

# Entrepreneurship, Knowledge, and the Industrial Revolution

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## Abstract

This paper constructs a two-sector unified growth model that explains the timing and the inevitability of an industrial revolution through entrepreneurs' role for the accumulation of useful knowledge. While learning-by-doing in agriculture eventually allows the preindustrial economy to leave its Malthusian trap, an industrial revolution is delayed as entrepreneurs of the manufacturing sector do not attempt invention if not much is known about natural phenomena. On the other hand, these entrepreneurs, as managers, serendipitously identify new useful knowledge in all times, and an industrial revolution inevitably starts at some time. The industrial revolution leads the economy to modern growth, the share of the agricultural sector declines, and the demographic transition is completed with a stabilizing level of population in the very long run. Several factors affect the timing of the industrial revolution in expected directions, but some factors that affect the optimal choice of fertility have ambiguous effects. The analysis almost completely characterizes the equilibrium path from ancient times to the infinite future, and the model economy successfully captures the qualitative aspects of the unified growth experience of England.

*Keywords:* unified growth theory, useful knowledge, industrial enlightenment, demographic transition, endogenous technological change.

*Journal of Economic Literature Classification:* O31, O33, O41, J13, N33.

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## 1 Introduction

Unified Growth Theory (UGT) of Galor and Weil (2000) and the literature following their methodology and motivation study the very long-run patterns of economic growth and development within dynamic general equilibrium environments. A unified model does not only feature the Malthusian stagnation and the modern growth equilibria; it also accounts for the factors that trigger and govern the endogenously occurring and the gradual transition from stagnation to growth.<sup>1</sup>

Since an industrial revolution is the major turning point in any economy's transition from stagnation to growth, the UGT literature is the very research program within which we would start developing a rigorous understanding of why the first Industrial Revolution in Britain started when it did and which factors did keep today's developed societies and others in a quasi-trap of poverty for several millennia. Some recent papers—including those of Desmet and Parente (2012), O'Rourke et al. (2013), Strulik et al. (2013), Peretto (2013), and Strulik (2014)—provide new insights regarding the causes of the first Industrial Revolution in Britain by focusing on endogenous technological change, and Madsen et al. (2010) show that British economic growth in the last couple of centuries can best be understood as a story of productivity growth resulting from inventive activities. Complementing this view, Clark (2014) argues that only the models that endogenize technological progress would be successful to explain England's transition from Malthusian stagnation to modern growth.

This paper constructs a unified growth model that explains the timing and the inevitability of an industrial revolution in a preindustrial economy through *entrepreneurs'* role for the accumulation of *useful knowledge*. The growth of living standards in the model is due to new inventions created by entrepreneurs who behave very much like Schumpeter's (1934) entrepreneur-inventor's, and the start of the industrial revolution is an endogenously occurring switch from an equilibrium regime of zero inventive effort to that of positive inventive effort.

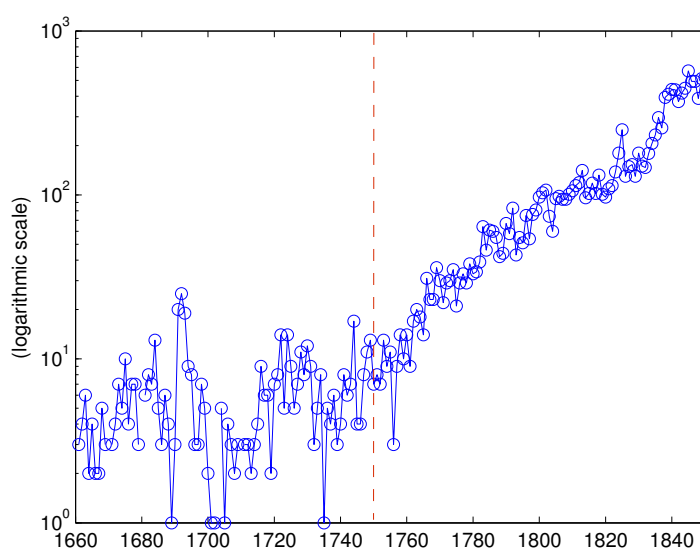
### 1.1 Premises

The model builds upon three premises that specify the role of entrepreneurs for the accumulation of useful knowledge:

Firstly, entrepreneurs in the model establish and manage their firms that produce and sell a consumption good in a perfectly competitive sector as in Hellwig and Irmen (2001) and Grossmann (2009). That entrepreneurs appropriate positive profits

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<sup>1</sup> Early contributions to the literature include those of Kögel and Prskawetz (2001), Jones (2001), Hansen and Prescott (2002), Lucas (2002), Tamura (2002), Hazan and Berdugo (2002), Boldrin and Jones (2002), Lagerlöf (2003), and Doepke (2004), and Galor (2005, 2010) provides two illuminating surveys of the UGT literature.



**Figure 1:** Patented Process Innovations in England, 1661-1850

*Data Source:* Sullivan (1989)

by managing their own firms implies that it may be optimal for entrepreneurs to allocate some of their scarce time endowment to inventive activity while decreasing the time they spend on routine management.

Next, building on Mokyr's (2002) theory of useful knowledge, *inventions* and *useful knowledge* are differentiated such that, for a given level of effort directed to inventive activity, it is less likely to be successful in achieving a given number of inventions if the stock of useful knowledge is smaller. By this premise, recently exploited by O'Rourke et al. (2013) as well, the model endogenizes the productivity term of a standard invention technology that exhibits constant-returns-to-scale with respect to its rival labor input.

Finally, the stock of useful knowledge expands in time through the process of *collective discovery*. Entrepreneurs serendipitously identify new useful knowledge about natural phenomena underlying the production processes and share what they discover with each other in their common social environment. This is a way to formalize, albeit imperfectly, what Mokyr (2002) calls *industrial enlightenment*, and it is motivated, among others, by Jacob (1997), Bekar and Lipsey (2004), and Landes (2006) who emphasize the creation and the diffusion of useful knowledge among British/European entrepreneurs and capitalists.

## 1.2 Motivation

What motivates the emphasis on entrepreneurial invention is Meisenzahl and Mokyr's (2012) prosopographical evidence on 759 British inventors born between 1660 and 1830. Among 598 inventors with a known business ownership status, only 88 inventors (around 15%) were employed as non-managers, and 467 of them (around 78%) were *business owners*. The latter statistic suggests that understanding the role of *inventors incentivized by profit motive* during the first Industrial Revolution may be of prime importance.

The motivation for focusing on the role of useful knowledge for inventions originates from Sullivan's (1989) evidence showing that the number of patented inventions in England started increasing around 1750 without any change in patent regulations and in the propensity to patent. Figure 1 pictures Sullivan's (1989) data on the number of process innovations patented in England, and the stock of useful knowledge reaching to a critical level around 1750 might indeed be the trigger of such a trend break without any change in other fundamentals.

## 1.3 Main Results

The model of this paper incorporates the mechanism of entrepreneurial invention, the distinction between useful knowledge and inventions, and the process of collective discovery with the simple unified growth model of Strulik and Weisdorf (2008)—a two-sector unified growth model with learning-by-doing and endogenous fertility that allows for a closed-form solution—to arrive at the following:

**Result 1** There exists an endogenously determined threshold level of the stock of useful knowledge: Entrepreneurs optimally choose to not to invest in inventive activities if the existing stock of useful knowledge is not larger than this threshold level.

**Result 2** An industrial revolution is inevitable: Collective discovery makes the stock of useful knowledge to continuously grow in preindustrial times and to cross its threshold level in finite time.

**Result 3** The preindustrial economy faces a “trade-off” for the timing of its industrial revolution: Two determinants of fertility—the preference parameter that increases the (marginal) utility from reproduction and the initial level of productivity in the manufacturing sector—affect the growth rate of useful knowledge before the industrial revolution through the mass of entrepreneurs, and the effects of these exogenous givens on the timing of the industrial revolution is ambiguous. A higher level of fertility implies a larger *mass* of population but a smaller *share* of

entrepreneurs, and the mass of entrepreneurs might increase or decrease to return a faster or a slower accumulation of useful knowledge.

#### 1.4 Contribution

This paper shows that thinking a bit seriously about the productivity term of an otherwise standard invention technology and bringing the entrepreneur-inventor back to the scene of economic development allow us to understand why purposeful invention may not be optimal for a very long episode of history and why an industrial revolution is inevitable. The central message of the paper is that, as the inventor is in the meantime the business owner, the industrial revolution starts only when the stock of useful knowledge is large enough to make inventive activity optimal while the pace at which the economy approaches to its industrial revolution depends on the supply of entrepreneurship. While the historical narratives of the Industrial Revolution strongly acknowledge these notions, existing unified growth models do not pay attention to this dual role of entrepreneur-inventor's.

In contrast with that of [Strulik and Weisdorf \(2008\)](#), the model of this paper tells a story of Schumpeterian innovation and makes explicit the role of incentives that govern the dynamics of technological progress in the long run. But building upon [Strulik and Weisdorf's \(2008\)](#) model with *demographic transition* and *structural transformation* enriches our understanding in another respect: The preferences towards fertility and the dynamics of the agricultural sector affect the timing of the industrial revolution, and this is not a trivial point as the timing results of this paper are in complete accordance with those obtained by [Desmet and Parente \(2012\)](#) and [Peretto \(2013\)](#).

Due to the uncertainty in the process of invention and ex post heterogeneity across firms in the manufacturing sector, the model provides a richer understanding of modern industrial structure than most unified growth models do. Specifically, it explains why more innovative firms on average are larger and why the size distribution of innovative firms are skewed. These are two well-known regularities most recently reiterated, respectively, by [Akcigit and Kerr \(2010\)](#) and [Klette and Kortum \(2004\)](#).

Finally, this paper complements some recent contributions—e.g., [O'Rourke et al. \(2013\)](#) and [Strulik \(2014\)](#)—that formalize [Mokyr's \(2002\)](#) notion that the creation and the diffusion of some sort of useful knowledge in preindustrial times was the key for the purposeful activation of innovation by business firms.

#### 1.5 Outline

Section 2 presents a brief discussion of the related literature. Section 3 introduces the model economy, and Section 4 defines and analyzes static, dynamic, and

asymptotic equilibria. Section 5 characterizes the model economy's equilibrium path in the very long run. Section 6 discusses the implications of the model, and Section 7 concludes. Proofs of all lemmas and propositions are presented in Appendix A.

## 2 The Related Literature

At least since Schumpeter (1934), entrepreneur-inventor's are leading actors of the narratives of the Industrial Revolution. Solo (1951), Baumol (1990), Murphy et al. (1991), and Mokyr (2010), among others, argue specifically that entrepreneurial invention was indeed the engine of technological progress during the first Industrial Revolution long before the rise of modern R & D lab. Peretto (1998) emphasizes the distinction between entrepreneurial invention and corporate R & D activities in a second generation Schumpeterian model. Michelacci (2003) studies the role of entrepreneurial skills in bringing inventions to markets. From an evolutionary perspective, Doepke and Zilibotti (2008) and Galor and Michalopoulos (2012) study the role of entrepreneurial traits for economic development in the very long run. This paper, differently from all these, incorporates both occupational choice and entrepreneurial invention within a unified growth setting. Two-occupation framework adapted is similar to, and even simpler than, those of Lucas (1978), Murphy et al. (1991), and Michelacci (2003), and the formulation of entrepreneurial invention under perfect competition shares similarities with the treatments of Hellwig and Irmen (2001), Grossmann (2009), and Haruyama (2009).

The conceptual framework of useful knowledge that this dissertation builds upon, with useful knowledge and inventions being distinct knowledge forms, is due to Mokyr (2002). In his theory, useful knowledge is propositional and does not have direct technological applications. The stock of useful knowledge is formed by laws and principles that answer "What?" questions about natural phenomena underlying the production processes. Inventions, in contrast, are prescriptive in the sense that they provide answers to "How?" questions; inventions take the forms of blueprints and recipes. Other than Mokyr (2002), the role of this distinction and the usefulness of some foundational knowledge for inventive activity are emphasized by Landes (1969), Rosenberg (1974), Nelson (1982), and Easterlin (1995). In one context, Weitzman (1998: 345) suggests that knowledge accumulation has distinct *recombinant* and *productivity* aspects where the former corresponds to the role of useful knowledge for inventions. With reference to *knowledge capital*, Lucas (2002: 12) asks "[w]hat can be gained by disaggregating into two or more knowledge-related variables." The model studied in this paper answers this question by showing that, when the productivity of inventive effort is endogenous to how large the stock of useful knowledge is, purposeful and costly invention may not be

optimal. The distinction is also emphasized by Haruyama (2009) in an endogenous growth model with perfectly competitive innovation. O'Rourke et al. (2013) incorporates it into the formal analysis of unified growth, and Strulik (2014) emphasizes existing knowledge and accessible knowledge where the access costs are decreasing in time as suggested by Mokyr (2002). However, the dual role of entrepreneurship for useful knowledge remains unexplored in a unified model with population growth and structural transformation. This is what this paper attempts to deliver.

