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Decision-Making under Radical Uncertainty: An Interpretation of Keynes' Treatise

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Abstract
Keynes' mathematical Treatise addresses what some call 'radical uncertainty', which he thought endemic in world affairs and whose appreciation underpinned much of his later work. In contrast, the mainstream view in economics, as elsewhere, has been that even if radical uncertainty exists, either there is in principle nothing that can ever be done about it, or that even if one could in theory do something about it then the institutions required would be unreliable, and one would be better off without them. Thus the mainstream has worked as if it were realistic to ignore even the possibility of radical uncertainty. But one needs some conceptualisation of radical uncertainty, such as Keynes', before one can make such judgments. This paper presents an interpretation, to inform debate. The viewpoint taken here is mathematical, but this is not to deny the value of other views.

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1 Introduction

Lord Turner's (2009) view of the financial crisis was that there was too much reliance on mathematics, as if the mainstream practice of thinking about uncertainty as being mere probability were mathematical[1]. Currently, the term 'radical uncertainty' is being used to represent non-probabilistic uncertainty, and some regard this as non-mathematical or even contra-mathematical. Yet the standard work on the subject is arguably still Keynes' mathematical Treatise on Probability (1921), which supplies a devastating critique of the 'uncertainty is just probability' dogma and the mathematics of some types of radical uncertainty, which underpinned his economics.

Keynes' work can be criticised on many grounds, but an understanding of Keynesian uncertainties seems relevant to many contemporary challenges. This introduction gives some examples of Keynesian uncertainties, a brief overview of Keynesian uncertainties and some implications for economics. The next section summarises Keynes' Treatise in more detail and in its context. Then Keynes' findings are related to more standard mathematics, as generalizations. These ideas are then put into the context of subsequent work taking forward Keynes' ideas, and then contemporary challenges. I end with comments and conclusions.

Gambling

Academics can argue particular examples endlessly, with many examples involving coins, balls and urns (e.g. Ellsberg 1961; Knight 1921). The same mathematics can be applied to horse-racing. If the bookmakers are openly and honestly trying to balance their books so that they make a profit whichever horse wins (Britannica 2015) then a gambler can tend to win in the long–run by being better at estimating the probabilities of winning than other gamblers, and they can be reasonably sure that they are better once they have established a relevant track record. Hence it may be 'rational' to gamble. But suppose that such a gambler visits Hong Kong for the first time. According to the usual theory of rationality they should still estimate the probabilities of the horses winning, compare them with the bookmaker's odds and then gamble just as they would with the same set of probabilities and odds in the UK. Anything else would be 'inconsistent'. This theory is 'mathematical' in the sense that it involves mathematical operators, but is it reasonable?

Behavioural economists claim that in some situations most people's choices reveal consistent biases. Thus a horse with a lucky name might become the favourite irrespective of its actual capability. If so, our gambler in Hong Kong might recognize such situations and bet against the crowd, rather than making his own probability estimates. This might be 'irrational' in the sense of the usual theory, but it should pay off if the biases are recognizable and large enough. Keynes' Treatise (1921: 315) points out the possibility of such 'productive irrationalities' and this insight supported some of his own stock market activities, but does not provide a good general guide to either horses or shares.

Thus even in gambling one might take more into account than just one's subjective probability or 'rational expectations'. One might also take account of what Keynes calls 'the weight of argument', as an indication of how reliable a guide one's subjective view might be. In this case, an argument based on a relevant experience is more 'weighty' than one based on an heuristic.

Trickery

Mathematicians generally discuss rather straightforward, even dull, examples, and Keynes' examples are hardly different. But his approach may be better motivated by a more challenging example.

[1] Some contemporary dictionary definitions (e.g. Oxford 2015) encourage a very narrow view of mathematics.

Imagine that we are invited to make a private bet in a situation where we might think we are experienced, but we suspect a trick.

Suppose that a magician has tossed a coin ten times, getting 'Heads' each time. We are invited to choose a side and bet £100 on it, for a gain of £200 if we are right. Conventionally we should consider the probability of Heads to be ½, so whichever side we choose we expect a gain of £50. From Keynes' point of view, however, we should consider the possibility that the coin is biased to Heads (1921: 170-171). The stronger the evidence for this the weaker the argument that the coin is fair and hence the less 'confidence' is justified in it, and the more cautious we should be about betting. In this case we might be tempted to choose Heads, since either Heads is more probable, or it is equally probable. In either case we expect to gain. But Keynes would advise us to consider the weight of argument for this, even though it is not narrowly mathematical.

From a game theory perspective (von Neumann and Morgenstern 1944: 49) the key question is 'is it possible for the magician to influence the outcome?' If this cannot be ruled out then it would not be possible to have a meaningful subjective probability or 'rational' expectation about the outcome independent of our choice: we should not expect that which we bet on. In Keynes' terms, there may be no possible choice for which there could be a positive weight of argument.

Keynesian Uncertainty

What follows is my view, informed by applications, including those at the UK's Bletchley Park. Sources are given later.

