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Relative Profit Maximization and Bertrand Equilibrium with Convex Cost Functions

Atsuhiro Satoh and Yasuhito Tanaka

Abstract
The authors study pure strategy Bertrand equilibria in a duopoly in which two firms produce a homogeneous good with convex cost functions, and they seek to maximize the weighted sum of their absolute and relative profits. They show that there exists a range of the equilibrium price in duopolistic equilibria. This range of the equilibrium price is narrower and lower than the range of the equilibrium price in duopolistic equilibria under pure absolute profit maximization, and the larger the weight on the relative profit, the narrower and lower the range of the equilibrium price. In this sense relative profit maximization is more aggressive than absolute profit maximization.

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Keywords Bertrand equilibrium; convex cost function; relative profit maximization

Authors
Atsuhiro Satoh, Faculty of Economics, Osaka University of Commerce, Higashi-Osaka, Osaka, 570-8505, Japan, chatsuhiro@gmail.com
Yasuhito Tanaka, ✉ Faculty of Economics, Doshisha University, Kamigyo-ku, Kyoto, 602-8580, Japan, yasuhito@mail.doshisha.ac.jp

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1 Introduction

Using a model developed by Dastidar (1995) we study pure strategy Bertrand equilibria in a duopoly in which two firms produce a homogeneous good with convex cost functions, and they seek to maximize the weighted sum of their absolute and relative profits instead of their absolute profits themselves. The relative profit of a firm is the difference between its absolute profit and the absolute profit of the rival firm.

For analyses about relative profit maximization please see Gibbons and Murphy (1990), Lu (2011), Matsumura, Matsushima and Cato (2013), Schaffer (1989), Vega-Redondo (1997) and Miller and Pazgal (2001)[1].

We think that seeking for relative profit or utility is based on the nature of human. Even if a person earns a big money, if his brother/sister or close friend earns a bigger money than him, he is not sufficiently happy and may be disappointed. On the other hand, even if he is very poor, if his neighbor is more poor, he may be consoled by that fact.

We show that there exists a range of the equilibrium price in duopolistic equilibria. This range of equilibrium price is narrower and lower than the range of the equilibrium price in duopolistic equilibria under pure absolute profit maximization[2], and the larger the weight on the relative profit, the narrower and lower the range of the equilibrium price. In this sense relative profit maximization is more aggressive than absolute profit maximization.

In another paper, Satoh and Tanaka (2013), we have shown a similar result in a case of linear demand functions and quadratic cost functions. In this paper we extend this result to a case of general demand functions and convex cost functions.

2 The model

There are two firms, A and B. They produce a homogeneous good. The price of the good of Firm A is p_A and the price of the good of Firm B is p_B. The outputs of Firm A and B are denoted, respectively, by x_A and x_B. The firms set the prices of their goods, and consumers buy the good from the firm whose price is lower. Let $p = \min\{p_A, p_B\}$. Consumers' demand is represented by the demand function $D(p)$. The cost functions of Firm A and B are $c_A(x_A)$ and $c_B(x_B)$.

[1] In Vega-Redondo (1997) it was shown that the equilibrium in a Cournot oligopoly with a homogeneous good under relative profit maximization is equivalent to the competitive equilibrium. But as shown in this paper the equilibrium in a Bertrand duopoly with a homogeneous good under relative profit maximization may not be equivalent to the competitive equilibrium

[2] Dastidar (1995) proved that there exists a range of the equilibrium price in duopolistic equilibria under absolute profit maximization.

Similarly to the model in Dastidar (1995) we make the following assumptions.

1. $D(p)$ is continuous and twice continuously differentiable.

2. There exists finite positive numbers p^{max} and x^{max} such that $D(p^{max}) = 0$ and $D(0) = x^{max}$, and $D'(p) < 0$ for $0 \leq p \leq p^{max}$.

3. $c(x_A)$ and $c(x_B)$ are continuous, twice continuously differentiable and strictly convex.

Further we assume that there is no fixed cost and two firms have the same cost function. Thus, $c_i(0) = 0$ for $i \in \{A, B\}$.

If $p_A = p_B$, each firm acquires a half of the demand, and two firms constitute a duopoly. Thus, if $p_A = p_B$, we have $x_A = x_B = \frac{1}{2}D(p)$. On the other hand if $p_A < p_B$ (or $p_B < p_A$) Firm A (or Firm B) acquires total demand, and it becomes a monopolist.

If $p_A < p_B$, the absolute profit of Firm A is

$$\pi_A^M(p) = pD(p) - c_A(D(p)).$$

M indicates *monopoly*. Of course the profit of Firm B is zero. Similarly if $p_B < p_A$, we have

$$\pi_B^M(p) = pD(p) - c_B(D(p)).$$

The profit of Firm A is zero.