The process of collective discovery by entrepreneurs is described by Landes (2006: 6) as “the seventeenth-century European mania for tinkering and improving.” Bekar and Lipsey (2004) go further to argue that the diffusion of Newtonian mechanics among British industrialists was the prime cause of the first Industrial Revolution. A similar argument on the diffusion of scientific culture, again with an emphasis on British success, is made by Jacob (1997). Kelly (2005) develops a network model that analyzes this type of collective learning for the industrial revolution. Lucas (2009) also emphasizes collective learning in a model that differentiates propositional knowledge from productivity. O'Rourke et al. (2013) and Milionis and Klasing (2009), with environments similar to that of Galor and Weil (2000), link the accumulation of propositional knowledge to human capital accumulation respectively through the number of high-skilled individuals and the individual-level stock of skills. Howitt and Mayer-Foulkes (2005) assume that the skill level of entrepreneurs is proportional to the average productivity associated with intermediate inputs of production. What differentiates this paper's formulation of industrial enlightenment is the role of *the mass of entrepreneurs*. More entrepreneurs create more useful knowledge given the quality of creating and diffusing it. This type of scale effect by which knowledge growth depends not on the mass of entire population but instead on a certain mass of urban population is emphasized by Crafts and Mills (2009).

### 3 The Model Economy

This section constructs the model economy. The first four subsections respectively introduce the demographic structure, endowments, preferences, and technologies, i.e., the model environment. Then described are occupations and the market structures. Decision problems are formally defined next, and the section concludes with the market clearing conditions.

Time in the model, denoted by  $t$ , is discrete and diverges to the infinite future:  $t \in \{0, 1, \dots\}$ . Following the UGT literature, the economy is closed, and there is no physical capital accumulation. The produced goods of the model are food and the manufactured good, and the primary inputs are land and labor.



### 3.1 The Demographic Structure

Consider two overlapping generations: Individuals who are *adults* in period  $t$  give birth to *children* at the beginning of period  $t$ . Their surviving children become adults at the beginning of period  $t + 1$ .

For simplicity, reproduction is asexual, and  $n_t \in \mathbb{R}_{++}$  denotes the number of surviving children a generic adult in period  $t$  optimally chooses, i.e., net fertility per adult. Note that  $n_t$ , not being an integer-valued variable, represents *average* net fertility among period- $t$  adults.

Denote by  $N_t \in \mathbb{R}_{++}$  the adult population in period  $t$ . Since there exists a common level of net fertility for all adults in equilibrium,  $N_{t+1}$  simply reads

$$N_{t+1} = n_t N_t \tag{1}$$

where  $N_0 > 0$  is exogenously given.

### 3.2 Endowments

Normalizing the length of a period to unity implies that a period- $t$  adult has a unit time endowment. This is the only source of homogeneous labor force in this economy. Children, not having a time endowment, remain idle until they become adults next period.

Land is a production factor of the agricultural technology that is used to produce food, and the total land endowment of the economy is fixed. As it is common in the UGT literature, there do not exist property rights over land, and who owns land and at what proportions are of no importance from an analytical point of view.

### 3.3 Preferences

An adult in period  $t$  derives lifetime utility from her consumption  $C_t$  of the manufactured good and her net fertility  $n_t$ . As in [Strulik and Weisdorf \(2008\)](#) and [de la Croix and Licandro \(2013\)](#), the utility function representing these preferences is quasi-linear and defined as in

$$U(C_t, n_t) \equiv C_t + \phi \ln(n_t) \quad \phi > 0 \tag{2}$$

with a boundary restriction in the form of

$$n_t \geq 1. \tag{3}$$

$U(C_t, n_t)$  takes its non-homothetic form for two reasons: First, linearity in  $C_t$  eliminates the direct income effect on fertility which leads to a fertility decline at the advanced stages of economic development. The second reason of adapting



risk neutral preferences with respect to  $C_t$  is to simplify the decision problem of entrepreneurs. As we shall see below, the decision toward entrepreneurial invention is an expected utility problem.

The inequality in (3) represents the parental preference for *reproductive success* in transmitting genes to the next generations.<sup>2</sup> (3) is the simplest way of introducing reproductive success in a model of fertility choice, but this suffices to produce a desired property:<sup>3</sup> The baseline level of net fertility is equal to unity as in Jones (2001), and this implies a stabilizing *level* of population in the long run.

### 3.4 Technologies

This subsection introduces the technologies of the model economy without reference to the ownership and the market structures.

#### Reproduction

The only input of the reproduction technology is food; reproduction requires only the food intake by children. For simplicity, adults do not consume food, and each surviving child requires one unit of food.<sup>4</sup> Then, the budgetary cost of having  $n_t$  surviving children is equal to  $n_t$  units of food.

#### Agriculture

Consider any farm  $f$  operating in the agricultural sector. The Cobb-Douglas technology of production for farm  $f$  is specified as in

$$Y_{ft} = (L_f X_{ft})^{1-\eta} H_{ft}^\eta \quad \eta \in (0, 1) \quad (4)$$

where  $Y_{ft}$  denotes output,  $L_f$  denotes land input,  $X_{ft}$  denotes productivity, and  $H_{ft}$  denotes the flow of worker hours.

The productivity  $X_{ft}$  of farm  $f$  changes in time due to learning-by-doing at the farm level. The technology of learning-by-doing is specified as in

$$X_{ft+1} = X_{ft} + \psi Y_{ft} \quad \psi > 0. \quad (5)$$

<sup>2</sup> Notice that, by construction, all surviving children become fecund at the beginning of their adulthood.

<sup>3</sup> de la Croix and Licandro (2013) and Strulik and Weisdorf (2014) incorporate reproductive success, respectively, into continuous and discrete time environments where parents simultaneously choose the number of children they have and the likelihood of these children's survival.

<sup>4</sup> This assumption, as in Strulik and Weisdorf (2008), simplifies the model to allow for a closed-form solution.

## Manufacturing

Consider any firm  $i$  operating in the manufacturing sector. The Cobb-Douglas technology of production for firm  $i$  reads

$$y_{it} = (X_{it}h_{wit})^\lambda h_{mit}^{1-\lambda} \quad \lambda \in (0, 1) \quad (6)$$

where  $y_{it}$  denotes output,  $X_{it}$  denotes productivity associated with worker hours  $h_{wit}$ , and  $h_{mit}$  denotes the flow of manager hours.

Recall that the labor endowment of individuals is homogeneous. Accordingly, what differentiates worker and manager hours in (6) is only the nature of the tasks in the question. Two distinct tasks are required to produce the good: Workers are the ones who actually produce the good in its finalized form with their eye-hand coordination, and managers are the ones who tell workers *what* to do and *how* to do it.

The growth of productivity in the manufacturing sector is due to inventions. Requiring research hours as its only rival input, an invention project generates a stochastic number of inventions, and each invention increases a baseline level of productivity by some fixed factor. As in [Desmet and Parente \(2012\)](#), this baseline level of productivity for firm  $i$  in period  $t$ , denoted by  $\bar{X}_t$ , is taken as the average operating productivity attained by the firms of the previous generation and defined as in

$$\bar{X}_t \equiv E_{t-1}^{-1} \int_0^{E_{t-1}} X_{it-1} di \quad (7)$$

where  $E_{t-1} > 0$  denotes the mass of firms operating in the manufacturing sector in period  $t - 1$ .

Define, now, firm  $i$ 's operating productivity  $X_{it}$  as in

$$X_{it} \equiv \sigma^{z_{it}} \bar{X}_t \quad (8)$$

where  $\sigma > 1$  is the step-size of inventions and  $z_{it} \in \{0, 1, \dots\}$  is the stochastic number of inventions satisfying

$$z_{it} \sim \text{Poisson}(a_{it}). \quad (9)$$

Here,  $a_{it} \in \mathbb{R}_+$  denotes the arrival rate of inventions for firm  $i$  and is tied to the research effort via

$$a_{it} = \theta f(K_t) h_{rit} \quad \theta > 0 \quad (10)$$

where  $h_{rit}$  denotes the flow of hours allocated to research by firm  $i$ . This invention technology features constant returns to scale with respect to its rival input  $h_{rit}$ .

The novelty here is the term  $\theta f(K_t)$ , i.e., the research productivity per unit of inventive effort, where  $K_t \in \mathbb{R}_+$  denotes the stock of useful knowledge and  $f(K_t) : \mathbb{R}_+ \rightarrow [0, 1]$  is a function that represents the role of useful knowledge for invention. What follows next is a discussion of the three restrictions on  $f(K_t)$ .

Suppose firstly that it is likely to be more successful in invention if a larger stock of useful knowledge is available. A plausible way to restrict the function  $f(K_t)$  to reflect this is to assume that  $f(K_t)$  is strictly increasing.

The second restriction is that useful knowledge not only is useful for invention but also is the essential input of the invention projects:

$$f(0) = 0. \tag{11}$$

In Mokyr's (2002: 13-14) words, "[t]he likelihood that a laptop computer would be developed in a society with no knowledge of computer science, advanced electronics, materials science, and whatever else is involved is nil."

Finally, there exists an upper limit of the research productivity  $\theta f(K_t)$  when  $K_t \rightarrow +\infty$  as in, say, Weitzman (1998: 342). Even if every single thing about the natural phenomena underlying the production processes is known, i.e.,  $K_t \rightarrow +\infty$ , the arrival rate  $a_{it}$  is bounded above because a unit of  $h_{rit}$  must have a finite capacity to process knowledge. One simple way to impose such a limit is to assume that

$$\lim_{K_t \rightarrow +\infty} f(K_t) = 1. \tag{12}$$

(12) basically implies that the case of  $K_t \rightarrow +\infty$  as the ultimate enlightenment eliminates what causes a unit of inventive effort to be less productive than its full potential of  $\theta$ . When firm  $i$  has access to the knowledge formed with practically everything about the natural phenomena, there is no need to spend resources to realize what part of knowledge is useful and what part is not. The firm simply generates an expected number  $a_{it} = \theta h_{rit}$  of inventions with constant (maximum) productivity  $\theta$  as it is usual in endogenous growth theory.<sup>5</sup>

### Collective Discovery

Collective discovery governs how  $K_t$  changes in time. The idea, as introduced earlier, is that entrepreneurs, owning the firms operating in the manufacturing

<sup>5</sup> Note that Strulik (2014) models the access cost to existing knowledge by firms with a similar increasing function described by (11) and (12) such that (i) knowledge diffusion accelerates if the stock of capital expands, (ii) knowledge does not diffuse if there is no capital, and (iii) knowledge diffusion is perfect if the stock of capital converges to positive infinity. The two postulations—those of Strulik (2009) and of this paper—are, however, on entirely different notions as  $f(K_t)$  described above governs the research productivity and  $K_t$  is collectively known by all period- $t$  individuals. Extending the present framework with the access costs as in Strulik (2014) is left for future research.

sector, collectively discover new pieces of propositional knowledge during their lifetime. They not only create new knowledge in this serendipitous way individually but also share what they create with each other in their common environment, e.g., in coffeehouses. This is a network effect that is most consistent with the public good characterization of useful knowledge.

The simplest way to formalize this is a linear knowledge production function of the form

$$K_{t+1} = K_t + \omega E_t \quad \omega > 0 \quad (13)$$

where  $E_t$ , the mass of the firms operating in the manufacturing sector in period  $t$ , denotes the total mass of all entrepreneurs in period  $t$  since, as we shall see in a moment, each manufacturing firm is owned by a single entrepreneur in equilibrium. Thus, the total mass  $E_t$  of all entrepreneurs is in fact the time input of collective discovery. The parameter  $\omega > 0$  in (13) represents the quality of the environment for creating and sharing useful knowledge. This constant thus represents geographical, cultural, and social determinants of collective discovery.<sup>6</sup>

### 3.5 Occupations and the Market Structures

There exist, in equilibrium, two occupational groups in this economy, *entrepreneurs* and *workers*, and the occupation is chosen optimally. An entrepreneur establishes a firm that produces and sells the manufactured good under perfectly competitive conditions, and a worker inelastically supplies her labor endowment to these firms and to the farms producing food. Thus, three things are traded in this economy: First, all adults consume the manufactured good produced and sold by the firms. This is the numéraire. Second, a worker hour is traded at the wage  $W_t > 0$  in a perfectly competitive labor market where firms run by entrepreneurs and farms producing food are the buyers. Finally, food is traded at the price  $P_t > 0$  in a perfectly competitive food market where all adults are, again, buyers.

The incentive for entrepreneurs to engage in inventive activities in the manufacturing sector originates from the presumption that each entrepreneur manages her own firm. This implies that the flow of profit to an entrepreneur is strictly positive under all circumstances. Instead of allocating all of her labor endowment to routine management, then, an entrepreneur may choose to allocate some of it to invention for an increased market share and profit. Firm  $i$  operating in period  $t$  shuts down when entrepreneur  $i$  dies at the end of period  $t$ . New firms are then established in the beginning of period  $t + 1$  by adults who choose to become entrepreneurs in period  $t + 1$ .

<sup>6</sup> Note that the main results of this paper are not sensitive to the linearity with respect to  $E_t$ . The qualitative nature of the main results does not change as long as  $K_{t+1} - K_t$  is an increasing function of  $E_t$ .

Closely following the usual treatment of the agricultural sector in the UGT literature, it is assumed that there exists a continuum  $[0, 1]$  of identical and perfectly competitive farms each owned by its workers with equal ownership shares. These imply, given the production technology (4) and the normalization of  $L_f = 1$  for all  $f \in [0, 1]$ , that a worker in the agricultural sector earns price times her average product

$$P_t \left( \frac{X_{ft}^{1-\eta} H_{ft}^\eta}{H_{ft}} \right)$$

if  $H_{ft}$  units of worker hours are allocated to production. Moreover, the perfect mobility of workers between agriculture and manufacturing dictates that

$$P_t \left( \frac{X_{ft}^{1-\eta} H_{ft}^\eta}{H_{ft}} \right) = W_t. \quad (14)$$

### 3.6 Decision Problems

There are three decision problems: the problem of the representative worker, the problem of the representative entrepreneur, and the problem of occupational choice for all individuals. The solution to the last problem builds upon the solutions of the first and the second problems in such a way that, in equilibrium, an individual is indifferent between becoming a worker and becoming an entrepreneur.