In the Treatise, Keynes supposes that knowledge derives from arguments from evidence, supplemented by judgments. With only limited evidence we may find that there are multiple distinct sets of possible rational expectations that could explain the same actions (as in Bray 1982). So when thinking about how they might respond to some future events, we might have to entertain multiple distinct possible beliefs. In which case, we might suppose that others can do likewise. For example, in a political squabble electors might think that one group is lying without having a view on who it is. Reasoning in such cases should take better account of the 'weights of argument' than usual.

As an example, consider the hypotheses H, T and F, that a coin is double-**H**eaded, double-**T**ailed or **F**air. I choose a coin twice and get two Heads. Applying Bayes' rule (e.g. Fenton 2015; Stone 2013) directly, this is evidence for H as against T or F.[2] But, applying Bayes' rule this is not considered to be evidence for 'the coin is the same on both sides' over H, since two Heads are equally likely for both. Keynes says that we should always consider such hypotheses separately, and not combine them into single hypotheses such as 'there may be a crash'.

There can never be a purely mathematical justification for the use of any method outside of pure mathematics. But Keynes argues that if we have established a way of characterising cases and have a variety of evidence (yielding strong weight) that the method gives reasonable results, then – by default – we might reasonably expect that the next application will yield results that are not very much worse, in a certain technical sense. In a typical application one would gather more evidence and confirm that weight of argument was still strong, for example by making a prediction and then confirming it, at least approximately. In this sense Keynes' method may be as good as it gets, or at least adequate for an important range of cases where not refining hypotheses sufficiently can give significant errors.

[2] Two Heads are certain for a double-headed coin.

The Implications for Economics

Keynes' Treatise is about understanding the current situation, but we need to go further, since even if we actually know everything about the current situation very precisely the future may still be radically uncertain (Turing 1952)[3]. Taking the economic concept of rational expectations, there is the potential for people to realise that there are other possibilities. For example an asset class may be under-valued and prices may boom, with assets becoming over-valued. This creates the possibility of an alternative set of expectations, with the potential losses increasing. Thus in addition to the conventional risk to continuing variability in prices, one has the 'radical risk' of a rapid fall in prices, due to the radical uncertainty going forward.

Of course, in economics being aware that there is a huge bubble is not enough to avoid a crash. More insight may be needed. But Keynes' approach seems to provide a useful element of the framework that would be needed to address this issue (cf King et al 2011).

2 Keynes' Treatise in Context

A difficulty of reading Keynes Treatise is that both the ideas and language of mathematics were still in flux. As now, most working mathematicians regarded their subject as a systematic collection of reliable methods for making calculations about number, quantity and space, capable of yielding precise results. They regarded mathematical theory as more rigorous than other subjects, and as setting the standard and in some cases supplying the tools. But the degree of rigour varied, with Geometry then (prior to Keynes) being regarded as the most rigorous. The view among academics and other leading thinkers was barely different.

Contemporary versus Classical Mathematics

Einstein (1922) noted the significance of the then new 'axiomatic' view of mathematics for the natural sciences, which he illustrates by reference to Geometry. My dictionary defines an axiom as a 'self-evident truth' whereas in modern mathematics the axioms are just those propositions about the truth of which mathematics is silent (cf Oxford 2015b). Thus the mathematical status of Geometry is independent of its validity for any applications. And the same holds for mathematical models and probability theories (Keynes 1921: 115). One has to check that the axioms are valid in any particular case.

Classical Probability

In the English speaking world Bayes (1763) is often credited with the initial mathematization of probability theory and in particular solving the inverse problem[4]. His paper is very reasonable, recognizing what we now call 'radical uncertainty'. But subsequently some mathematicians (e.g. Ramsey 1926; von Neumann and Morgenstern 1944: 17-18; Savage 1972: 33-46) have been interpreted as saying that one should always disregard radical uncertainty. But they later clarified this, in line with Keynes (e.g. Ramsey 1928, 1929; von Neumann and Morgenstern 1944: 17-20; Savage 1972: 13-17, 82-91). In any case the mathematics of Smith (1961) seems decisive in showing that radical uncertainty should be considered[5].

[3] Ray Bradbury later called this 'the Butterfly Effect'.
[4] E.g.: given a sequence of Heads and Tails, what is the bias of the coin?
[5] Cf. the gambling example in the introduction.

Keynes' Objective

In his thesis, Keynes (1921: v-vi) was attempting to create a mathematical version of classical probability and hence had to address the question of whether classical probability was an accurate model of common concepts of uncertainty, and whether it corresponded to reality in any other sense. He appears not to have started with any appreciation that radical uncertainty was important, but was led to it by considering the problem mathematically.

Early Radical Uncertainty

There had long been wide-spread and respectable dissent (as outlined by Keynes 1921: 79-91) from the view that people necessarily thought in terms of numeric probabilities, but the classical theory was still widely regarded as the norm, and any inconsistent reasoning was often labelled as 'irrational'. George Boole, in his 'Laws of Thought' (1854) set out to establish the then conventional practices of logic and probability on a firm footing, but in so doing identified short-comings.

Boole noted that logical deduction applied to formal concepts is completely rigorous and reliable, but in applications one can never be sure of the correspondence between reality and its representation and the scope for error increases with each step (1854: 3,4). When reasoning in an unfamiliar situation there will be 'radical uncertainty', and so it is advisable to check any deductions. For Boole, probability theory is not an exception to this rule.