On the other hand, if $p_A = p_B$, the absolute profits of Firm A and B are

$$\pi_A^D(p) = \frac{1}{2}pD(p) - c(x_A),\ x_A = \frac{1}{2}D(p),$$

and

$$\pi_B^D(p) = \frac{1}{2}pD(p) - c(x_B),\ x_B = \frac{1}{2}D(p).$$

D indicates *duopoly*. In this case $p = p_A = p_B$.

The objective of Firm A is the weighted sum of its absolute profit and its relative profit. In a duopoly it is expressed as follows.

$$\Pi_A^D = (1-\alpha)\pi_A^D + \alpha(\pi_A^D - \pi_B^D) = \pi_A^D - \alpha\pi_B^D,$$

and the objective of Firm B is

$$\Pi_B^D = (1-\alpha)\pi_B^D + \alpha(\pi_B^D - \pi_A^D) = \pi_B^D - \alpha\pi_A^D,$$

where

$$0 < \alpha < 1.$$

Call a firm in a duopoly a duopolist. Since, at a duopolistic equilibrium $\pi_A^D = \pi_B^D$, we have

$$\Pi_A^D = \Pi_B^D = (1-\alpha)\pi_A^D.$$

We assume

$$\max_p \pi_i^M(p) > 0, \text{ and } \max_p \pi_i^D(p) > 0 \text{ for } i \in \{A, B\}.$$

In a monopoly the absolute profit of a firm other than the monopolist is zero. Thus, the absolute profit and the relative profit of the monopolist are equal, and the objective of the monopolist is its absolute profit, that is, if Firm A is a monopolist,

$$\Pi_A^M = \pi_A^M,$$

and if Firm B is a monopolist,

$$\Pi_B^M = \pi_B^M.$$

Without loss of generality we assume $p_A \leq p_B$.

3 Preliminary results

According to Dastidar (1995) for $i \in \{A, B\}$ we define

$$\hat{p}_i \text{ such that } \pi_i^D(\hat{p}_i) = 0,$$

$$\tilde{p}_i \text{ such that } \pi_i^M(\tilde{p}_i) = 0,$$

$$\bar{p}_i \text{ such that } \pi_i^D(\bar{p}_i) = \pi_i^M(\bar{p}_i).$$

By symmetry of the model $\hat{p}_A = \hat{p}_B$, $\tilde{p}_A = \tilde{p}_B$ and $\bar{p}_A = \bar{p}_B$. So, we denote them, respectively, by $\hat{p}$, $\tilde{p}$ and $\bar{p}$. In Dastidar (1995) the following results have been proved.

1. There exists a unique $\hat{p}$ in $[0, p^{max})$. (Lemma 1 in Dastidar (1995))

2. There exists a unique $\tilde{p}$ in $[0, p^{max})$. (Lemma 4 in Dastidar (1995))

3. There exists a unique $\bar{p}$ in $[0, p^{max})$. (Lemma 5 in Dastidar (1995))

4. $\hat{p} < \tilde{p} < \bar{p}$. (Lemma 6 in Dastidar (1995))

Now we define another critical price p_i^* by

$$\Pi_i^D(p_i^*) = \Pi_i^M(p_i^*).$$

Also by symmetry we have $p_A^* = p_B^*$, and so denote them by p^*. We show the following lemmas.

Lemma 1. *There exists a unique* p^* *in* $[0, p^{max})$*, and* $\tilde{p} < p^* < \bar{p}$ *for* $0 < \alpha < 1$.

Proof. Note that

$$\Pi_i^D(p) = (1-\alpha)\pi_i^D(p),$$

and

$$\Pi_i^M(p) - \Pi_i^D(p) = \pi_i^M(p) - (1-\alpha)\pi_i^D(p).$$

If $p = \bar{p}$, $\pi_i^M(p) - \pi_i^D(p) = 0$, and so

$$\Pi_i^M(\bar{p}) - \Pi_i^D(\bar{p}) = \alpha\pi_i^D(\bar{p}).$$

If $p = \tilde{p}$, $\pi_i^M(p) = 0$, and so

$$\Pi_i^M(\tilde{p}) - \Pi_i^D(\tilde{p}) = -(1-\alpha)\pi_i^D(\tilde{p}).$$

Now

$$\frac{\partial \pi_i^D(p)}{\partial p} = \frac{1}{2}\left\{ D(p) + D'(p)\left[p - c_i'\left(\frac{1}{2}D(p)\right)\right]\right\}. \tag{1}$$

When $\pi_i^D(p) \leq 0$, we have $-c_i\left(\frac{1}{2}D(p)\right) \leq -\frac{1}{2}pD(p)$.