#### Workers

Let  $C_{wt}$  and  $n_{wt}$  respectively denote consumption and net fertility for the worker. Since the worker earns only the wage income  $W_t$  regardless of the sector she supplies her labor endowment, the worker's deterministic problem, given (2), is to maximize

$$U_{wt} \equiv C_{wt} + \phi \ln(n_{wt}) \quad (15)$$

subject to the budget constraint

$$C_{wt} + P_t n_{wt} = W_t \quad (16)$$

and the boundary constraint  $n_{wt} \geq 1$ . The problem is then to

$$\max_{n_{wt} \geq 1} W_t - P_t n_{wt} + \phi \ln(n_{wt}). \quad (17)$$

## Entrepreneurs

With a slight abuse of notation, let entrepreneur  $i$  be the representative entrepreneur who establishes firm  $i$ . This entrepreneur's income is equal to the flow  $\Pi_{it}$  of profit, and  $\Pi_{it}$  is, in the general case, stochastic due to the uncertainty in the process of invention. (2) and entrepreneur  $i$ 's budget constraint

$$C_{it} + P_t n_{it} = \Pi_{it} \quad (18)$$

imply the expected utility of entrepreneur  $i$  as in

$$\mathbf{E}[U_{it}] \equiv \mathbf{E}[\Pi_{it}] - P_t n_{it} + \phi \ln(n_{it}) \quad (19)$$

where  $n_{it} \geq 1$  is her net fertility, and  $\mathbf{E}[\bullet]$  is the expectation operator.

Entrepreneur  $i$  maximizes  $\mathbf{E}[U_{it}]$  by choosing an optimal level of  $n_{it}$  and by maximizing  $\mathbf{E}[\Pi_{it}]$ . To achieve the latter, operating with the production technology (6) and the invention technology (10), she chooses an optimal level  $h_{wit}$  of the demand for worker hours and allocates her time between management and invention under the resource constraint

$$h_{mit} + h_{rit} = 1. \quad (20)$$

Suppose that entrepreneur  $i$  chooses  $h_{wit}$  contingent upon  $(X_{it}, h_{mit}, W_t)$ .<sup>7</sup> Given (6), the profit function is defined as in

$$\Pi_{it} \equiv \Pi(h_{wit}, X_{it}, h_{mit}, W_t) \equiv (X_{it} h_{wit})^\lambda h_{mit}^{1-\lambda} - W_t h_{wit},$$

and the problem of maximizing  $\Pi(h_{wit}, X_{it}, h_{mit}, W_t)$  by choosing  $h_{wit} \geq 0$  has a unique interior solution satisfying

$$h_{wit} = \frac{\lambda^{\frac{1}{1-\lambda}} X_{it}^{\frac{\lambda}{1-\lambda}} h_{mit}}{W_t^{\frac{1}{1-\lambda}}} \quad (21)$$

and implying

$$\Pi_{it} = (1 - \lambda) \lambda^{\frac{\lambda}{1-\lambda}} \left( \frac{X_{it}}{W_t} \right)^{\frac{\lambda}{1-\lambda}} h_{mit}. \quad (22)$$

<sup>7</sup> This is not an uncommon assumption in the quality ladder framework adapted here—see, e.g., Aghion and Howitt (2009). Forcing the entrepreneur to choose  $h_{wit}$  and  $h_{rit}$  simultaneously results in a “second-best” result for the entrepreneur such that she cannot perfectly insure herself against too low or too high levels of  $h_{wit}$  relative to  $X_{it}$  ex post. The main results of the paper, however, are not altered.

Using (8), (9), (10), (20), and (22), the expected profit  $\mathbf{E}[\Pi_{it}]$  of entrepreneur  $i$  can now be defined as in

$$\mathbf{E}[\Pi_{it}] \equiv \sum_{z=0}^{+\infty} \left[ \frac{a_{it}^z \exp(-a_{it})}{z!} \right] \left[ (1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}} \left( \frac{\sigma^z \bar{X}_t}{W_t} \right)^{\frac{\lambda}{1-\lambda}} \left( 1 - \frac{a_{it}}{\theta f(K_t)} \right) \right]. \quad (23)$$

The term in the first brackets on the right-hand side denotes the Poisson probability of generating  $z$  inventions given  $a_{it}$ , and the term in the second brackets is the level of optimal profit when the entrepreneur generates  $z$  inventions given  $a_{it}$ .

**Lemma 1.** *The expected profit in (23) can be rewritten as*

$$\mathbf{E}[\Pi_{it}] = \exp(\Sigma a_{it}) \Lambda \left( \frac{\bar{X}_t}{W_t} \right)^\Gamma \left( 1 - \frac{a_{it}}{\theta f(K_t)} \right) \quad (24)$$

where the parameters

$$\Gamma \equiv \frac{\lambda}{1-\lambda} > 0, \quad \Lambda \equiv (1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}} > 0, \quad \text{and} \quad \Sigma \equiv \sigma^{\frac{\lambda}{1-\lambda}} - 1 > 0$$

are defined for notational ease.

(24) identifies the return to and the cost of inventive activity.  $\mathbf{E}[\Pi_{it}]$  increases with  $\exp(\Sigma a_{it})$  that depends, through  $\Sigma$ , on the step-size  $\sigma$  of invention and the elasticity  $\lambda$  of output with respect to  $X_{it}$ . The last term in the last parentheses,  $a_{it}/[\theta f(K_t)]$ , is the time cost of inventing with an expected number  $a_{it}$  of inventions. Clearly, (24) specifies the *deterministic* level of profit when no inventive activity is undertaken ( $a_{it} = 0$ ). In this case, the entire labor endowment of the entrepreneur is spent to management, i.e.,  $h_{mit} = 1$ , and the realization of  $z$  is equal to zero.  $X_{it}$  is thus equal to  $\bar{X}_t$  if  $a_{it} = 0$ .

Using (19) and (24), entrepreneur  $i$ 's problem of maximizing  $\mathbf{E}[U_{it}]$  can now be formally stated as in

$$\max_{n_{it} \geq 1, a_{it} \in [0, a_{it}^{\max}]} \exp(\Sigma a_{it}) \Lambda \left( \frac{\bar{X}_t}{W_t} \right)^\Gamma \left( 1 - \frac{a_{it}}{\theta f(K_t)} \right) - P_t n_{it} + \phi \ln(n_{it}) \quad (25)$$

where  $a_{it}^{\max} \equiv \theta f(K_t)$  is the upper bound of  $a_{it}$  associated with the case of  $h_{rit} = 1$ . Notice that, since all entrepreneurs face the same set  $\{\Sigma, \Lambda, \bar{X}_t, W_t, \Gamma, \theta, K_t, P_t, \phi\}$  of givens, any solution to (25) implies a unique  $\mathbf{E}[U_{it}]$ .

### Occupational Choice

Given (15) and (19), the optimality condition of the occupational choice is simply

$$\mathbf{E}[U_{it}] = U_{wt} \quad (26)$$

so that all adults are indifferent between becoming an entrepreneur and becoming a worker at the beginning of period  $t$ .



### 3.7 Market Clearing Conditions

This subsection closes the model through the market clearing conditions where the left-hand sides denote the quantity supplied and the right-hand sides denote the quantity demanded.

Firstly, the food market clears via

$$Y_{ft} = (N_t - E_t)n_{wt} + \int_0^{E_t} n_{it} di. \quad (27)$$

Next, the market for the manufactured good clears via

$$\int_0^{E_t} y_{it} di = (N_t - E_t)C_{wt} + \int_0^{E_t} C_{it} di. \quad (28)$$

Finally, the market for worker hours, where the equilibrium mass  $E_t \in (0, N_t)$  of entrepreneurs is determined residually, clears via

$$N_t - E_t = H_{ft} + \int_0^{E_t} h_{wit} di. \quad (29)$$

## 4 Static, Dynamic, and Asymptotic Equilibria

This section defines and analyzes the equilibria of the model economy. The main purpose is to establish the analytical foundations of the model economy's equilibrium path from some initial period to the infinite future.

### 4.1 Static General Equilibrium (SGE)

**Definition 1.** A SGE of the model economy, for any  $t \in \{0, 1, \dots\}$ , is a collection

$$\{n_{wt}, C_{wt}, H_{ft}, Y_{ft}, E_t, \{n_{it}, a_{it}, h_{rit}, h_{mit}, z_{it}, X_{it}, h_{wit}, y_{it}, \Pi_{it}, C_{it}\}_{i \in [0, E_t]}\}$$

of quantities and a pair  $\{P_t, W_t\}$  of relative prices such that, given the state vector  $(N_t, \bar{X}_t, X_{ft}, K_t)$ ,

- $n_{wt}$  solves the worker's problem (17),
- $(n_{it}, a_{it})$  solves the entrepreneur's problem (25),
- all adults are indifferent between becoming a worker and becoming an entrepreneur through (26),

- the food market and the market for worker hours clear respectively via (27) and (29),<sup>8</sup> and
- the equations (4), (6), (8), (9), (10), (14), (16), (18), (20), (21), and (22) are satisfied.

**Proposition 1.** *There exists a unique SGE at which  $n_{it} = n_{wt} = n_t \geq 1$  and  $a_{it} = a_t \geq 0$ . Furthermore, depending on the given state vector  $(N_t, \bar{X}_t, X_{ft}, K_t)$ , this unique SGE features either*

- Regime 1:  $n_t = 1$  and  $a_t = 0$  or
- Regime 2:  $n_t > 1$  and  $a_t = 0$  or
- Regime 3:  $n_t > 1$  and  $a_t > 0$  or
- Regime 4:  $n_t = 1$  and  $a_t > 0$ .

Proposition 1 lists all four equilibrium regimes that the unique SGE might feature depending on  $(N_t, \bar{X}_t, X_{ft}, K_t)$ , and it is constructive to discuss these regimes under two separate headings; firstly, with regard to  $a_t \geq 0$ , the equilibrium in manufacturing, and, then, with regard to  $n_t \geq 1$ , the equilibrium in agriculture.

### Invention and the Manufacturing Sector

The industrial revolution *in the model* is defined as the endogenously occurring switch from the equilibrium regime of  $a_t = 0$  to that of  $a_t > 0$ , and the main result following from the proof of Proposition 1 is the following:

**Corollary 1.** *The arrival rate  $a_t$  of inventive activity is characterized by a unique threshold such that*

$$a_t = \begin{cases} 0 & \text{if } K_t \leq \hat{K} \\ \theta f(K_t) - \left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1} & \text{otherwise} \end{cases} \quad (30)$$

where the threshold level  $\hat{K}$  is defined as in

$$\hat{K} \equiv f^{-1} \left[ \theta^{-1} \left( \sigma^{\frac{\lambda}{1-\lambda}} - 1 \right)^{-1} \right] > 0 \quad (31)$$

<sup>8</sup> Note that the market clearing condition (28) for the manufactured good is satisfied via Walras' Law in general equilibrium, and it is therefore not an equilibrium-defining equation.

Corollary 1 directly follows from the solution (A.2) to the entrepreneur's problem. Notice from (24) that the return to and the cost of invention are not additively separable: The marginal cost of increasing the expected number of inventions from zero to an infinitesimally small amount is a strictly positive number that may well exceed its marginal return basically because the entrepreneur has to decrease her management input to increase her inventive effort. This is not trivial as the entrepreneur's dual role as manager-inventor, i.e., entrepreneurial invention, is essential in implying this non-separability.

Naturally, then, the stock  $K_t$  of useful knowledge determines, through the time cost of inventive effort, whether invention is optimal or not given  $(\sigma, \lambda, \theta)$ . Given  $K_t$ , on the other hand, higher values of  $\sigma$  and  $\lambda$  increase the return to inventive effort, and a higher value of  $\theta$  decreases the time cost of it.

The real wage  $W_t$  at the unique SGE, along with other things, determines the level of economic development in this economy. Specifically, it affects the size of the agricultural sector and the optimal level of net fertility. Thus, it is useful to remark how  $W_t$  is tied to  $\bar{X}_t$ .

**Corollary 2.** *The mapping from productivity to wage in manufacturing is*

$$W_t = \delta(a_t, K_t) (1 - \lambda)^{1-\lambda} \lambda^\lambda \bar{X}_t^\lambda \quad (32)$$

where  $\delta(a_t, K_t) \geq 1$  is an auxiliary function defined as in

$$\delta(a_t, K_t) \equiv \exp \left[ (1 - \lambda) \left( \sigma^{\frac{\lambda}{1-\lambda}} - 1 \right) a_t \right] \left( 1 - \frac{a_t}{\theta f(K_t)} \right)^{1-\lambda}. \quad (33)$$

When inventive activity is not optimal ( $a_t = 0$ ), we have  $\delta(0, \bullet) = 1$  that implies  $W_t = (1 - \lambda)^{1-\lambda} \lambda^\lambda \bar{X}_t^\lambda$ . This simply corresponds to the unit price of a worker hour that would prevail in a competitive model of occupational choice with Cobb-Douglas technology and *without* entrepreneurial invention.

