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Reserve Price When Bidders are Asymmetric

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Abstract
The authors analyze the optimal reserve price in a second price auction when there are N types of bidders whose valuations are drawn from different distribution functions. The seller cannot determine the specific type of each bidder. First, the authors show that the number of bidders affects the reserve price. Second, they give the sufficient conditions for the uniqueness of the optimal reserve price. Third, the authors find that if a bidder is replaced by a stronger bidder, the optimal reserve price may decrease. Finally, they give sufficient conditions that ensure the seller will not use a reserve price; hence, the auction will be efficient.

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Keywords Auction; reserve (reservation) price; asymmetric bidders

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1 Introduction

Auctions are widely used by business, government and citizens to buy and sell goods. For example, firms use auctions for initial public offerings, governments use auctions to sell spectrum licenses and the general public has been increasingly using auction sites such as E-bay. In many of these auctions, single goods are sold and sellers use reserve prices. Accordingly, there has been much research on the effect of reserve prices in single object auctions (e.g., Maskin and Laffont 1980, Riley and Samuelson 1981, Englebrecht-Wiggans 1987, McAfee and McMillan 1987, Levin and Smith 1996, McAfee and Vincent 1997, Lu 2010, Hu et. al. 2010). However, all of these papers assume symmetric bidders. In this paper, we relax this commonly-used assumption by allowing the independent private valuations to be drawn from different distribution functions, then analyze how this will affect optimal reserve prices.

In our model, there is one good that is auctioned off using a second price auction. The seller knows that there are N types of bidders whose valuations are drawn from different distribution functions but he cannot determine the specific type of each bidder,[1] or equivalently we assume that the seller has to use a unique reserve price.[2] We give conditions that determine whether or not the seller will use a reserve price. If a reserve price is used, we analyze its properties for cases where the bidders' valuations are drawn from common supports. Our assumptions differ from Myerson (1981:59-72) who studies the revenue maximizing mechanism under the assumption that seller can determine the types.[3] As a result, he finds a "discriminatory reserve price"; i.e., an optimal reserve price for each type. Our motivation comes from the fact that in practice, almost all auctions have a single reserve price (or no reserve price at all).[4] We show that the optimal reserve price is determined when bidders' weighted average of virtual valuations is equal to the seller's valuation. In Myerson (1981:59-72), the optimal reserve price is found by equating the virtual valuation of each bidder to the seller's valuation; hence, there is a different reserve price for **each** type of bidder.[5]

[1] This is an assumption commonly used in industrial organization. Firms are aware of the distribution of types across consumers but they cannot price discriminate because they do not know which specific consumers have each given type.

[2] For example, when government agencies cannot use different reserve price for different bidders; they may be accused of corruption, if they do so.

[3] Myerson (1981) also shows that there exists a unique optimal reserve price when bidders are symmetric.

[4] Englmaier and Schmöller (2012) document that many players do not use reserve price while playing the online game HATTRICK.

[5] It must be clear by writing "bidders who have different types" we mean that their valuations are drawn from different distribution functions. Of course, bidders who have the same type will have different valuations with probability measure 1 since support is continuous.

Our first contribution is showing that, under these assumptions, the number of bidders affects the optimal reserve price. This is in contrast with Maskin and Laffont (1980) and Riley and Samuelson (1981). This result is in line with Levin and Smith (1996) but their model assumes affiliated valuations. In a private valuation model, a reserve price affects revenue when there is exactly one agent bidding above the reserve price. In the case of symmetric bidders, the marginal cost and marginal revenue of changing the reserve price will be the same regardless of the identity of this bidder since all bidders are identical. In the case of asymmetric bidders, however, the identity of the bidders will have an effect. Changing the number of agents of each type will influence the probability of who may be that critical agent and, as a result, the reserve price will be affected.