Boole (1854: 10-12) thinks of a possible probability assignment as a mathematical variable subject to various constraints, and solves the resultant equations. This can yield many possible solutions, and hence a form of radical uncertainty, as in interval-valued probabilities (e.g. Walley 1991).

Keynes' Approach

Keynes considers the general conditions under which one could use the precise theory, and how to interpret the results, starting from Boole. He particularly considers scientific practice.

Keynes notes that scientific methods depend on probability theory and hence one cannot use science to justify probability theory. Keynes (1921: 406-428) constructs a logical argument (supporting Boole's intuition) that 'normal science' applied in similar circumstances to those where the theory was developed is relatively reliable and rigorous. Thus he characterises those circumstances in which radical uncertainty is minimised, and in so doing illuminates more problematic cases. Turing (1950) later emphasised this.

Keynes' Conclusions

Part I. Fundamental Ideas

The two most familiar types of probability are subjective and objective. A subjective probability is an opinion, and the normative theory only concerns the consistency of the opinions[6]. An objective probability is regarded as true to reality, and hence is hard or impossible to know[7]. Keynes (1920: 3-9) followed Boole (1854: 187) in considering logical probabilities, in which the assessment of probability depends in some principled way on the evidence available and the assumptions made. These are comparative probabilities, where one simply says 'more probable than' or 'less probable than', but allowing for conventional numeric probabilities where they are justified, as for a fair coin (Keynes 1921: 38-40).

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[6] For a coin, the normative theory only requires that $P(\text{Heads})+P(\text{Tails})=0$. It would be a reasonable subjective view that $P(\text{Heads})=0.5$ even if the coin were actually biased.

[7] If a trickster offers me a coin, I may have no way to know what $P(\text{Heads})$ is.

Keynes (1921: 71) also introduces the notion of 'weight of arguments'. The more the evidence the greater the weight, and the greater the weight that counts against a hypothesis the less likely it is to be true. Otherwise the evidence is consistent with the hypothesis, but that doesn't mean that it is true.

If one has two urns containing black and white balls, and one has inspected the first urn, but not the second, and found it to contain an equal number of black and white balls, then in both cases one might suppose that the probability of a random draw yielding a white ball was 0.5, but the 'weight of argument' and hence confidence would be greater for the first urn (cf. Ellsberg 1961). Keynes (1921: 315) suggests that an increased weight of argument for an action should increase the desirability of that action, even if the increase is not measureable[8].

Part II. Fundamental Theorems

This is largely technical, providing the ground-work for the later parts. Keynes (1921: 144-157) discusses at length how to combine results from different sources to make an overall assessment of weight of argument.

Part III. Induction and Analogy

Induction is the basis of the scientific method, and the core of the Treatise is a discussion of the conditions under which it is valid, which Keynes supposes are reasonable assumptions for the natural sciences (1921: 406-428). But induction, and statistical inference more generally, never allows us to conclude that a particular theory is correct, only that the testable predictions made using it have survived certain tests, and hence induction suggests that 'very probably' future predictions for the same range of circumstances will also tend to survive. If induction based on observations in England has suggested that 'all swans are white' then we are only really justified in predicting that swans will continue to be white in England, not Australia (e.g. Keynes 1921: 417).

Part IV. Some Philosophical Applications of Probability

Chapter XXVI on 'The Application of Probability to Conduct' concludes Part IV. In it Keynes says:

> "If … the question of right action is under all circumstances a determinate problem, it must be in virtue of an intuitive judgement directed to a situation as a whole, and not in virtue of an arithmetical deduction derived from a series of separate judgements directed to the individual alternatives each treated in isolation. (1921: 312)
>
> …
>
> The old assumptions, that all quantity is numerical and that all quantitative characteristics are additive, can no longer be sustained. Mathematical reasoning now appears as an aid in its symbolic rather than its numerical character." (1921: 316)

As he suggested in the Preface (1921: v), the old mathematics of Leibnitz, Poisson and Pascal are no longer appropriate and should be replaced by the mathematics of Johnson, Moore and Russell. Thus, the problem in finance and economics may not be that mathematics and mathematical models have been used, but that they have been of the wrong type or that their conclusions have been interpreted wrongly.

Part V. The Foundations of Statistical Inference

In the final paragraph of his Treatise (1921: 428), Keynes noted that

[8] It might be that the increase is hugely significant, even if we lack a means of measuring it.

> "Professors of probability have often and justly been derided for arguing as if nature were an urn containing black and white balls in fixed proportions."

He critiqued some previous work (including 'the principle of indifference') and established some credible axioms that have strong intuitive appeal in some cases and which logically justify the usual axioms (1921: 133-138). For example, he notes that the sciences of his day were tending to find finitely axiomatizable models of the kind that economists call 'mathematical models'. He ended his Treatise by opining that 'it may turn out to be true' that nature is an urn.

Using Keynes' 'Consequences' to Rationalize Parts IV and V

We can rationalize parts IV and V by noting that in Keynes' time human behaviour and economics would not have been regarded as 'natural' and that by 'science' was meant 'natural science'.