When inventive activity is optimal ( $a_t > 0$ ), management input is tied to invention technology through the optimal use of entrepreneurs' time.  $W_t$  in competitive equilibrium thus embeds this effect via  $\delta(\bullet, \bullet)$  function. Another important result for the equilibrium regime of  $a_t > 0$  is the following:

**Corollary 3.** *Ex ante symmetry across entrepreneurs translates into ex post heterogeneity such that, for any arrival rate  $a_t > 0$ , the ex ante probability of generating  $z$  inventions is equal to the ex post fraction of entrepreneurs with  $z$  inventions under (Borel's version of) the law of large numbers. Thus, the unique cross-section distribution of any element of the set  $\{X_{it}, h_{wit}, y_{it}, \Pi_{it}, C_{it}\}_{i \in [0, E_t]}$  is the Poisson distribution with the parameter  $a_t > 0$ .*

Corollary 3 emphasizes that the stochastic nature of inventive activity creates winners and losers among ex ante symmetric entrepreneurs as in Galor and Michalopoulos (2012). Consequently, more innovative entrepreneurs/firms attain higher productivities, higher firm sizes, and higher market shares.

The final result to be noted regarding the manufacturing sector is on the equilibrium supply of entrepreneurship. Recall that the mass  $E_t$  of entrepreneurs is central to the equilibrium path of the model economy as the growth rate of  $K_t$  is a function of  $E_t$ , and it is in this respect necessary to highlight the equilibrium solution of  $E_t$ :

**Corollary 4.** *At the unique SGE,  $E_t$  satisfies*

$$E_t = (1 - \lambda) \left( 1 - \frac{H_{ft}}{N_t} \right) N_t \quad (34)$$

where  $H_{ft}/N_t$  is the labor share of the agricultural sector.

Note that  $E_t$  satisfies (34) regardless of  $a_t \geq 0$  and  $n_t \geq 1$ . Here,  $(1 - \lambda)$  is the Cobb-Douglas exponent of the manager hours in (6), and the fraction  $(1 - \lambda)$  of all adult individuals occupying the manufacturing sector, i.e.,  $N_t - H_{ft}$ , become entrepreneurs.  $E_t$  thus negatively depends on the size of the agricultural sector as expected.

### Fertility and the Agricultural Sector

**Corollary 5.** *At the unique SGE, net fertility ( $n_t$ ) satisfies*

$$n_t = \begin{cases} \left( \frac{\phi}{\bar{W}_t} \right)^\eta \left( \frac{X_{ft}}{N_t} \right)^{1-\eta} & \text{if } X_{ft} > \frac{N_t}{\left( \frac{\phi}{\bar{W}_t} \right)^{\frac{\eta}{1-\eta}}} \\ 1 & \text{otherwise} \end{cases} \quad (35)$$

and the labor share of the agricultural sector ( $H_{ft}/N_t$ ) satisfies

$$\frac{H_{ft}}{N_t} = \begin{cases} \frac{\phi}{\bar{W}_t} & \text{if } X_{ft} > \frac{N_t}{\left( \frac{\phi}{\bar{W}_t} \right)^{\frac{\eta}{1-\eta}}} \\ \left( \frac{X_{ft}}{N_t} \right)^{-\frac{1-\eta}{\eta}} & \text{otherwise} \end{cases} \quad (36)$$

given  $W_t = \delta(a_t, K_t) (1 - \lambda)^{1-\lambda} \lambda^\lambda \bar{X}_t^\lambda$ .

Corollary 5 follows from (A.1) and the solutions of  $P_t$  and  $H_{ft}$  for the cases of  $\phi > P_t$  given (A.4) and of  $\phi \leq P_t$  given (A.5). Almost exactly as in Strulik and Weisdorf (2008),  $n_t$  changes positively with  $X_{ft}$  and negatively with  $N_t$  and  $\bar{X}_t$ . These follow from the equilibrium value of the price  $P_t$  of food that responds

negatively to agricultural productivity  $X_{ft}$  and positively to adult population  $N_t$  and to manufacturing productivity  $\bar{X}_t$  (see the proof of Proposition 1). Also as in [Strulik and Weisdorf \(2008\)](#),  $n_t$  converges to unity when  $N_t$  and  $\bar{X}_t$  get sufficiently high relative to  $X_{ft}$  given  $(\eta, \phi)$ . Differently than [Strulik and Weisdorf \(2008\)](#), however, the regime of  $n_t = 1$  prevails mainly because of the reproductive success constraint  $n_t \geq 1$  that forces adults to have the minimum number of surviving children for the maximization of lifetime utility.

The labor share  $H_{ft}/N_t$  of the agricultural sector is a decreasing function of  $W_t$  (and hence of  $\bar{X}_t$ ) for the regime of  $n_t > 1$ . This is the pull effect of the manufacturing sector. In the regime of  $n_t = 1$ , the role of  $W_t$  vanishes completely, and the labor share of agriculture depends negatively on  $X_{ft}$  since higher productivity releases labor out of agriculture and positively on  $N_t (= N_{t+1} = \bar{N})$  because food remains essential for reproduction where  $\bar{N}$  is some fixed level of adult population.

For completeness, it is necessary to state the conditions required for both sectors to operate in equilibrium, i.e.,  $H_{ft}/N_t < 1$ . In the regime of  $n_t > 1$ ,  $\phi < (1 - \lambda)^{1-\lambda} \lambda^\lambda \bar{X}_0^\lambda$  for  $t = 0$  is sufficient to imply  $\phi/W_t < 1$  for all  $t$  since  $\delta(a_t, K_t)$  is bounded below by unity and  $\bar{X}_t$  is non-decreasing. In the regime of  $n_t = 1$  where  $H_{ft}/N_t = (X_{ft}/N_t)^{-(1-\eta)/\eta}$ , the sufficient condition simply reads  $X_{ft} > N_t (= N_{t+1} = \bar{N})$  since  $X_{ft}$  increases without bound given (5).

## 4.2 Dynamic General Equilibrium (DGE)

To define a DGE, how the vector  $(N_t, \bar{X}_t, X_{ft}, K_t)$  of state variables evolves from  $t$  to  $t + 1$  should be specified. The laws of motion for  $N_t$ ,  $X_{ft}$ , and  $K_t$  are respectively (1), (5), and (13). To derive the law of motion for  $\bar{X}_t$ , iterate (7) to obtain

$$\bar{X}_{t+1} = E_t^{-1} \int_0^{E_t} X_{it} di.$$

Substituting  $X_{it}$  with  $\sigma^{z_{it}} \bar{X}_t$  and noting, once again, that the *ex post* fraction of entrepreneurs with  $z$  inventions is equal to the *ex ante* probability of generating  $z$  inventions result in

$$\bar{X}_{t+1} = \bar{X}_t \sum_{z=0}^{+\infty} \left( \frac{a_t^z \exp(-a_t)}{z!} \right) \sigma^z.$$

This law of motion reduces into the following after some arrangements as in the proof of Lemma 1:

$$\bar{X}_{t+1} = \bar{X}_t \exp [(\sigma - 1) a_t]. \quad (37)$$

Thus, as in [Aghion and Howitt \(1992\)](#) and others, the growth rate of (average) productivity in the innovating sector is explained by the step-size  $\sigma$  and the arrival rate  $a_t$  of inventions.

**Table 1:** SGE Regimes

Regime	Conditions	Invention	Fertility
1	$K_t \leq \hat{K} \ \& \ v_t \leq 1/x_{ft}$	$a_t = 0$	$n_t = 1$
2	$K_t \leq \hat{K} \ \& \ v_t > 1/x_{ft}$	$a_t = 0$	$n_t > 1$
3	$K_t > \hat{K} \ \& \ v_t > 1/x_{ft}$	$a_t > 0$	$n_t > 1$
4	$K_t > \hat{K} \ \& \ v_t \leq 1/x_{ft}$	$a_t > 0$	$n_t = 1$

**Definition 2.** Given the vector  $(N_0, \bar{X}_0, X_{f0}, K_0) \in \mathbb{R}_{++}^4$  of initial values, a DGE is a sequence of SGE, for the entire history from  $t = 0$  to  $t \rightarrow +\infty$ , together with the sequences  $\{N_t, \bar{X}_t, X_{ft}, K_t\}_{t=1}^{+\infty}$ , that satisfies the laws of motion (1), (5), (13), and (37).

**Proposition 2.** There exists a unique DGE.

### 4.3 Global Dynamics and the Asymptotic Equilibrium

Since the model economy's dynamical system cannot be transformed into an autonomous dynamical system of (normalized) state variables, the analysis of the asymptotic equilibrium builds upon a conditional dynamical system as in Galor and Weil (2000) and others. This subsection constructs this conditional dynamical system, defines an asymptotic equilibrium of the model economy, and shows that the unique asymptotic equilibrium is globally stable.

To ease the exposition in what follows, define two endogenous state variables,  $x_{ft}$  and  $v_t$ , as in

$$x_{ft} \equiv \frac{X_{ft}}{N_t} \quad \text{and} \quad v_t \equiv \left( \frac{\phi}{W_t} \right)^{\frac{\eta}{1-\eta}},$$

and let  $G_{Wt} \equiv W_{t+1}/W_t \geq 1$  denote the gross growth rate of  $W_t$ . Since  $\bar{X}_{ft}$  is the prime determinant of the growth of living standards in this economy in the very long run, the rest of the analysis focuses on  $G_{Wt}$  as the main indicator of economic growth.<sup>9</sup>

<sup>9</sup> Real GDP per worker, with  $t = 0$  being the base period, is defined as in

$$y_t \equiv N_t^{-1} \left( P_0 Y_{ft} + \int_0^{E_t} y_{it} di \right)$$

where the integral term denotes the total volume of output in the manufacturing sector. Because (i) this total volume is proportional to  $E_t$  given the cross-section Poisson distribution and (ii)  $Y_{ft}/N_t$  declines in the long run as we see below, the secular growth of  $y_t$  originates from the growth of  $\bar{X}_t$ .

With the new notation introduced, (35) now implies

$$n_t = \begin{cases} (x_{ft}v_t)^{1-\eta} & \text{if } (x_{ft}v_t)^{1-\eta} > 1 \\ 1 & \text{otherwise} \end{cases} \quad (38)$$

given  $(x_{ft}, v_t)$ , and Table 1 summarizes the equilibrium regimes the unique SGE at  $t$  might feature given  $(x_{ft}, v_t, K_t)$ .

**Lemma 2.**  $G_{W_t}$  is increasing in  $t$  for  $a_t > 0$  with

$$\lim_{t \rightarrow +\infty} G_{W_t} = G_W^* \equiv \exp \left[ \lambda(\sigma - 1) \left( \theta - (\sigma^{\frac{\lambda}{1-\lambda}} - 1)^{-1} \right) \right] > 1. \quad (39)$$

**Lemma 3.** Given  $G_{W_t} \geq 1$ , there exists a conditional dynamical system of  $(x_{ft}, v_t)$  that satisfies

$$\frac{x_{ft+1}}{x_{ft}} = \begin{cases} \frac{1}{x_{ft}^{1-\eta} v_t^{1-\eta}} + \frac{\psi}{x_{ft}} & \text{if } n_t > 1 \\ 1 + \frac{\psi}{x_{ft}} & \text{if } n_t = 1 \end{cases}, \text{ and} \quad (40)$$

$$\frac{v_{t+1}}{v_t} = G_{W_t}^{-\frac{\eta}{1-\eta}}. \quad (41)$$

Figure 2 pictures the global dynamics of  $x_{ft}$  and  $v_t$  on  $(x_{ft}, v_t)$  plane for the cases of  $G_{W_t} = 1$  and  $G_{W_t} > 1$ . For  $G_{W_t} = 1$ , we simply have  $v_{t+1} = v_t$  for all  $t$ , and, for  $G_{W_t} > 1$ , Lemma 2 implies that  $v_t$  is decreasing towards zero as  $W_t$  grows without bound.

Define, now, the  $N^*$  locus using (38) as

$$N^* \equiv \left\{ (x_{ft}, v_t) : n_t = (x_{ft}v_t)^{1-\eta} = 1 \text{ or, equivalently, } v_t = \frac{1}{x_{ft}} \right\}$$

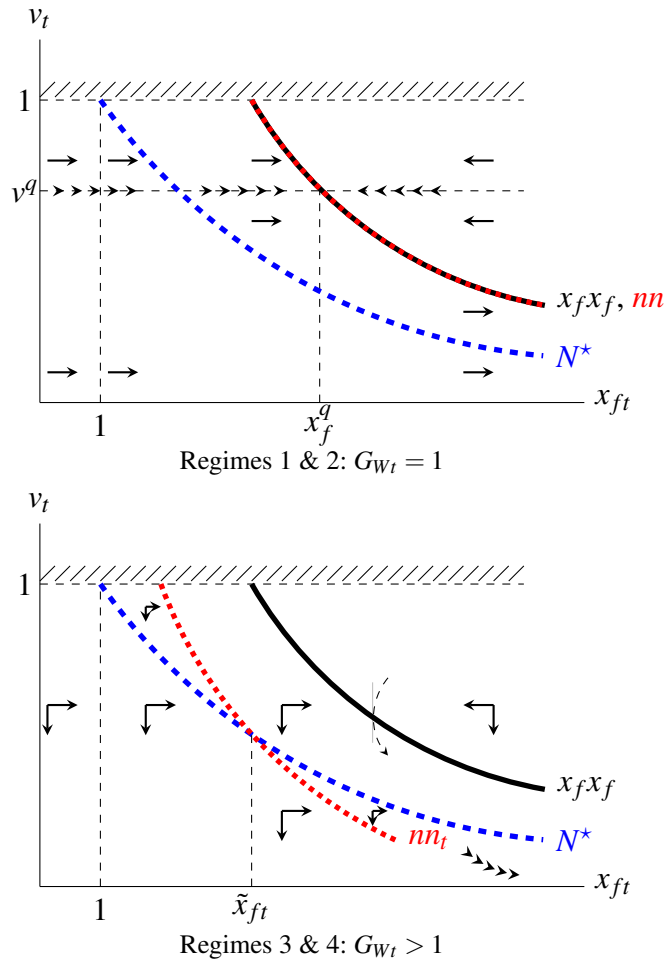
where we have  $n_t > 1$  above and  $n_t = 1$  below and over the  $N^*$  locus. Thus, the  $N^*$  locus divides the plane into Regimes 1 and 2 for  $G_{W_t} = 1$  and into Regimes 3 and 4 for  $G_{W_t} > 1$ .