Our second contribution is to show that if a weak bidder is replaced with a stronger bidder ("strongness" is determined by the first order stochastic dominance), the optimal reserve price may decrease. Hence, there is no monotonic relation between first order stochastic dominance and the reserve price. Hu et. al. (2010:1188-1202) show that the optimal reserve price will decrease if the seller becomes more risk averse since the seller does not like the risk of not selling the object. In our model, the seller is risk-neutral but lowers the reserve price due to a change in the type of bidders.

Our third contribution is giving sufficient conditions for the uniqueness of the optimal reserve price. We prove this by showing that the revenue function is quasi-concave when there are bidders that can be ranked via likelihood ratio dominance.

Our fourth contribution is showing when the seller will use (effective) reserve prices, and when she will not.[6] We show that when the virtual valuation of each bidder at the lower bound of the support is greater than the seller's valuation and Myerson's regularity condition holds, then the seller will not use an effective reserve price. While there are other papers in the literature that find the same result (e.g. Englebrecht-Wiggans (1987), McAfee and McMillan (1987), and Lu (2010)), they all assume that *symmetric* bidders have to pay some participation (or information) cost/fee before entering the auction or non quasi linear utility functions (Dastidar (2010)). In our model, *asymmetric* bidders do not have cost of participation nor do they buy information. We find sufficient conditions for cases when the seller uses and does not use an (effective) reserve price. When the seller does not use reserve price, the auction is efficient. The implicit assumption in the literature (e.g Myerson (1981)) is that the seller is committed not to re-auction the object if he fails to sell it in his first attempt (McAfee and Vincent (1997) is one

[6] By effective reserve price, we mean a reserve price strictly greater than the lower bound of buyer's valuations.

of the exceptions in the literature). By finding the conditions that make using the reserve price redundant, we ensure that the commitment problem is not an issue.

There are a few papers in the literature that assume asymmetric bidders other than Myerson (1981). Plum (1992) and Lebrun (1999) show the existence of equilibrium bids. Maskin and Riley (2000) and Kirkegaard (2010) compare revenues of the English (second price) auction and the high-bid (first price) auction. Cantillon (2008) investigates the effect of bidder asymmetries on revenue of different auction formats compared to a symmetric case (geometric average of the asymmetric distributions). Kirkegaard (2005) shows that using reserve price is better than using entry fees. None of these papers analyze the optimal reserve price.

2 Asymmetric Distributions with Common Support

Assume that there are $N \geq 2$ risk neutral bidders, and one risk-neutral seller with one object. Bidders' valuations are independently distributed with cumulative distribution functions F_i, $i = 1,2,..N$ with support $[\underline{v},\overline{v}]$ (with $\underline{v} \geq 0$). All distribution and density functions are continuously differentiable and density function f_i are positive everywhere.[7] The distributions are common knowledge but the valuations are private information. The seller cannot determine the specific type for each bidder.[8] We assume that we have at least one i such that $F_i \neq F_j$; hence, we have asymmetric bidders. The value of the object to the seller is $x_0 < \overline{v}$. If the inequality does not hold, the seller will never sell the object.

We will calculate the optimal reserve price that will be set by the seller in a second-price auction. It is well-known that the bidders bid truthfully in a second price auction even when they know the existence of asymmetric bidders (e.g. see Krishna (2009:13-26)). First, we have to calculate the seller's revenue function, $E\Pi$, when facing asymmetric bidders. For this, we will fix a bidder i, and calculate the first (i.e., highest) order statistic of the remaining bidders; $H_i(x) = \prod_{j \neq i} F_j(x)$. We let h_i denote the corresponding density function. The expected revenue function is given by:

$$E\Pi = x_0 \prod_i^N F_i(r) + \sum_i^N [(1 - F_i(r))H_i(r)r + \int_r^{\overline{v}} (\int_r^{x_i} y h_i(y)dy) f_i(x_i)dx_i]$$

[7] This implies continuous differentiability of the expected revenue function.