The Treatise was revised and published just after Keynes' 'Economic Consequence of the Peace' (1920: 1). It begins:

> "Very few of us realise with conviction the intensely unusual, unstable, complicated, unreliable, temporary nature of the economic organisation by which Western Europe has lived for the last half century. We assume some of the most peculiar and temporary of our late advantages as natural, permanent and to be depended on, and we lay our plans accordingly. On this sandy and false foundation we scheme … ."

This clearly relates to part IV of the Treatise, which goes some way to providing a 'logic' that is appropriate to such challenges, recognizing radical uncertainty.

Keynes' Legacy

Keynes' clearest legacy is in the work of Whitehead (1929: 314), Russell (1948: 390-397), Turing (1952) and Good (1983: x, 160-162) concerning the nature of science, mathematics and knowledge, and in particular the achievements of Bletchley Park (Good 1983: x; MacKay 2003: 265). Unfortunately the trend since then has been to over-simplify things. For example, the use of weights of argumments become fusion of probabilities and Bayesian nets (MacKay 2003: 293; Khaleghi et al 2013), which rely on making precise assumptions beforehand. But even in the conventional theory it is a key assumption that one is considering all the possibilities, so it is obviously true that the resultant probabilities are conditional on nothing radically new happening. In this sense radical uncertainty is at least implicit in 'Bayesian probability': Keynes makes it more explicit.

Implications for Science

In current usage science can simply mean "reliable and teachable knowledge about a topic". This hardly favours an appreciation of radical uncertainty or of the approaches of Keynes, Einstein, Whitehead or Russell.

Suppose, for example, that I have a theory 'All swans are white'. I go to New Zealand for the first time and see a large black bird. In the modern, broad, sense it would be considered scientific to apply my theory to deduce that the bird is not a swan. But in Keynes' sense I should realize that my theory was based on evidence from the England alone, and so I should be 'radically uncertain' about whether the black bird is a swan. I just don't know.

Here I attempt to present an accessible theory inspired by the work of Keynes and his colleagues and contacts.

3 A Mathematical View of Radical Uncertainty

Probability as Logic

Keynes (1921: 3-4) thinks that to be meaningful probabilities should relate to some general background context, G. In Good's notation (1983: 124), for supposed unconditional or conditional probabilities[9], P(A) or P(A|B), there is an implied G for which these are really P(A|G) or P(A|B.G). Radical uncertainty arises from uncertainty about the appropriate G. For example, the probability of 'Heads' is certain for a fair coin, but one has radical uncertainty for an unknown coin tossed by a magician. Similarly, in 1700 we may have had good grounds for an estimate of P(Black|Swan.England) and still have be uncertain about P(Black|Swan.World).

The logic is necessarily incomplete, at least where one has multiple possible rational expectations (Bray 1982), which can develop due to 'circular logic'. But if one treats these 'emergent properties' (Smuts 1931: 7-9, 13, 14) as 'brute facts', then one can have logical probabilities that relate to these. For example, it is logical to expect house prices to continue to rise during a long-running boom, but only relative to a 'brute facts' – including speculation – that may not persist. Thus radical uncertainty arises not because the assessed probability is wrong or illogical, but because it depends on something that is itself uncertain.

Probability as a Measure

Keynes (1921: 133-138) provided an incomplete axiomatization for his concept of probability. Later Kolmogorov (1936) developed a simplified axiomatization for the case where one has a well-defined measure. This has the advantage of being complete in the sense that where – as is often the case – one has some base events, the axioms determine precise probabilities for all events of interest.[10] Kolmogorov's axiomatization is now widely regarded as the definitive axiomatization, and yet as Bayes (1763), Boole (1854: 10-12), Keynes (1921: 20) and Ramsey (1928) noted, not all probabilities are definitely measureable and so the incompleteness may be unavoidable.

Keynes (1921: 10,11) emphasises that the measure is only ever defined relative to some context, and that even then there may be many possible measures that satisfy the axioms. Thus instead of considering 'the' probability, one can consider a set of possible measures and hence a set of possible probabilities. For each proposition/event one can derive the inner and outer measures and hence the lower and upper probabilities, P_{-} and P^{-}. These satisfy axioms like Kolmogorov's:

- Positivity: $0 \leq P_{-}(E) \leq P^{-}(E)$, for all events E.
- Unitarity: $P_{-}(\Omega) = P^{-}(\Omega) = 1$, where Ω is always the case.
- Additivity: $P_{-}(\cup_i E_i) = \sum_i P_{-}(E_i)$, where the E_i are mutually exclusive, and similarly for P^{-}.

The conventional assumption is that $P_{-} \equiv P^{-}$. More generally the upper and lower probabilities of two events may overlap, and in this sense the events may have overlapping and incomparable probabilities.

Keynes (1921: 428) noted that practice within the natural sciences was tending to make the bounds more and more precise. It seemed that natural scientists were becoming increasingly expert at setting up standardised experimental conditions in which all relevant factors had been identified and

[9] As usual, P(A) denotes 'the probability of A', P(A|B) 'the probability of A given that B holds'. (These are not necessarily numeric.)