Next, the  $x_f x_f$  locus, for the case of  $n_t > 1$ , is defined as in

$$x_f x_f \equiv \left\{ (x_{ft}, v_t) : x_{ft+1} = x_{ft} \text{ or, equivalently, } v_t = \left( \frac{x_{ft}^\eta}{x_{ft} - \psi} \right)^{\frac{1}{1-\eta}} \right\}$$

given (40), where  $x_{ft}$  is decreasing above and increasing below the  $x_f x_f$  locus. For the case of  $n_t = 1$ , on the other hand, there does not exist a pair  $(x_{ft}, v_t)$  that implies  $x_{ft+1} = x_{ft}$ , and  $x_{ft}$  is increasing for any  $(x_{ft}, v_t)$ .





**Figure 2:**  $(x_{ft}, v_t)$  dynamics for  $G_{Wt} = 1$  (top) and  $G_{Wt} > 1$  (bottom)

Finally, using (38), (40) and (41), the  $nn$  locus—below which  $n_t$  is increasing and above which  $n_t$  is decreasing—is defined as in

$$nn \equiv \left\{ (x_{ft}, v_t) : n_{t+1} = n_t \text{ or, equivalently, } v_t = \left( \frac{x_{ft}^\eta}{G_{Wt}^{\frac{\eta}{1-\eta}} x_{ft} - \psi} \right)^{\frac{1}{1-\eta}} \right\}$$

and overlaps with the  $x_f x_f$  locus for  $G_{Wt} = 1$ . For  $G_{Wt} > 1$ , Lemma 2 implies that the  $nn$  (or  $nn_t$ ) locus moves towards the origin as  $G_{Wt}$  increases.

Notice that the  $N^*$  locus resides below the  $x_f x_f$  locus for any  $\psi > 0$ ; these two loci do not intersect for  $x_{ft} > 1$  since  $\psi > 0$ . Also notice that the  $x_f x_f$  and the  $nn$  loci either do not intersect (for the case of  $G_{Wt} > 1$ ) or overlap for any  $(x_{ft}, v_t)$  (for the case of  $G_{Wt} = 1$ ). On the other hand, the  $N^*$  locus and the  $nn$  locus intersect at

a unique point  $\tilde{x}_{ft}$  defined as in

$$\tilde{x}_{ft} = \frac{\Psi}{G_{Wt}^{\frac{\eta}{1-\eta}} - 1} \quad (42)$$

given  $G_{Wt} > 1$ , and the  $nn$  locus resides below the  $N^*$  locus for any  $x_{ft} > \tilde{x}_{ft}$ .

**Proposition 3.** *Given any initial value  $\bar{X}_0$  of  $\bar{X}_t$ , there exists a unique quasi-steady-state equilibrium  $(x_f^q, v^q)$  in Regime 2 such that*

$$v^q = v_0 = \left[ \frac{\phi}{(1-\lambda)^{1-\lambda} \lambda \bar{X}_0^\lambda} \right]^{\frac{\eta}{1-\eta}} \quad \text{and} \quad (43)$$

$$x_f^q = \arg \text{solve}_x \left\{ v^q - \left( \frac{x^\eta}{x - \Psi} \right)^{\frac{1}{1-\eta}} = 0 \right\}. \quad (44)$$

The quasi-steady-state  $(x_f^q, v^q)$  in Regime 2 is an equilibrium state to which the economy may converge in finite time as long as the unique SGE features  $a_t = 0$ . This quasi-steady-state is a balanced growth equilibrium where adult population  $N_t$  and agricultural productivity  $X_{ft}$  grow at the same gross growth rate

$$n_t = n^q = \left( x_f^q v^q \right)^{1-\eta} > 1,$$

and the growth of  $X_{ft}$  allows the economy to sustain a growing  $N_t$  with a constant price  $P_t$  of the food.

**Proposition 4.** *Suppose that  $K_0 \leq \hat{K}$ , i.e., the economy is either in Regime 1 or in Regime 2 initially. Then, there exists a period  $\hat{t} > 0$  such that  $a_{\hat{t}-1} = 0$  and  $a_{\hat{t}} > 0$ : If the economy starts its evolution in a period at which invention is not optimal, an industrial revolution inevitably starts at some future period.*

That  $K_t$  is increasing for all  $t$  (once it exceeds its threshold of  $\hat{K}$ ) and that  $W_t$  is increasing for all  $t$  after the industrial revolution motivate the following definition of the asymptotic equilibrium:

**Definition 3.** *An asymptotic equilibrium of the model economy, for  $t \rightarrow +\infty$ , is the (unique) limiting SGE of the model economy in Regime 4 with*

$$n_t = n^* = 1 \quad \text{for } t \geq t^* \quad \text{and} \quad a_t \rightarrow a^* = \theta - \left( \sigma^{\frac{\lambda}{1-\lambda}} - 1 \right)^{-1} > 0.$$

At this asymptotic equilibrium, the labor share of the agricultural sector declines towards zero for  $t \rightarrow +\infty$  as implied by (36) and (40), and adult population  $N_t$  is stabilized at some  $N^* > 0$  for  $t \geq t^* + 1$  given  $n_{t^*} = 1$ . As the growth of  $K_t$  implies  $f(K_t) \rightarrow 1$  through (12),  $a_t$  converges to  $a^*$ . Thus, the asymptotic equilibrium is the one with the perpetual growth of  $\bar{X}_t$ —and, therefore, of  $W_t$  and  $y_t$ .

**Proposition 5.** *The unique asymptotic equilibrium of the model economy is (asymptotically) globally stable. That is, for any  $(N_0, \bar{X}_0, X_{f0}, K_0) \in \mathbb{R}_{++}^4$ , the model economy's SGE converges to the asymptotic equilibrium for sufficiently large  $t$ .*

## 5 The Equilibrium Path from Stagnation to Growth

While the economy's unique asymptotic equilibrium is globally stable, this does not guarantee that the model economy's transition from stagnation to growth exhibits a demographic transition. An industrial revolution inevitably starts at some finite  $t$  as stated in Proposition 4, but the economy may in general transit directly from Regime 1 to Regime 4 with  $n_t = 1$  for all  $t$ . However, if the model economy starts its evolution at  $t = 0$  in Regime 1 and if  $\hat{t}$  is sufficiently large, then the model economy's unique DGE features an equilibrium path from stagnation to growth with an industrial revolution and a demographic transition. Specifically, under these two assumptions, the model economy exhibits regime transitions in the order of their numbering.

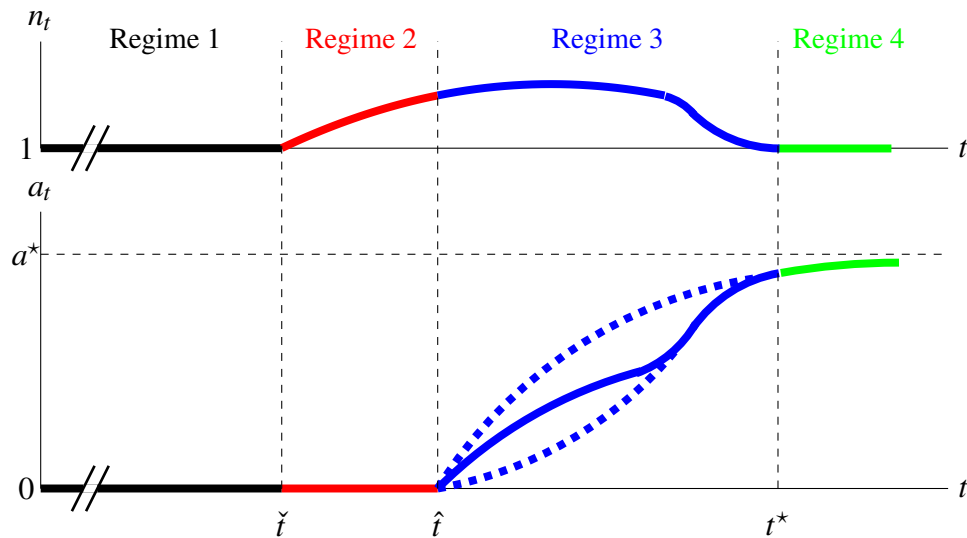
**Proposition 6.** *Suppose that (i)  $v_0 \leq 1/x_{f0}$  and  $K_0 \leq \hat{K}$ , i.e., the economy is in Regime 1 initially, and that (ii)  $\hat{t}$  is sufficiently large, i.e., the industrial revolution is to start sufficiently late. Then, the economy enters Regime 2 at some  $\check{t} > 0$ , then enters Regime 3 at some  $\hat{t} > \check{t}$ , and eventually enters Regime 4 at some  $t^* > \hat{t}$ , where  $\check{t}$ ,  $\hat{t}$ , and  $t^*$  are all endogenously determined.*

**Remark 1.** *Both assumptions of Proposition 6 are in complete accordance with historical evidence. The first one requires that the preindustrial economy to have a sufficiently low level  $X_{f0}$  of agricultural productivity, given  $N_0$  and  $\bar{X}_0$ , such that historically low values of  $N_0$  and  $\bar{X}_0$  imply a low level of upper bound for  $X_{f0}$  for Regime 1 to prevail, i.e.,*

$$v_0 \leq \frac{1}{x_{f0}} \Leftrightarrow X_{f0} \leq N_0 \left( \frac{(1-\lambda)^{1-\lambda} \lambda^\lambda \bar{X}_0^\lambda}{\phi} \right)^{\frac{\eta}{1-\eta}}.$$

*The economy without a sufficiently advanced agricultural sector, then, cannot support fertility above replacement. Moreover, since  $x_{f0}$  is itself sufficiently low, the labor share of the agricultural sector is at its historical maximum at  $t = 0$ .*

*The second assumption builds upon the fact that the first Industrial Revolution in Britain started around 50,000 years later than the rise of modern human populations, i.e., populations that share cultural universals such as language, art, religion, and toolmaking. The takeoff to modern growth through an industrial revolution is a recent phenomenon if one takes a very long-run perspective.*



**Figure 3:** The Equilibrium Path from Stagnation to Growth

*Note:* This figure pictures the equilibrium path under the assumption that the industrial revolution starts before the economy converges to the quasi-steady-state equilibrium. The dashed lines of the bottom panel indicate alternative paths of  $a_t$  in Regime 3 depending on the alternative parameterizations of  $f(K_t)$ .

Note that the convergence to Regime 4 through Regimes 2 and 3 can be of two types. If the growth rate of  $x_{ft}$  in Regime 2 is large enough given the growth rate of  $K_t$ , then the economy enters Regime 3 after completing the convergence to the quasi-steady-state in Regime 2. In this scenario, fertility remains stable at its historical maximum some periods before the industrial revolution. This shape of the equilibrium path is arguably consistent with the 20th century experience of today's least developed economies that remain agrarian but sustain an increasing level of population with a high and a stable level of fertility.

The second possibility is the one where the growth rate of  $x_{ft}$  in Regime 2 is not large enough given the growth rate of  $K_t$ . Then, the industrial revolution starts before the economy converges to its quasi-steady-state, and the fertility decline therefore starts at a level of fertility that is less than its quasi-steady-state level. This scenario is largely consistent with the unified growth experience of early industrialized countries such as England and France as we do not observe long episodes of population growth remaining stable at a high level for these countries before the growth takeoffs.

Figure 3 pictures the economy's transition from Malthusian stagnation to modern growth for the case of the industrial revolution starting before the convergence to the quasi-steady-state is completed. A narrative of this transition is now in order:

At the very beginning of history, productivity measures in both sectors and population are at historically lowest levels, and the labor share of the agricultural sector is at its historical maximum to sustain a stable population level under poverty. Since knowledge about natural phenomena underlying the manufacturing technology—in the form of useful knowledge—is limited, entrepreneurs allocate their entire labor endowment to routine management. While this implies stagnating productivity in manufacturing, agricultural productivity grows in time through learning-by-doing and results in a declining labor share of agriculture.

This expansion eventually allows the economy to sustain a level of fertility above replacement, and population starts growing. The ongoing growth of agricultural productivity implies an increasing rate of population growth in this regime. The economy starts converging to a quasi-steady-state at which growing population is sustained only by growing agricultural productivity, and the labor share of the agricultural sector is now constant because of stagnating productivity in the manufacturing sector.