Since $c_i(\cdot)$ is strictly convex,

$$c_i\left(\frac{1}{2}D(p)\right) - c_i(0) = c_i\left(\frac{1}{2}D(p)\right) < \frac{1}{2}D(p)c_i'\left(\frac{1}{2}D(p)\right),$$

or

$$-c_i\left(\frac{1}{2}D(p)\right) > -\frac{1}{2}D(p)c_i'\left(\frac{1}{2}D(p)\right).$$

This means

$$-\frac{1}{2}D(p)c_i'\left(\frac{1}{2}D(p)\right) < -\frac{1}{2}pD(p).$$

Therefore,

$$p < c_i'\left(\frac{1}{2}D(p)\right).$$

Since $D'(p) < 0$, from (1) we find that when $\pi_i^D(p) \leq 0$, $\frac{\partial \pi_i^D(p)}{\partial p} > 0$ holds. Thus, the continuity of $\pi_i^D(p)$ and the uniqueness of $\hat{p}$ means that $\pi_i^D(\bar{p}) > 0$, and so $\Pi_i^M(\bar{p}) - \Pi_i^D(\bar{p}) > 0$ for $\alpha > 0$ because $\hat{p} < \bar{p}$.

Similarly $\Pi_i^M(\tilde{p}) - \Pi_i^D(\tilde{p}) = -(1-\alpha)\pi^D(\tilde{p}) < 0$ for $0 < \alpha < 1$ because $\hat{p} < \tilde{p}$.

Therefore, by the continuity of $\pi_i^M(p)$ and $\pi_i^D(p)$ there exists p^* such that $\Pi_i^M(p^*) - \Pi_i^D(p^*) = 0$ between $\tilde{p}$ and $\bar{p}$, that is, $\tilde{p} < p^* < \bar{p}$.

We show uniqueness of p^*. Note that

$$\begin{aligned}\Pi_i^M(p) - \Pi_i^D(p) =& pD(p) - c_i(D(p)) - (1-\alpha)\left[\frac{1}{2}pD(p) - c_i\left(\frac{1}{2}D(p)\right)\right] \\ =& \frac{1+\alpha}{2}pD(p) - c_i(D(p)) + (1-\alpha)c_i\left(\frac{1}{2}D(p)\right).\end{aligned}$$

Now

$$\begin{aligned}\frac{\partial}{\partial p}\left[\Pi_i^M(p) - \Pi_i^D(p)\right] =& \frac{1+\alpha}{2}D(p) + D'(p)\left\{\frac{1+\alpha}{2}\left[p - c_i'(D(p))\right]\right. \\ &\left. + \frac{1-\alpha}{2}\left[c_i'\left(\frac{1}{2}D(p)\right) - c_i'(D(p))\right]\right\}\end{aligned} \tag{2}$$

When $\Pi_i^M(p) - \Pi_i^D(p) \leq 0$, we have $-c_i(D(p)) + (1-\alpha)c_i\left(\frac{1}{2}D(p)\right) \leq -\frac{1+\alpha}{2}pD(p)$.

Since $c_i(\cdot)$ is strictly convex,

$$c_i(D(p)) - c_i(0) = c_i(D(p)) < D(p)c_i'(D(p)),$$

$$c_i(D(p)) - c_i\left(\frac{1}{2}D(p)\right) < \frac{1}{2}D(p)c_i'(D(p)),$$

or

$$-D(p)c_i'(D(p)) < -c_i(D(p)),$$

$$-\frac{1}{2}D(p)c_i'(D(p)) < -c_i(D(p)) + c_i\left(\frac{1}{2}D(p)\right),$$

From them

$$-\frac{1+\alpha}{2}D(p)c_i'(D(p)) < -c_i(D(p)) + (1-\alpha)c_i\left(\frac{1}{2}D(p)\right).$$

It means

$$p < c_i'(D(p)).$$

Also we have

$$c_i'\left(\frac{1}{2}D(p)\right) < c_i'\left(D(p)\right).$$

Since $D'(p) < 0$, from (2) we find that when $\Pi_i^M(p) - \Pi_i^D(p) \le 0$, $\frac{\partial}{\partial p}\left[\Pi_i^M(p) - \Pi_i^D(p)\right] > 0$ holds. Since $\Pi_i^M(p)$ and $\Pi_i^D(p)$ are continuously differentiable, this fact implies that p^* is unique.

