	3.57%	3.16%
Simulated	**0.997**	4.19%	4.51%	4.62%	4.67%	4.71%
Simulated	**1.000**	4.21%	4.53%	4.65%	4.71%	4.74%
Theoretical	**1.000**	4.33%	4.65%	4.77%	4.83%	4.86%

We can see that Gollier was right: when there is no autocorrelation, the term structure of CERs is flat, whereas when autocorrelation is perfect, the acceleration effect of CERs over time can be observed.

What is interesting, is that even with the very high degree of autocorrelation of 0.9 the term structure of CERs is still flat. This is not so surprising, however, as year to year autocorrelation would decay rapidly for low levels of autocorrelation, just as the sequence of powers of a number less than one declines. With a coefficient of 0.993 we see CERs becoming higher, but not yet monotonically increasing. An autocorrelation of 0.997 or higher is required to observe the effect assumed by Weitzman's (1998) model.

Such high degrees of autocorrelation are unlikely to be observed in real life for long periods of time. Therefore it is hard to escape the conclusion the Weitzman-Gollier paradox was moot all along.