An industrial revolution starts since a sufficiently larger stock of useful knowledge is now available for entrepreneurs to optimally allocate some of their scarce time endowment to inventive activities. Increasing with a declining labor share of agriculture in Regime 1 and with an expanding population in Regime 2, the growth of the stock of useful knowledge is now increasing with both factors, and more time allocated to invention by each generation means faster growth of manufacturing productivity. Besides, an increasing share of entrepreneurs' time allocated to invention implies a more skewed cross-section distribution of productivity and firm size in the manufacturing sector. Some time after the industrial revolution starts, population and manufacturing productivity get sufficiently high to imply a sufficiently high price for the food and, hence, to imply a declining population growth. In the meantime, since the declining labor share of agriculture feeds back to the growth of the stock of useful knowledge and since population gets even larger in time, the growth of manufacturing productivity may accelerate at some period before it decelerates and converges to its asymptotic equilibrium.

When it is too costly to have more than one child, the economy enters the final stage of its demographic transition. Population level and fertility are stabilized while the decline of the agricultural sector continues. Since the total mass of entrepreneurs is also stabilized, the growth rate of the stock of useful knowledge starts decreasing. At the very end of history, humanity has access to every bit of knowledge about natural phenomena underlying the manufacturing technology and keeps investing into new technology for higher prosperity.

To conclude, Proposition 6 shows that the unified growth model of this paper, *if located at the correct initial position in historical time*, successfully replicates the equilibrium path of an economy, say, England, in the very long run. After suffering for several thousand years in a Malthusian trap, population in England starts increasing in mid-1600s, at the period  $\check{t}$ , and the first Industrial Revolution starts at around 1750, at the period  $\hat{t}$ . According to the most recent population projections of the [United Nations \(2012\)](#), population in the United Kingdom will stabilize at around 77 million at the end of this century, at the period  $t^* + 1$ .

## 6 Discussion

Entrepreneurs of a special generation find it optimal to direct resources into risky inventive activities unlike those of past generations. These entrepreneurs are special because the stock of useful knowledge they have access to is large enough to signal a higher expected level of profit for them if they are to decrease the time they spend to routine management. They benefit from standing on the shoulders of dead entrepreneurs who collectively created all of this knowledge in a serendipitous way.

The invention threshold in the model leads to a kinked time-series of labor productivity in manufacturing, and this in turn implies a kinked time-series of the real wage that exhibits exponential growth starting with the industrial revolution. The Industrial Revolution in history is matched by an invention revolution in the model. After this invention revolution, exerting inventive effort to appropriate an increasing profit remains optimal throughout the history.

### 6.1 The Industrial Revolution: Break or Continuity?

Whether the first Industrial Revolution, roughly covering the period from 1760 to 1830, is a break from the past or a continuity remained controversial among some economic historians. The *gradualist* view of [Crafts and Harley \(1992\)](#) suggests that there was little economic growth in England until the early 19th century in per capita terms and that the scope of fast technological progress was limited with the textile sector before the diffusion of the steam technology. However, an industrial revolution as a structural break characterized by very slow growth in per capita terms is not controversial. As [Pereira \(2004\)](#) documents, several variables of interest—including total industrial output and population—exhibit endogenously determined upward trend breaks during the first Industrial Revolution in Britain, and [Mokyr \(2004\)](#) and others suggest that what kept output per capita at a very low level during the first Industrial Revolution was indeed the fast expansion of English population.

The model of this paper, as a unified model, captures exactly this type of dynamics between population and technology. The model predicts that an industrial revolution may start when population growth is accelerating and that the productivity gains due to inventions would be modest during these early stages. These naturally imply that the acceleration of the rate of economic growth in per capita terms would be slow during the first Industrial Revolution.

## 6.2 The Timing of the Industrial Revolution

Desmet and Parente (2012) and Peretto (2013) ask a specific *timing question*: What factors do affect/explain the period at which the industrial revolution starts? This timing question is basically asking for a solution of  $\hat{t}$ —the period at which the industrial revolution starts—in terms of the model’s exogenous givens—parameter values and the initial values of endogenous state variables.

The model of this paper is not simple enough to allow for a closed-form solution of  $\hat{t}$ . However, some concrete answers to the timing question can be obtained by inspecting the economy’s evolution in Regimes 1 and 2.

The timing effects are of two types: the *threshold effects* where the fixed threshold value  $\hat{K} > 0$  of  $K_t$  determines *how far away the industrial revolution is*, and the *growth effects* where the gross growth rate  $G_{K_t} \equiv K_{t+1}/K_t$  of  $K_t$  determines *how fast the economy moves towards its invention threshold*.

Since  $\hat{K}$ , defined in (31), is determined by the exogenous component  $\theta > 0$  of research productivity, the step-size  $\sigma > 1$  of inventions, and the labor exponent  $\lambda \in (0, 1)$  of manufacturing production such that

$$\frac{\partial \hat{K}}{\partial \theta} < 0, \quad \frac{\partial \hat{K}}{\partial \sigma} < 0, \quad \text{and} \quad \frac{\partial \hat{K}}{\partial \lambda} < 0,$$

the higher values of these parameters imply a lower value of  $\hat{t}$ . Clearly, a higher initial value  $K_0$  of  $K_t$  also implies a lower value of  $\hat{t}$ .

More complicated is the analysis of growth effects where (13), (34), and (36) imply  $G_{K_t}$  in Regimes 1 and 2 as in

$$\begin{aligned} G_{K_t}^{\text{Regime 1}} &= 1 + \omega(1 - \lambda) \left( 1 - x_{ft}^{-\frac{1-\eta}{\eta}} \right) \left( \frac{N_0}{K_t} \right) \\ G_{K_t}^{\text{Regime 2}} &= 1 + \omega(1 - \lambda) \left( 1 - \frac{\phi}{\lambda^\lambda (1 - \lambda)^{1-\lambda} \bar{X}_0^\lambda} \right) \left( \frac{N_t}{K_t} \right) \end{aligned}$$

Notice that  $\omega > 0$ , representing the quality of the process of collective discovery, unambiguously increases  $G_{K_t}$ —for any  $t$  and in both regimes—and, hence, unambiguously decreases  $\hat{t}$ . A higher initial value  $N_0$  of adult population also



implies a higher value for  $G_{Kt}$  unambiguously in both regimes. For  $\lambda$ , the growth effect is ambiguous because we have  $\partial G_{Kt}^{\text{Regime 2}}/\partial \lambda \geq 0$ .

A faster growth of  $x_{ft}$  with a higher value of  $\psi > 0$  increases the share of entrepreneurs in adult population and implies a higher level of  $G_{Kt}^{\text{Regime 1}}$ . Besides, since faster growth of  $x_{ft}$  increases  $n_t$  without affecting the labor share of the agricultural sector in Regime 2, we have  $\partial G_{Kt}^{\text{Regime 2}}/\partial \psi > 0$ . Thus, a faster growth of agricultural productivity unambiguously hastens the industrial revolution.

The preference parameter  $\phi$  and the initial value  $\bar{X}_0$  of manufacturing productivity affect both  $n_t$  and  $H_{ft}/N_t$  in Regime 2 and have ambiguous growth effects. In Regime 2, we have

$$\begin{aligned} n_t^{\text{Regime 2}} &= \left( \frac{\phi}{\lambda^\lambda (1-\lambda)^{1-\lambda} \bar{X}_0^\lambda} \right)^\eta x_{ft}^{1-\eta} \quad \text{and} \\ \left( \frac{H_{ft}}{N_t} \right)^{\text{Regime 2}} &= \frac{\phi}{\lambda^\lambda (1-\lambda)^{1-\lambda} \bar{X}_0^\lambda}. \end{aligned}$$

A higher level of  $\phi$  and a lower level of  $\bar{X}_0$  increase the labor share of agriculture—for the economy needs to sustain faster population growth with a higher level of agricultural production—and, hence, decrease the share of entrepreneurs contributing to collective discovery:  $G_{Kt}^{\text{Regime 2}}$  is decreasing with  $\phi$  and increasing with  $\bar{X}_0$ . But a higher level of  $\phi$  and a lower level of  $\bar{X}_0$  also imply a larger population  $N_t$  for all  $t$  in Regime 2 through a higher level of  $n_t$ , and this leads to a higher value of  $G_{Kt}^{\text{Regime 2}}$ . In other words, the economy faces a “trade-off” for the timing of the industrial revolution as faster population growth, through higher  $\phi$  or lower  $\bar{X}_0$ , implies both a lower labor share of the manufacturing sector and a larger mass of adult population.

These results on threshold and growth effects perfectly overlap with the results obtained by [Desmet and Parente \(2012\)](#) and [Peretto \(2013\)](#). In [Desmet and Parente’s \(2012\)](#) model, the parameters that are most closely related with institutional quality and policies are the parameters that represent the cost of innovative activities by firms. Higher values of these parameters imply a delayed industrial revolution similarly to the ambiguous hastening effect of higher  $\theta$ . [Desmet and Parente \(2012\)](#) also document the hastening timing effect of improving infrastructure represented here by a higher value of  $\omega$  that ambiguously increases  $G_{Kt}$  in both regimes. Regarding the (exogenous and fixed) growth of agricultural productivity before the industrial revolution in their model, [Desmet and Parente \(2012\)](#) find an unambiguous effect where a faster growth of agricultural productivity hastens the industrial revolution. [Peretto’s \(2013, Prop.s 6-7\)](#) analytical results on the timing of the industrial revolution also indicate that population growth has an ambiguous effect under all scenarios, a higher initial level of population implies a sooner

industrial revolution, and a higher level of fixed operating cost of firms delays the takeoff.

### 6.3 England vs. China

While there are difficulties in comparing preindustrial economies of different geographical sizes, as Pomeranz (2000) emphasizes, the model of this paper has interesting implications for the question of “Why England, but not China?”

As noted earlier,  $\omega$  as a structural parameter represents the quality of the environment in which entrepreneurs create and share useful knowledge. England here had the advantage of being a small country in terms of its geographical size. Also advantages of England, as noted by Mokyr (2002) and others, were (i) the gentlemanly behavior and the technological motivation of business owners and (ii) the efficiency of social networks and informal institutions.

Next, if we assume that  $\lambda$ , as a technological parameter, is not radically different for England and China, the prime determinant of the share of entrepreneurs in adult population would be the labor share of the agricultural sector. The limited data here indicates that England in preindustrial times had a higher rate of urbanization than China (Voigtländer and Voth, 2006).

In general, any rival use of time endowment is important in determining the supply, or the *lack*, of entrepreneurship, and the labor shares of occupations that do not contribute to collective discovery would have delaying growth effects for the timing of the industrial revolution. One such occupation regarding which England had arguably an advantage compared to China is state bureaucracy. A larger state bureaucracy would imply a lower level of  $G_{Kt}$ , *ceteris paribus*, because the mass of entrepreneurs who collectively discover would be smaller. England might indeed have benefited from avoiding a large professional bureaucracy before the Industrial Revolution, as noted by Mokyr (1998), and China’s potential might indeed have been restricted by its large and ineffective bureaucracy, as emphasized by Landes (2006). In the terminology of Rodríguez-Pose (1999), England in mid-1700s was an *innovation prone* society whereas China was an *innovation averse* one.

### 6.4 Serendipitous Inventions

That the rate of technological progress in manufacturing *before* the Industrial Revolution is zero is counter-factual to what we observe in the data: The real wage series in England has an increasing trend after mid-1600s as Clark (2010) documents, and a minuscule rate of growth in the real wage before the Industrial Revolution is also consistent with the patent data of Sullivan (1989) pictured in Figure 1. A question of interest is thus whether the model can be extended to account for such haphazard type of technological progress.

The simplest extension along this line of thought is to allow for *serendipitous inventions* to exogenously increase the baseline productivity  $\bar{X}_t$ . Serendipitous inventions can be thought of as resulting exogenously without altering the optimal behavior of entrepreneurs regarding the inventive activity. The law of motion for  $\bar{X}_t$  can simply be extended to include serendipitous inventions as in

$$\bar{X}_{t+1} = \bar{X}_t \exp[(\sigma - 1)(a_t + a_s)]$$

where  $a_s > 0$  represents the arrival rate of serendipitous inventions. Clearly, whenever  $a_t = 0$ , the gross growth rate of  $\bar{X}_t$  reduces into  $\exp[(\sigma - 1)a_s]$ .

### 6.5 Adult Longevity

The simplest way of capturing the role of adult longevity within this framework is to assume, as in [Hazan and Zoabi \(2006\)](#), that all period- $t$  adults live a *fraction*  $\ell_t \in [0, 1]$  of period  $t$ . For simplicity,  $\ell_t$  is exogenous, common across period- $t$  adults, and known by period- $t$  adults with certainty.

This extension generalizes the main results in two respects: Firstly, the threshold value  $\hat{K}$  now depends on  $\ell_t$  because a longer life, as an endowment, implies a higher level of profit for entrepreneurs, and (31) reads

$$\hat{K}_t \equiv f^{-1} \left[ \theta^{-1} \left( \sigma^{\frac{\lambda}{1-\lambda}} - 1 \right)^{-1} \ell_t^{-1} \right] > 0$$

where we have  $\partial \hat{K}_t / \partial \ell_t < 0$ . Next, since the total lifetime of entrepreneurs now reads  $E_t \ell_t$ , the process of collective discovery in (13) is generalized to

$$K_{t+1} = K_t + \omega E_t \ell_t$$

with  $\partial G_{K_t} / \partial \ell_t > 0$ . Since a higher level of  $\ell_t$  decreases  $\hat{K}_t$  and increases  $G_{K_t}$ , adult longevity unambiguously hastens the industrial revolution by implying a lower value of  $\hat{t}$ .