□

Lemma 2. *p^* is decreasing with respect to α.*

Proof. p^* satisfies

$$\Pi_i^M(p^*) - \Pi_i^D(p^*) = p^*D(p^*) - c_i(D(p^*)) - (1-\alpha)\left[\frac{1}{2}p^*D(p^*) - c_i\left(\frac{1}{2}D(p^*)\right)\right] = 0.$$

Differentiating this with respect to α,

$$\left(\frac{\partial}{\partial p}\left[\Pi_i^M(p^*) - \Pi_i^D(p^*)\right]\right)\Bigg|_{p=p^*}\frac{dp^*}{d\alpha} = -\left[\frac{1}{2}p^*D(p^*) - c_i\left(\frac{1}{2}D(p^*)\right)\right].$$

This is negative because $\pi_i^D(p^*) = \frac{1}{2}p^*D(p^*) - c_i\left(\frac{1}{2}D(p^*)\right) > 0$ for $\hat{p} < p^*$.

□

4 Pure strategy Bertrand equilibrium

We verify the following result.

Lemma 3. *For $p > \tilde{p}$ we have*

$$\Pi_i^M(p) = \pi_i^M(p) > 0,$$

and for $p < \tilde{p}$

$$\Pi_i^M(p) = \pi_i^M(p) < 0.$$

Proof. Now

$$\frac{\partial \pi_i^M(p)}{\partial p} = \left\{D(p) + D'(p)\left[p - c_i'\left(D(p)\right)\right]\right\}. \tag{3}$$

When $\pi_i^M(p) \le 0$, we have $-c_i\left(D(p)\right) \le -pD(p)$.

Since $c_i(\cdot)$ is strictly convex,

$$c_i(D(p)) < D(p)c_i'(D(p)),$$

or

$$-c_i(D(p)) > -D(p)c_i'(D(p)).$$

This means

$$-D(p)c_i'(D(p)) < -pD(p).$$

Therefore,

$$p < c_i'(D(p)).$$

Since $D'(p) < 0$, from (3) we find that when $\pi_i^M(p) \leq 0$, $\frac{\partial \pi_i^M(p)}{\partial p} > 0$ holds. Thus, the continuity of $\pi_i^M(p)$ and the uniqueness of $\tilde{p}$ means that $\pi_i^M(p) > 0$ for $p > \tilde{p}$ and $\pi_i^M(p) < 0$ for $p < \tilde{p}$. □

First we show non-existence of monopolistic equilibrium.

Theorem 1. *There is no monopolistic equilibrium.*

Proof. A monopolistic equilibrium is an equilibrium where Firm A is the monopolist. Suppose that $p_A < p_B$ and $p_A > \tilde{p}$. Then, Firm B can set p_B slightly lower than p_A and earn the positive profit. If $p_A < p_B$ and $p_A = \tilde{p}$, Firm A can set p_A slightly higher than $\tilde{p}$ but lower than p_B and earn the positive profit, or Firm B can set $p_B = p_A$ and earn the positive profit in a duopoly ($\hat{p} < \tilde{p}$). Of course $p_A < \tilde{p}$ is not profitable for Firm A. □

Next we show

Theorem 2. *There exists a range of the equilibrium price* $[\hat{p}, p^*]$ *in a duopoly.*

Proof.
1. Suppose $p_A = p_B = p$ and $p^* < p < \bar{p}$. The relative profits of the firms are zero. Firm B (or A) can set p_B (or p_A) slightly lower then p, and earn the positive absolute profit as a monopolist. Although that absolute profit is smaller than its absolute profit in a duopolistic equilibrium (because $p < \bar{p}$), its relative profit is positive and it is equal to its absolute profit because the profit of the rival firm is zero, and we have

$$\Pi_A^M > \Pi_A^D.$$

.

2. Suppose $p_A = p_B = p$ and $\hat{p} \leq p \leq p^*$. If Firm B (or A) sets p_B (or p_A) lower than p, it becomes a monopolist, but in this case we have

   $$\Pi_A^D \geq \Pi_A^M.$$

   Thus, there is no incentive to deviate from the equilibrium.

3. Of course if $p_A = p_B = p$ and $p < \hat{p}$, the absolute profits of the firms are negative and their relative profits are zero, so each firm can set its price higher than the price of the rival firm and make its absolute profit zero and its relative profit positive since the absolute profit of the rival firm is negative because $\hat{p} < \tilde{p}$.

□

The range of the equilibrium price in a duopoly under absolute profit maximization is $[\hat{p}, \bar{p}]$ (Proposition 1 in Dastidar (1995)), and $p^* < \bar{p}$. Therefore, the range of the equilibrium price in a duopoly under relative profit maximization is lower and narrower than that under absolute profit maximization.

Lemma 2 means that the larger the weight on the relative profit, the narrower and lower the range of the equilibrium price.

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