[8] This is analogous to a firm facing two or more different types that it cannot distinguish. One can also assume that the seller can distinguish the types but is required to use a unique reserve price by law. For example, a government agency cannot treat bidders differently and use different reserve price for different bidders.

The first term with x_0 shows that the case where all bidders bid below the reserve price r. The first term in the brackets of summation shows the case where a given bidder i wins the auction and all other bidders bid below the reserve price r, or equivalently, the other bidders do not enter the auction. The second term is the expected payment of bidder i given that he wins the auction, and at least one other bidder bids above r. Finally, to get the expected revenue function, we have to sum over all N bidders.

By changing the order of integration, we can write this as:

$$E\Pi = x_0 \prod_i^N F_i(r) + \sum_i^N [(1 - F_i(r))H_i(r)r + \int_r^{\overline{v}} y h_i(y)(1 - F_i(y))dy]$$

By taking the derivative of this function with respect to r, we have the following equation:

$$\begin{aligned} E\Pi' &= x_0 \sum_i^N f_i(r)H_i(r) + \sum_i^N H_i(r)(1 - F_i(r) - f_i(r)r) \\ &= \sum_i^N H_i(r)(1 - F_i(r) - f_i(r)(r - x_0)) \end{aligned} \tag{1}$$

We use Myerson's virtual valuation which is $J_i(x) = x - \frac{1-F_i(x)}{f_i(x)}$ in the equation, then we have:

$$E\Pi' = -[\sum_i^N H_i(r) f_i(r)(J_i(r) - x_0)] \tag{2}$$

So we find an equation that the optimal reserve price must satisfy. Our first lemma summarizes this result.

Lemma 1 *Any interior optimal reserve price must satisfy the implicit equation:*

$$\sum_i^N H_i(r)(1 - F_i(r) - f_i(r)(r - x_0)) = -\Big(\sum_i^N H_i(r) f_i(r)(J_i(r) - x_0)\Big) = 0 \tag{3}$$

The first order condition shows that bidders' weighted average of virtual valuations is x_0 at the optimal reserve price, since one can re-write the first order condition as $\frac{\sum_i^N H_i(r)f_i(r)(J_i(r)-x_0)}{\sum_i^N H_i(r)f_i(r)} = 0 \Rightarrow \frac{\sum_i^N H_i(r)f_i(r)(J_i(r))}{\sum_i^N H_i(r)f_i(r)} = x_0$. In Myerson (1981), when bidders are asymmetric but their distributions are known, a different reserve price is determined for each bidder by setting their virtual valuation equal to the seller's valuation. This may result in an inefficient allocation since the bidder who values the object most may lose the auction.[9] As we discussed before, setting a different reserve price for different bidders is not practical. For example, government agencies cannot set different reserve price for different bidders. If one uses a unique reserve price, the inefficient allocation result of Myerson (1981) also disappears (at the expense of seller's revenue). That is, if the object is sold, the winner would be the bidder who values it most in our second price “unique" reserve price auction.

If we had symmetric bidders (i.e, $F_i = F$, $H_i = H$, and $J_i = J$ for all i), equation 3 would be equal to $(1 - f(r)(r - x_0)) = F(r)$ or $J(r) = x_0$, and hence, the optimal reserve price would not depend on the number of bidders (Maskin and Laffont (1980), Riley and Samuelson (1981), and Krishna (2009)). However, in our case, the optimal reserve price does depend on the number of bidders.

Why does the result change? One has to understand that the reserve price matters when there is exactly one bidder bidding above the reserve price. Suppose that with **symmetric** bidders, we increase the reserve price slightly (by one very small unit). The marginal cost of increasing r is the net expected revenue forgone by not selling the object. In equation 3, this is the part $(r - x_0)f(r)H(r)$ where the seller loses r with probability $f(r)H(r)$ but still owns the object so enjoys x_0. The net marginal benefit of increasing r is $(1 - F(r))H(r)(r - x_0)$. The winning bidder pays the new higher reserve price when she is the only person bidding higher than r and no other players bids above r. The probability of this is $(1 - F(r))(H(r))$. Since the loser sells the object, he loses x_0 so this amount is subtracted. With symmetric bidders, it does not matter which bidder was bidding above the reserve price. The marginal cost and benefit is the same for any given bidder.