[10] Even in 1933 Kolmogorov seemed to think this an essential property for an axiomatization.

controlled, thus establishing standardised contexts in which it was reasonable to suppose that precise probabilities, as envisaged by Kolmogorov, would exist. But not all sciences are like this.

One has radical uncertainty where the upper and lower probabilities differ, and yet more radical uncertainty where they cannot be determined precisely. In practice, it can be helpful to develop the above system further, but there seems to be no universal way to do this. Keynes 'points the way' but does not always show a final destination.

The Weight of Arguments

Keynes (1921: 71-78) partially developed a notion of 'weight of argument' between two hypotheses[11]. The essential idea is that the more evidence the greater the 'weight', and the less weight the more radical uncertainty. In the Ellsberg Paradox (Keynes 1921: 71; Ellsberg 1961), for example, one has two urns for which the probability of a black ball is the same, but the weights of argument are very different. This leads some mathematicians to behave in ways that appear irrational (Ellsberg 1961), but only against a standard of rationality that takes no account of radical uncertainty.

As with the concept 'probability', it should not simply be assumed that the 'weight' of an argument necessarily corresponds to any particular mathematical structure. Rather the main concern – as with probability – is to compare weights between arguments and with some standard. For example, in a UK civil legal case the standard is 'balance of probability'. But one should perhaps have a sufficient weight of argument to justify the estimated probabilities, not just rely on prejudice. Similarly in a UK criminal case 'beyond reasonable doubt' should perhaps imply not just a high probability, but an adequate weight of argument for the estimated lower probability.

Likelihood Ratio Test

Jack Good (1983: x, 133-137, 332), working under Alan Turing, applied the notion of weights at Bletchley Park, which was vital to its successes. For stochastic hypotheses, he showed that the appropriate weight was the ratio of log likelihoods. Thus the weight of argument that evidence E yields for H as against K, is:

$$W(\mathrm{H}/\mathrm{K}{:}\mathrm{E}) \equiv \log\big(P(\mathrm{E}|\mathrm{H})/P(\mathrm{E}|\mathrm{K})\big),$$

provided that the two probabilities are well-defined. It is natural to take K = ¬H, the complement of H, to derive a simple weight for H of $W(\mathrm{H}/\neg\mathrm{H}{:}\mathrm{E})$, but this requires that we know $P(\mathrm{E}|\neg\mathrm{H})$. Good notes that for a composite hypothesis[12] $\neg\mathrm{H} = \vee_i \mathrm{H}_i$ it is tempting – as is now common practice – to perform a Bayesian calculation, such as:

$$P(\mathrm{E}|\neg\mathrm{H}) = \textstyle\sum_i P(\mathrm{E}|\mathrm{H}_i).P(\mathrm{H}_i|\neg\mathrm{H})$$

Although "it is usually difficult to specify the probabilities … with much precision", in many cases, such as those at Bletchley Park or those arising from the natural sciences or well understood situations in which we really are experts, this seems – as Good argues – to be reasonable. More generally, though, Good points to the likelihood ratio test, for compound hypotheses (1983: 137). For hypothesis $\vee_i \mathrm{H}_i$ that refines a hypothesis $\vee_j \mathrm{H}'_j$ the likelihood ratio statistic is

$$\max_i P(\mathrm{E}|\mathrm{H}_i)/\max_j P(\mathrm{E}|\mathrm{H}'_j).$$

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[11] In structured applications assumptions are typically fixed and only the evidence varies, so the weight of argument is often called 'the weight of evidence'.

[12] $\vee_i \mathrm{H}_i$ holds whenever some H_i holds.

This is suggestive of the idea that since weights of argument for composite hypotheses are unreliable, we should split them into hypotheses that are approximately statistical in the sense that we have good grounds for making reasonably precise estimates of their Bayesian likelihoods, and then maximizing over those hypotheses.

Often the weight is high for some compound hypothesis $\vee_i H_i$ but low for some constituent, H_k. In this case it is expedient to refine $\vee_i H_i$ by deleting all low-weight constituents, leaving only those where the weight is reasonably high. Similarly if the weight is high for both H and K then the usual Bayesian heuristic (Good 1983: 133) would yield a weight of argument supporting H, but from a likelihood ratio perspective we should instead say that the weight supports $H \vee K$. Thus we should not claim that 'all swans are white' unless we can be sure to have considered an adequate sample.

Synopsis

Keynes (1921: 71-78, 180-185) does not provide a generic, straightforward, formula from which one can derive weights of argument, but does provide a useful discussion of the issues. We need to consider statistics such as

$$\max_i P(E|H_i.G),$$

or bounds such as

$$\max_i P^-(E|H_i).$$

The result is necessarily imprecise, except where the likelihoods $P(E|H_i)$ are precise for definitive contexts, G. In practice one can often represent radical uncertainty using explicit hypotheses, uncertain contexts or imprecise likelihoods. Instead of assigning a number to a hypothesis, we need to determine the hypothesis that is supported by the evidence, under reasonable assumptions.