### 6.6 Mortality Shocks and the Loss of Useful Knowledge

While the law of motion (13) implies  $K_{t+1} > K_t$  for any  $t$ , mortality shocks in an extended model would lead to the loss of useful knowledge before the industrial revolution. Since knowledge resides at least partially in the minds of people, a mortality shock such as the Black Death affects the growth of  $K_t$  as in [Bar and Leukhina \(2010\)](#).<sup>10</sup>

<sup>10</sup> Strictly speaking, the type of knowledge that is lost due to mortality shocks in [Bar and Leukhina \(2010\)](#) is prescriptive knowledge in the terminology of [Mokyr \(2002\)](#) and corresponds to  $\bar{X}_t$  of the model of this paper.

Suppose, then, that  $m_t$  is some random measure of mortality for adult individuals, presumably with a deterministic component, such that

- a fraction  $1 - s(m_t) \in [0, 1]$  of entrepreneurs  $E_t$  dies at some interim point between  $t$  and  $t + 1$  (with  $s' < 0$ ) before participating to collective discovery but after completing reproduction, and that
- a fraction  $d(m_t) \in [0, 1]$  of useful knowledge  $K_t$  depreciates, again, at some interim point between  $t$  and  $t + 1$  (with  $d' > 0$ ).

$G_{K_t}$  in Regimes 1 and 2 then reads

$$G_{K_t} = 1 + \omega(1 - \lambda) \left(1 - \frac{H_{ft}}{N_t}\right) \left(\frac{N_t}{K_t}\right) s(m_t) - d(m_t)$$

where  $G_{K_t} > 1$  requires

$$\frac{d(m_t)}{s(m_t)} < \omega(1 - \lambda) \left(1 - \frac{H_{ft}}{N_t}\right) \left(\frac{N_t}{K_t}\right).$$

The growth of  $K_t$  in some long run before the Industrial Revolution thus necessitates that the deterministic component of  $m_t$  decreases to lower  $d(m_t)/s(m_t)$  in time as in the case of England. In general, one can conclude that sizable mortality shocks such as the Black Death unambiguously delay the industrial revolution because of the loss of useful knowledge.

## 6.7 The Fishing-Out Effect

Productivity growth in the manufacturing sector is not subject to the so-called *fishing-out* effect: The (unit) productivity of labor directed to inventive activity does not decrease with the level of baseline productivity  $\bar{X}_t$ . No matter how high  $\bar{X}_t$  is, the arrival rate  $a_{it}$  of inventions is fixed for given levels of  $K_t$  and  $h_{rit}$ . The endless expansion of  $K_t$  makes inventive effort always more productive in time as postulated, but the only limit realistically imposed is on the *level* of  $f(K_t)$  when  $K_t \rightarrow +\infty$ .

The fishing-out effect can be incorporated by extending  $f(K_t)$  into  $f(K_t, \bar{X}_t)$  with

$$\frac{\partial f(K_t, \bar{X}_t)}{\partial \bar{X}_t} < 0.$$

This extension would have two implications for the model economy: Firstly, while the threshold  $\hat{K}_t$  would now be increasing in  $\bar{X}_t$ , the economy starting in Regime 1 or Regime 2 at which  $a_t = 0$  eventually enters Regime 3 with an industrial

revolution as we have  $\bar{X}_t = \bar{X}_0$  in Regimes 1 and 2. There is, therefore, *technological lock-in* as a higher level of  $\bar{X}_0 > 0$  delays the industrial revolution, but the industrial revolution is still an inevitable outcome of the process of collective discovery.

Secondly, the shape of the equilibrium path would be affected by the evolution of  $(K_t, \hat{K}_t, \bar{X}_t)$ . The growth of  $\bar{X}_t$  after the date  $\hat{t}$  of the industrial evolution may eventually imply  $K_{t_1} \leq \hat{K}_{t_1}$  at some  $t_1 > \hat{t}$  to stop the wheel of invention and fix  $\bar{X}_t$  at  $\bar{X}_{t_1}$ . Then, it would take some number of periods for the growth of  $K_t$  to catch up with now fixed  $\hat{K}_{t_1}$ , and the arrival rate  $a_t$  would become positive again at some  $t_2 > t_1$ . There might, then, exist a “punctuated” equilibrium of  $\bar{X}_t$  over alternating episodes of  $a_t = 0$  and  $a_t > 0$  depending on the parameterization of  $f(K_t, \bar{X}_t)$ . However, the shape of the equilibrium path, once smoothed, would not be dramatically different because  $a_t$  would be periodically positive, or, provided that a limit-cycle of  $a_t$  can be constructed analytically, its periodicity would be at least one.<sup>11</sup>

## 7 Concluding Remarks

The turning point of the transition from Malthusian stagnation to modern growth was the Industrial Revolution—a structural break in the sense that technological progress was no longer simply due to serendipitous inventions. Schumpeter’s (1934) entrepreneur-inventor’s, seeking increased market shares and profits, took the stage instead, and the world was not the same when the first corporate R & D lab was opened by Thomas Edison in 1876.

This paper studies a view of the Industrial Revolution that promotes the dual role of entrepreneurship for inventions and useful knowledge; the serendipitous expansion of the latter eventually leads to the purposeful activation of the former. No such thing as an industrial revolution occurred for a very long episode of history because not enough was known about natural phenomena. Yet the type of useful knowledge relevant to production processes was created by and diffused among entrepreneurs. In one sense, this had to be the case because these agents were managing the firms utilizing these production processes.

<sup>11</sup> There may exist parameterizations of  $f(K_t, \bar{X}_t)$  that lead  $\hat{K}_t$  to converge to a constant for some large  $\bar{X}_t$ . In such a case, the model tells a more involved story of technological progress in the long run: There exist episodes before the industrial revolution over which  $\bar{X}_t$  grows and then stabilizes, and the industrial revolution now starts at the period where the fishing-out effect of  $\bar{X}_t$  is no longer binding with  $K_t > \hat{K}_t$  for large enough  $t$ . Mokyr (2002, Ch. 3), in fact, suggests such an interpretation: Before the times of the Industrial Revolution, the waves of invention were subject to diminishing returns because not enough was known about the natural phenomena underlying the production techniques given the achieved sophistication represented by  $\bar{X}_t$ .

The endogenous timing of the industrial revolution is affected by several structural parameters and initial values, and this prediction of the model is in accordance with the observation that countries realize their takeoffs to modern growth regime at different dates.

Several issues are left to future research. Among these are the role of the demand-side determinants of inventive activity, the role of patents for the industrial revolution, the importance of professional scientists for the *second* Industrial Revolution, the effect of the growth of useful knowledge on the decline of mortality, and, the last but not the least, the mechanisms by which *the enlightenment of the economy through useful knowledge* affects the rises of democracy and formal education.

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## Appendix A Proofs

### Proof of Lemma 1:

(23) can be arranged as in

$$\mathbf{E}[\Pi_{it}] = \exp(-a_{it}) (1 - \lambda) \lambda^{\frac{\lambda}{1-\lambda}} \left( \frac{\bar{X}_t}{W_t} \right)^{\frac{\lambda}{1-\lambda}} \left( 1 - \frac{a_{it}}{\theta f(K_t)} \right) \sum_{z=0}^{+\infty} \left( \frac{a_{it}^z \sigma^{z(\frac{\lambda}{1-\lambda})}}{z!} \right)$$

where the summation on the right-hand side is the Taylor series expansion of  $\exp\left(\sigma^{\frac{\lambda}{1-\lambda}} a_{it}\right)$  around  $a_{it} = 0$ . Thus, we simply have

$$\mathbf{E}[\Pi_{it}] = \exp(-a_{it}) (1 - \lambda) \lambda^{\frac{\lambda}{1-\lambda}} \left( \frac{\bar{X}_t}{W_t} \right)^{\frac{\lambda}{1-\lambda}} \left( 1 - \frac{a_{it}}{\theta f(K_t)} \right) \exp\left(\sigma^{\frac{\lambda}{1-\lambda}} a_{it}\right)$$

and this can be rewritten as

$$\mathbf{E}[\Pi_{it}] = \exp\left[\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right) a_{it}\right] (1 - \lambda) \lambda^{\frac{\lambda}{1-\lambda}} \left( \frac{\bar{X}_t}{W_t} \right)^{\frac{\lambda}{1-\lambda}} \left( 1 - \frac{a_{it}}{\theta f(K_t)} \right)$$

Defining  $\Gamma \equiv \frac{\lambda}{1-\lambda}$ ,  $\Lambda \equiv (1 - \lambda) \lambda^{\frac{\lambda}{1-\lambda}}$ , and  $\Sigma \equiv \sigma^{\frac{\lambda}{1-\lambda}} - 1$  completes the proof. *Q.E.D.*

### Proof of Proposition 1:

Note that the unique solutions of (17) and (25) satisfy

$$n_{wt} = n_{it} = n_t = \begin{cases} \frac{\phi}{P_t} & \text{if } \phi > P_t \\ 1 & \text{otherwise} \end{cases} \quad (\text{A.1})$$

and

$$a_{it} = a_t = \begin{cases} 0 & \text{if } f(K_t) \leq \left[\theta \left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)\right]^{-1} \\ \theta f(K_t) - \left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1} & \text{otherwise} \end{cases} \quad (\text{A.2})$$

where  $P_t$  depends on  $(N_t, \bar{X}_t, X_{ft})$ . At these solutions, (26) reduces into  $W_t = \mathbf{E}[\Pi_{it}]$  to yield

$$W_t = \exp(\Sigma a_t) \Lambda \left( \frac{\bar{X}_t}{W_t} \right)^\Gamma \left( 1 - \frac{a_t}{\theta f(K_t)} \right)$$

which implies

$$W_t = \delta(a_t, K_t) (1 - \lambda)^{1-\lambda} \lambda^\lambda \bar{X}_t^\lambda \quad (\text{A.3})$$

where  $\delta(a_t, K_t)$  is an auxiliary function defined as in

$$\delta(a_t, K_t) \equiv \exp\left[(1 - \lambda) \left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right) a_t\right] \left( 1 - \frac{a_t}{\theta f(K_t)} \right)^{1-\lambda}.$$

Since  $a_t \geq 0$  is a function of  $K_t$  when it is strictly positive and does not depend on other endogenous state variables, (A.2) and (A.3) solve  $W_t$ .  $a_t$  from (A.2) also solves  $h_{rit} = h_{rt}$  via (10), and this solves  $h_{mit} = h_{mt}$  via (20). Given  $a_t \geq 0$ , the realization of  $z_{it}$  for entrepreneur  $i$  follows from (9), and (8) solves  $X_{it}$  for entrepreneur  $i$  given  $z_{it}$  and  $\bar{X}_t$ . Given these solutions, the unique values of  $h_{wit}$ ,  $y_{it}$ , and  $\Pi_{it}$  follow respectively from (21), (6), and (22). Four equations that solve  $n_t$ ,  $P_t$ ,  $H_{ft}$  and  $Y_{ft}$  given  $W_t$  and  $N_t$  are (4) with  $L_f = 1$ , (14), (27), and (A.1) where (27) reduces into  $Y_{ft} = N_t n_t$  under  $n_{it} = n_{wt} = n_t$ . These solutions satisfy the following: For the case of

$$\phi > P_t \Leftrightarrow X_{ft} > \frac{N_t}{\left(\frac{\phi}{W_t}\right)^{\frac{\eta}{1-\eta}}}, \quad (\text{A.4})$$

we have

$$\begin{aligned} n_t &= \left(\frac{\phi}{W_t}\right)^\eta \left(\frac{X_{ft}}{N_t}\right)^{1-\eta}, \quad P_t = \frac{\phi}{\left(\frac{\phi}{W_t}\right)^\eta \left(\frac{X_{ft}}{N_t}\right)^{1-\eta}}, \quad H_{ft} = \left(\frac{\phi}{W_t}\right) N_t, \quad \text{and} \\ Y_{ft} &= X_{ft}^{1-\eta} \left(\frac{\phi}{W_t}\right)^\eta N_t^\eta. \end{aligned}$$

For the case of

$$\phi \leq P_t \Leftrightarrow X_{ft} \leq \frac{N_t}{\left(\frac{\phi}{W_t}\right)^{\frac{\eta}{1-\eta}}}, \quad (\text{A.5})$$

on the other hand, we have

$$n_t = 1, \quad P_t = W_t \left(\frac{X_{ft}}{N_t}\right)^{-\frac{1-\eta}{\eta}}, \quad H_{ft} = \left(\frac{X_{ft}}{N_t}\right)^{-\frac{1-\eta}{\eta}} N_t, \quad \text{and} \quad Y_{ft} = N_t.$$

Now, given  $n_t$ ,  $P_t$ ,  $W_t$  and  $\Pi_{it}$ , (16) and (18) respectively solve  $C_{wt}$  and  $C_{it}$ . Thus, only  $E_t$  remains to be solved.