Decision Making

Keynes (1921: 313, 315) suggested that the weight of argument ought to be taken into account in decision making, as a kind of 'degree of confidence'. Conventional rationality is the special case where one has a sufficient weight of argument that the probability estimates are precise and reliable. A common, idealistic, approach is to ignore any radical uncertainty and to proceed as if one had adequate arguments for one's estimates. Much of the criticism of the use of mathematics in finance (e.g. Turner 2009) has really been of such unrealistic use of mathematics. It might be better to develop some of Keynes' thinking. [13]

Whereas classical rationality offers the prospect of 'objectively' justifiable decisions, Keynes' mathematics has shown this to be impossible, so that some logical subjectivity is needed, except where there really is no radical uncertainty. Both the determination and application of weights of argument depend on the context, with no universal method.

4 Complementary Concepts

Keynes' Treatise taken in isolation is not very easy to interpret and apply. It is helpful to consider it in a broader context.

[13] The approach resembles abduction, but without the emphasis on simplicity.

Whitehead and Russell

Russell (1948: 390-397) agrees with Keynes that the conventional idea of probability as applying to propositions or events (as in Bayes 1763; Boole 1854; Ramsey 1926; Kolmogorov 1936) is very limited, and that probability may be better regarded as a relation between propositions or – better still – as a relation between propositional functions. Thus Keynes' Treatise is not 'the last word' on the subject.

Whitehead had commented heavily on Keynes' (1921) draft and later cited it in his 'Process and Reality' (1929: 314), so one may reasonably suppose some coherence between the two works. He introduces the notion of an 'epoch' within which one has ordinary uncertainty but beyond which one has radical uncertainty. Ordinary rationality proceeds as if the epoch were everything. But it isn't. We might see different rational expectations as characterising different latent epochs or 'modes', so if we can identify them we can see what changes might happen. This insight is perhaps implicit in both Keynes' Treatise and his 'Consequences' (1920: 1-6, 236-239, 261-265, 277-279).

Consequences of the Peace

In his *Economic consequences of the peace* Keynes (1920: 1) had pointed out how the usual rationalities of Versailles were – taking a broader view – likely to lead to disaster, as they did. Experience from before the Great War was misleading, as was applying findings from finance within nations to finance between nations. Moreover, the belief that all would be well was contributing to the problems, and some opening of minds to the potential for disaster was needed to avoid it. The upper probability of disaster was too great to be ignored while the lower probability was low enough to justify some action to avoid it. A key aspect of Keynes' 'Consequences' was that some of the factors behind the conventional thinking were no longer the case.

Keynes worked with J.C. Smuts and – later – A.M. Turing. Smuts (1931: 8-9, 13-14) noted that what we now call evolutionary stable strategies are dependent on context. Turing (1952) showed how dynamical systems governed by differential equations and incorporating random disturbances had 'critical instabilities'. In both cases systems could be stable for long periods but then de-stabilized by endogenous changes, possibly un-observed. Explanations of the crises 2007/8 essentially identify factors that had previously been overlooked, such as finance, health, energy, security and higher education.

Learning in Markets

The notion of rational expectations has played a key role in economics. Much of Keynes' discourse is relevant (e.g. 1936: 96-106). Bray (1982) developed a model in which, assuming a great deal of true common beliefs, for some values of a 'stability parameter' there is a unique rational expectations equilibrium to which credible market-driven learning will tend to converge. This is in line with the spirit of Keynes' 'constructive theory' (1921: 406-428). But otherwise "instability is a real possibility". In this model stability relies on most demand within the market coming from traders who have some fundamental knowledge and rational expectations based on what they observe, as against those who merely trade statistically. This is consistent with Keynes' views:

- "Our so-called 'permanent causes' are always changing a little and are liable at any moment to radical alteration." (1921: 419).
- It may be impractical to attempt genuine long-term investments, so that speculation tends to pay off much better –until a crisis (1936: 102).
- As the organisation of an investment market improves, the risk of predominance of speculation increases (1936: 103).

- When speculation predominates the long-term investor will seem rash (1936: 102).
- The position is serious when enterprise becomes a bubble on a whirlpool of speculation (1936: 103).
- That the stability of the system is also dependent on their being a variety of views about the future (1936: 111).
- Nonetheless, classical rationality is compatible with considerable continuity and stability, so long as we can rely on the convention (1936: 99).

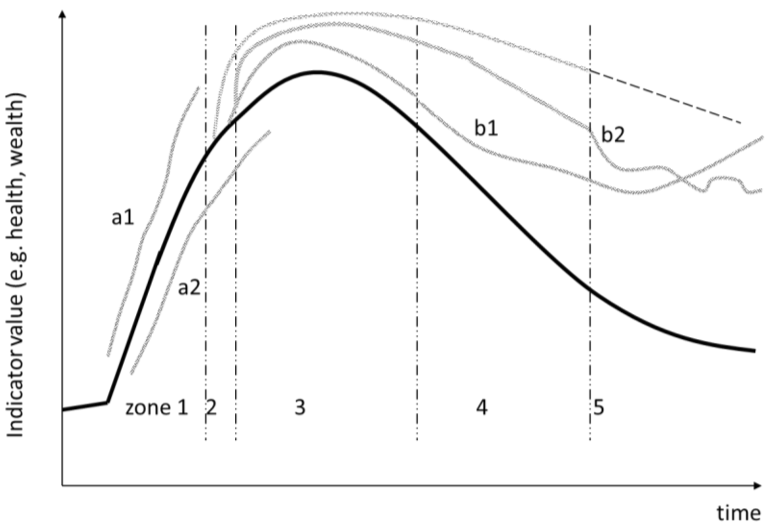
This part of Keynes' General Theory is explicitly dependent on the Treatise. What Bray (like Turing 1952) adds is an acknowledgement that there may be multiple possible equilibria, or none. What she lacks is a sense that the very idea of a rational expectation could be nonsense. It may also be worth pointing out that if stability depends on traders having knowledge, it may be better for them if the knowledge is soundly based, and not reliant on a misuse of probability theory. More broadly, in collaborations one has radical uncertainty by definition, so there may not be a unique agreement.

The Nature of Crises

In equilibrium the long-run behaviour is predetermined and with rational expectations it is common knowledge, so only the short-run matters. There are no decisive times. But actually – as shown quite generally by Turing (1952) – things might never settle even temporarily into a stable epoch, but might keep seeming to settle, only to diverge off to a new solution. The period after 2008 seems somewhat like this, politically, financially and economically.

Crisis Management

Prior to the crises of 2007/8 I attempted, with others, to develop an educational visual aid for policy makers, crisis managers and their advisers from across Europe (Hudson 2009). The working figure was:



The axes are purely nominal, without any specific scale. Initially (zone 1) there is apparent stability and conventional utilitarianism seems to be working, but then (2) radical uncertainty becomes more significant. When the critical instability is realised (3) there is typically a 'shock', a realisation that conventional methods are not working, loss of confidence and potentially incoherence and loss of value (4). Conventionally this may lead to a new stability, either a depression (bold) or renewed growth (b1) following a 'creative' recession. But a sustained period of incoherence and long-term uncertainty and instability (b2) is also possible. A focus on short-term value (a1) can lead to earlier and deeper shocks, while slower growth without reform (a2) may only delay the inevitable.

The visual aid was helpful in facilitating a debate across domains. For the participants, the main value in this model was in realising the need to look out for new factors emerging, to consider how they might interact and to be prepared for there to be no coherent view. This aid may be an over-interpretation of Keynes, but the main point is that, contrary to much opinion, recognizing Keynes' radical uncertainty need not require weird reasoning, and is compatible with effective action, as it was for him.

Contrast with Conventional Concepts

Keynes contrasts his view with those of frequentism (1921: 92-110, 406-418) and of what we now call Bayesianism (1921: 33-35, 312-313). Extreme Bayesians assign probabilities to everything, making whatever assumptions are required. If Keynes thought these assumptions reasonable he might make them explicit and provide some supporting arguments, with weights of arguments. Thus where there is an unknown objective probability he (Keynes 1921: 71) might agree with the Bayesians about the probabilities but associate different weights of arguments, thus acknowledging radical uncertainty. Where frequentists assign probabilities, Keynes would tend to agree, but there would again be some assumptions to make explicit and use to assign weights of arguments and inform interpretations.

A key difference between Bayesianism and frequentism is their attitude to 'prior probabilities'. Jack Good (1977) showed how one could discount different sources of evidence, thus interpolating between fully using or not using priors. Hence frequentism and Bayesianism might be regarded as extreme, dogmatic, cases of Keynes' theory. On the other hand, both frequentists and Bayesians tend to extrapolate without regard to critical instabilities. Keynes is more realistic. Where Bayesianism and frequentism have standardised methods, embodying fixed assumptions, for Keynes the assumptions are things to be tailored to the circumstances, reflecting any radical uncertainties. They would come from domain experts, not technocrats. It is not so much that Bayesianism and frequentism are wrong but that they are often mis-applied (Russell 1948: 403).

Bletchley Park's success depended on recognizing and managing its uncertainties. Under Turing and Good (1983: x), it used 'weights of arguments' effectively. Their approach has been described as 'Bayesian', but it is important to recognize that radical uncertainty was not so vital:

- They were largely dealing with a special case (where there is a lot of data, and good reason to suppose the mechanism fixed, at least for 24hrs).
- They had many cunning tricks to shorten searches.
- They only used probabilities as a guide: any findings were tested.

Also of relevance is game theory. It is sometimes said that this 'proves' that the usual concept of rationality is appropriate to economics. But this is only true for fixed coalitions (von Neumann & Morgenstern 1944: 43, 564-573), and hence does not apply when the economic balances of preferences, ideas and power are shifting, as they are.

5 Relevance to Contemporary Challenges

The calling notice for this special issue notes that conventionally:

> "Decision science … has fostered the development of … models in which decision-makers can be modelled as calculating machines, optimising subjective expected utility under constraints."

And asks:

> "Can we think of other ways of proceeding and still produce rigorous models capable of empirically validated prediction?"

This question has both micro and macro aspects. Keynes work could be adapted as a framework in which decision-makers adopt their own habits and 'attitudes to risk', including satisficing (Simon 1955). But the real concern for economics is not individual decision-makers but their macro impact.