What solves  $E_t$  is basically (29). To see this, first recall that the arrival rate  $a_t$  is common across entrepreneurs. This and the fact that invention events are independent across entrepreneurs imply, via (Borel's version of) the law of large numbers, that the ex post fraction of entrepreneurs with  $z \geq 0$  inventions for any given  $a_t$  is equal to the ex ante Poisson probability  $[a_t^z \exp(-a_t)]/z!$  of achieving  $z \geq 0$  inventions. This property allows us to write

$$\int_0^{E_t} h_{wit} di = E_t \sum_{z=0}^{+\infty} \left[ \frac{a_t^z \exp(-a_t)}{z!} \right] h_{wt}(z) \quad (\text{A.6})$$

where  $h_{wt}(z)$  reads

$$h_{wt}(z) \equiv \left(\frac{\lambda}{1-\lambda}\right) \exp\left[-\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right) a_t\right] \sigma^{\left(\frac{\lambda}{1-\lambda}\right) z}$$



as implied by (21) and the solution of  $W_t$ . Applying now the reasoning of the proof of Lemma 1 to the right-hand side of (A.6) implies

$$\int_0^{E_t} h_{wit} di = E_t \left( \frac{\lambda}{1-\lambda} \right).$$

This last equation and (29) then solve  $E_t$  as in (34). *Q.E.D.*

**Proof of Proposition 2:**

The existence and the uniqueness of period- $t$  SGE from Proposition 1 and that the laws of motion for endogenous state variables, i.e. (1), (5), (13), and (37), are all real-valued functions imply the existence and the uniqueness of the DGE for the entire history from  $t = 0$  to  $t \rightarrow +\infty$ . *Q.E.D.*

**Proof of Lemma 2:**

Starting with the claim that  $G_{W_t}$  is increasing in  $t$  for  $a_t > 0$ , (32) implies

$$G_{W_t} = \left[ \frac{\delta(a_{t+1}, K_{t+1})}{\delta(a_t, K_t)} \right] \left( \frac{\bar{X}_{t+1}}{\bar{X}_t} \right)^\lambda.$$

Since (37) with  $a_t > 0$  implies  $\bar{X}_{t+1}/\bar{X}_t > 1$ , it is sufficient to show that  $\delta(a_t, K_t)$  is increasing in  $t$ . Substituting  $a_t > 0$  from (30) in (33) implies

$$\delta_t \equiv \delta(a_t, K_t) = \left[ \frac{\exp \left[ \left( \sigma^{\frac{\lambda}{1-\lambda}} - 1 \right) \theta f(K_t) - 1 \right]}{\left( \sigma^{\frac{\lambda}{1-\lambda}} - 1 \right) \theta f(K_t)} \right]^{1-\lambda}, \quad (\text{A.7})$$

and it follows, given  $\Sigma = \sigma^{\frac{\lambda}{1-\lambda}} - 1$ , that

$$\frac{\partial \delta_t}{\partial f(K_t)} = (1-\lambda) \delta_t^{-\frac{\lambda}{1-\lambda}} \left[ \frac{\Sigma \theta \exp[\Sigma \theta f(K_t) - 1] (\Sigma \theta f(K_t) - 1)}{(\Sigma \theta f(K_t))^2} \right] > 0$$

since  $a_t > 0$  requires  $\Sigma \theta f(K_t) - 1 > 0$ . As  $f(\bullet)$  is a strictly increasing function, showing that  $K_t$  is also increasing in  $t$  completes this part of the proof. To see this, rewrite (13) with (34) to get

$$K_{t+1} = K_t + \omega E_t = K_t + \omega(1-\lambda) \left( 1 - \frac{H_{f_t}}{N_t} \right) N_t$$

where  $H_{f_t}/N_t < 1$  since both sectors operate in equilibrium. Thus,  $K_{t+1} > K_t$  for  $a_t > 0$ .

Next, regarding the limit  $G_{W_t}^*$ , notice from (A.7) that  $\delta_t$  converges to a positive constant since we have  $K_{t+1} > K_t$  for all  $t$  and  $f(K_t) \rightarrow 1$  from (12). This implies  $G_{W_t} \rightarrow \lim_{t \rightarrow +\infty} (\bar{X}_{t+1}/\bar{X}_t)^\lambda$ , and (37) leads to

$$G_{W_t}^* = \lim_{t \rightarrow +\infty} \left( \frac{\bar{X}_{t+1}}{\bar{X}_t} \right)^\lambda = \lim_{t \rightarrow +\infty} \{ \exp[(\sigma - 1)a_t] \}^\lambda = \{ \exp[(\sigma - 1)a^*] \}^\lambda$$



where  $a^* \equiv \lim_{t \rightarrow +\infty} a_t = \theta - (\sigma^{\frac{\lambda}{1-\lambda}} - 1)^{-1}$  given (30) and (12). *Q.E.D.*

**Proof of Lemma 3:**

(41) simply follows from  $v_t = (\phi/W_t)^{\frac{\eta}{1-\eta}}$ . For  $x_{ft+1}/x_{ft}$ , we have

$$\frac{x_{ft+1}}{x_{ft}} = \frac{X_{ft+1}/N_{t+1}}{X_{ft}/N_t} = \frac{X_{ft+1}/X_{ft}}{N_{t+1}/N_t} = \frac{X_{ft+1}/X_{ft}}{n_t}$$

by definition. For  $n_t > 1$ , (4), (5), (35) and (36) imply

$$\frac{X_{ft+1}}{X_{ft}} = 1 + \frac{\psi X_{ft}^{1-\eta} H_{ft}^\eta}{X_{ft}} = 1 + \psi \left( \frac{\phi}{W_t} \right)^\eta x_{ft}^{1-\eta} \left( \frac{1}{x_{ft}} \right) = 1 + \frac{\psi n_t}{x_{ft}}$$

which implies the desired result given (38). For the case of  $n_t = 1$ , the result follows from  $Y_{ft} = N_t$ .

**Proof of Proposition 3:**

Since  $v^q$  in (43) uniquely exists given any  $\bar{X}_0$ , the only task is to show that the equation within the arg solve term of (44) is solved for a unique and a strictly positive  $x$  given  $v^q$ . Rewrite this equation as in

$$x = (v^q)^{-(1-\eta)} x^\eta + \psi, \tag{A.8}$$

and notice that, since the right-hand side of (A.8) is strictly increasing, strictly concave, and equal to  $\psi$  for  $x = 0$ , there exists a unique  $x > 0$  that solves the equation.

**Proof of Proposition 4:**

Since we have  $K_{t+1} > K_t$  for all  $t$  from the proof of Lemma 2, the growth of  $K_t$  throughout Regimes 1 and 2 eventually implies  $K_t > \hat{K}$  at some  $\hat{t}$ . *Q.E.D.*

**Proof of Proposition 5:**

The task is to show that, if the economy starts its evolution at  $t = 0$  in Regimes 1, 2 or 3, it eventually enters Regime 4, and that, if the economy is in Regime 4, it stays in Regime 4 to converge to the unique asymptotic equilibrium.

Starting in Regime 1 (with  $n_0 = 1$  and  $a_0 = 0$ ),  $v_t = v^q = v_0$  is constant, and  $x_{ft}$  and  $K_t$  are growing. There are, then, three possibilities for a regime change at a future period: If the growth of  $x_{ft}$  is sufficiently larger than the growth of  $K_t$ , the economy enters Regime 2 with  $x_{ft} > 1/v^q$  and  $K_t \leq \hat{K}$ . Conversely, if the growth of  $x_{ft}$  is not sufficiently larger than the growth of  $K_t$ , the economy enters Regime 4 with  $K_t > \hat{K}$  and  $x_{ft} \leq 1/v^q$ . Finally, if  $x_{ft}$  and  $K_t$  grow in such a way that the conditions  $K_t > \hat{K}$  and  $x_{ft} > 1/v^q$  are satisfied for the same  $t > 0$ , then the economy enters Regime 3.

Next, starting in Regime 2 (with  $n_0 > 1$  and  $a_0 = 0$ ),  $v_t = v^q = v_0$  is constant,  $K_t$  is growing, and  $x_{ft}$  is either increasing or decreasing towards its quasi-steady-state value of  $x_f^q > 0$  depending on  $x_{f0}$  as the point  $(x_{f0}, v_0)$  may reside either below or above the  $x_f x_f$  locus. The economy then can only transit to Regime 3 when  $K_t$  is large enough to imply  $K_t > \hat{K}$  since we have  $n_t > 1$  for all  $t$  in Regime 2.

Starting in Regime 3 (with  $n_0 > 1$  and  $a_0 > 0$ ) the economy does not transit to Regime 1 or Regime 2 because the growth of  $K_t$  implies  $a_t > 0$ . This leaves only two possibilities, transiting to Regime 4 or staying in Regime 3. The claim is that the economy enters Regime 4 at some finite  $t$ . Since we have  $x_{ft} v_t > 1$  in Regime 3 and  $x_{ft} v_t \leq 1$  in Regime 4, the transition requires  $x_{ft} v_t$  to decrease and to intersect with the  $N^*$  locus of the bottom panel of Figure 2. Denoting by  $G_{x_{fv},t}$  the gross growth rate of  $x_{ft} v_t$ , the transition may occur in two ways:

- Firstly, the state of the economy in Regime 3, i.e.,  $(x_{ft}, v_t)$ , may reside above the  $nn_t$  locus so that

$$v_t > \left( \frac{x_{ft}^\eta}{G_{W_t}^{\frac{\eta}{1-\eta}} x_{ft} - \Psi} \right)^{\frac{1}{1-\eta}} \Leftrightarrow G_{W_t}^{-\frac{\eta}{1-\eta}} \left( \frac{1}{x_{ft}^{1-\eta} v_t^{1-\eta}} + \frac{\Psi}{x_{ft}} \right) = G_{x_{fv},t} < 1.$$

$x_{ft} v_t$  then decreases towards unity to lead  $(x_{ft}, v_t)$  to intersect with the  $N^*$  locus, and the economy enters Regime 4.

- Secondly,  $(x_{ft}, v_t)$  in Regime 3 may reside below the  $nn_t$  locus to imply  $G_{x_{fv},t} > 1$ . In this case, the state of the economy does not (directly) move towards the  $N^*$  locus because  $x_{ft} v_t$  is growing. But since  $x_{ft}$  is increasing and  $v_t$  is decreasing, the state of the economy can move only towards the  $nn_t$  locus while the  $nn_t$  locus itself is gradually shifting towards the origin given  $G_{W_t}$  is increasing in  $t$  (see Lemma 2). Basically because the  $nn_t$  locus and the  $N^*$  locus intersect for a unique and a strictly positive  $x_{ft}$ , denoted by  $\tilde{x}_{ft}$ , the state of the economy satisfies  $G_{x_{fv},t} < 1$  after some finite  $t$ . Then,  $x_{ft} v_t$  decreases towards unity as in the previous case, and the economy enters Regime 4.

Finally, starting in Regime 4 (with  $n_0 = 1$  and  $a_0 > 0$ ), the economy does not transit to Regime 1 or Regime 2 because the growth of  $K_t$ , again, implies  $a_t > 0$ . There exist two scenarios through which the economy ends up staying in Regime 4 (with  $x_{ft} v_t \leq 1$ ) to converge to the asymptotic equilibrium:

- Firstly, if we have  $x_{ft} > \tilde{x}_{ft}$ , (42) implies that

$$G_{x_{fv},t} = G_{W_t}^{-\frac{\eta}{1-\eta}} \left( 1 + \frac{\Psi}{x_{ft}} \right) < 1$$

and the state of the economy resides below the  $N^*$  locus. That is, the economy stays in Regime 4 for  $t \rightarrow +\infty$ .

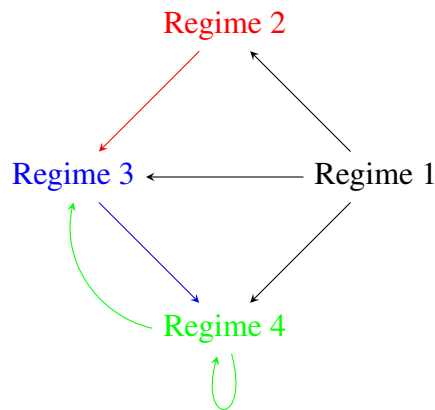


Figure A.1: Regime transitions

- If, however, we have  $x_{ft} \leq \tilde{x}_{ft}$ , (42) now leads to  $G_{x_f v, t} \geq 1$  and the economy might enter Regime 3 at some finite  $t$  if it does not stay in Regime 4. But if the economy enters Regime 3, it eventually enters Regime 4 as discussed above.

Consequently, the economy always ends up in Regime 4 for large  $t$ —as it transits only to Regime 3 while in Regime 2 and only to Regime 4 while in Regime 3 (see Figure A.1). In Regime 4,  $v_t$  keeps decreasing towards zero, and  $x_{ft}$  keeps increasing towards  $+\infty$ . The economy thus converges to its asymptotic equilibrium with  $a_t \rightarrow a^* > 0$ ,  $G_{Wt} \rightarrow G_W^*$ ,  $n_t = n^* = 1$ ,  $N_t = N^* > 0$ , and  $H_{ft}/N_t \rightarrow 0$ . *Q.E.D.*

### Proof of Proposition 6:

Starting in Regime 1, the economy does not enter Regime 3 or Regime 4 since, as argued in the proof of Proposition 5 above,  $\hat{t}$  is assumed to be sufficiently large. There exists, then,  $\check{t} > 0$  at which the economy enters Regime 2. Then, again as argued in the proof of Proposition 5, the economy enters only Regime 3 at some  $\hat{t} > \check{t}$ , and transits from Regime 3 to Regime 4 at some  $t^* > \hat{t}$ . *Q.E.D.*

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