Keynes (1921:406-408) discusses the extent to which a model could be 'empirically validated', arguing that logic matters. For example, (as in de Antoni 2010) individual probabilistic rationality en-masse can lead to a crisis, contrary to the notion of a rational 'representative agent'. Thus the 'rational expectations hypothesis' (Muth 1961) could never be empirically validated, however long it seemed to hold. I speculate that crises will continue so long as Keynes' logical critique remains valid, and that even post-hoc crises cannot be rationalized conventionally, but are the result of undue idealization.

The calling notice asks:

> *When should the EU permit sales of particular GM crops, if at all?*

This raises broader issues. Is there any radical uncertainty, and does government science adequately take account of it? In the UK, with the BSE fiasco in mind, one might think not (Phillips et al 2000: 264-266). Briefly, cycles in the food chain allowed a new disease to emerge. While one could not necessarily have predicted this in advance, there was a very clear cause of radical uncertainty that could easily have been regulated, but wasn't. For GM there might be similar risks. The challenges are firstly analysis and then effective communication. This would seem implausible without some language for discussing radical uncertainty and hence radical risks. Similarly for the other challenges.

6 Comments

The mathematical axiomatic approach to uncertainty can seem radically different to the usual idealistic approach. The difference is not in the fundamental mathematical theory, but that – as Einstein pointed out – realistic mathematicians and idealists typically treat axioms quite differently. Whereas idealists regard something as true unless and until proven otherwise, mathematicians regard something as possibly false, unless and until proven otherwise. What we need is not just different mathematics, but to take a more mathematical attitude.

In conventional probability theory, it is axiomatic that probabilities can be represented by numbers. Idealists treat axioms as universal truths, but to a mathematician axioms are building components, to be used as appropriate – or not. This is often by decomposing statements of interest into statements to which the axioms do apply (Oxford 2015b). Perhaps we might characterise what is often called 'radical uncertainty' by saying that it is uncertainty to which the usual axioms do not apply. Whereas utility maximization is often described as being 'risk neutral', we might regard it as uncertainty blind. If probability applies when a situation has developed far enough to constrain probabilities adequately, then an important special case of radical uncertainty arises where we there are different ways in which the situation could have or could be developed. For example, the possible futures in which one had different unique possible rational expectations might be identifiable without their probabilities being adequately constrained.

This is mathematically valid, but is it useful? If the events of interest are reactions to unpredictable external (exogenous) events, then even the best theory will not support adequate forecasting and policy making. But if something like rational expectations or spiralling debt (Minsky moments) are significant factors in events of interest, such as crashes, then the theory provides a means of analysing the issues, which might be useful.

The Bank of England (2015), for example, publishes probabilistic fan charts, as if there were no decisive times or uncertainty. But sometimes it might be more informative to identify different decisive factors and publish separate fan charts. For example, from a game-theoretic point of view, a break-up of the UK, an exit from the EU or the eclipse of the US by China would all need considering. Similarly for events such as wars and oil crises. There is no sound way of reducing the uncertainty to a probability distribution.

Keynes notes that conventional reasoning often proceeds as if the subject were state-determined[14], and a 'proof' of his theory would have to assume this. Yet Keynes does not think that economies, for example, are. Moreover, radical uncertainty could be quite different outside of free markets, such as when a hegemon provided stability, but even then coalition formation can play a key role and the approach seems relevant. Thus nothing in Keynes (or this paper), beyond the mathematics, should be taken as a 'universal truth'.

7 Conclusion

Keynes was arguably the first person to put probability theory on a sound mathematical footing. If hypotheses constrain events sufficiently, in a statistical sense, then the usual theories apply (e.g. Kolmogorov 1936). But often events are not sufficiently constrained. In this case one can split the hypothesis into cases each of which is adequately constrained to apply the usual theory – conditional on the case. Radical uncertainty concerns the case. One cannot logically assign probabilities to cases, but – in principle – one can assign weights of arguments. For example, if one has repeatedly sampled an urn and found a black ball half the time, that is a much better weight of argument for a probability estimate than if one has simply applied an heuristic. Keynes also develops a theory of interval-valued probabilities, which is technical utility, but in my experience the weights of arguments are more important.

Learnt judgements, such as rational expectations, play a key role in adaptive systems, including economics. New evidence should only make great difference if its weight outweighs that of the old, but then it could lead to a cascade of changes. But despite common misconceptions, new evidence can lead to a great change in the 'probability', and evidence does not always tend to make people's subjective probabilities converge. (Think politics.)

Keynes' insights can be helpful in identifying and characterising decisive times. My hypothesis is that it would be helpful to deploy this approach more widely in economics. But one needs to follow Einstein and not what Keynes (1936: 187) called 'pseudo-mathematics'. Perhaps the commonest failing of the conventional approach is to provide an answer solely in terms of the clients' question. Often, much radical uncertainty can be avoided by refining or extending the implicit hypotheses, giving ranges for probabilities, and summarising arguments.

[14] As in classical physics. Not to be confused with the Soviet economy having been state-planned.

But whatever one takes from Keynes, bear in mind that:

> “The object of our analysis is, not to provide a machine, or method of blind manipulation, which will furnish an infallible answer, but to provide ourselves with an organised and orderly method of thinking out particular problems … .”
> (Keynes 1936: 187)

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