Now consider our asymmetric case and for the sake of example assume there are two (types of) bidders. Increasing the reserve price will give different marginal costs and benefits since it now matters whether the bidder bidding above the reserve price is the first or the second type. As the number of bidders changes, it affects the probability that the bidder bidding above the reserve price is the first or second type, and the reserve price is affected. However, note that if we double the number of all types of bidders, we find the same optimal reserve price. On the other hand, if we add more of one type of bidder, the optimal reserve price still changes.[10]

Example 2 *Suppose there are N bidders whose valuations are drawn from a uniform distribution F, and M bidders whose valuations are drawn from the*

[9] The inefficiency in Myerson's paper can be, other than the reserve price, due to the allocation rule; one with the highest virtual valuation gets the object. His mechanism favors the weak.

[10] Assume that an E-bay seller faces bidders from two countries (i.e types). If she expects that bidders from one country would increase, then she should set a reserve price accordingly.

distribution function $F_j(x) = F^{1/2}$ *with support on* $[0,1]$*, the seller values the object at* $x_0 = 0$*. The optimal unique reserve price is determined by the following equation:* $(2N+M+M/2)r - M(r)^{1/2} - N = 0$*. When* $N = M = 1$*, the optimal reserve price is* 0.4846*; when* $N = 2$*,* $M = 1$*, the optimal reserve price is* 0.4910*.*

Levin and Smith (1996), under the affiliated valuations assumption show that the number of bidders affects the reserve price. In their paper, the probability of everyone bidding below r, and hence, not selling the good may not go to zero in the limit (as N increases) due to the affiliated values. In our private value settings, the probability of not selling the good is zero in the limit. Yet, we find the same result.

Now, we will find conditions under which the seller will set an effective reserve price $r > \underline{v}$, and when he will not. In the theorem below, note that we do not assume regularity condition in part a; i.e, we do not assume that J_i functions are increasing.

Theorem 3 *Let* $t_i(\underline{v}) = lim_{x \downarrow \underline{v}} \frac{H_i(x) f_i(x)}{\Sigma_i^N H_i(x) f_i(x)}$ *exist.*

a) If $\Sigma_i^N t_i(\underline{v})(J_i(\underline{v}) - x_0) < 0$ *then* $r > \underline{v}$*.*

b) Assume J_i *is increasing for each i. If* $\Sigma_i^N t_i(\underline{v})(J_i(\underline{v}) - x_0) \geq 0$ *then there will be no effective reserve price.*[11]

Proof Part a) Suppose $m = \Sigma_i^N t_i(\underline{v})(J_i(\underline{v}) - x_0) < 0$. Note that by the properties of the limit function and J is a continuous function (f and F are continuous and f is positive everywhere) we have

$$m = \Sigma_i^N t_i(\underline{v})(lim_{x \downarrow \underline{v}} J_i(x) - x_0) = \Sigma_i^N t_i(\underline{v})(J_i(\underline{v}) - x_0) < 0$$

Then, by definition of the limit, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in (\underline{v}, \underline{v} + \delta)$, $m - \varepsilon < \frac{\Sigma_i^N H_i(x) f_i(x)}{\Sigma_i^N H_i(x) f_i(x)} (J_i(x) - x_0) < m + \varepsilon$. Since $m < 0$, we can pick ε small enough so that $\Sigma_i^N H_i(x) f_i(x)(J_i(x) - x_0) < 0$, and thus $-\Sigma_i^N H_i(x) f_i(x)(J_i(x) - x_0) > 0$ for all $x \in (\underline{v}, \underline{v} + \delta)$. But this last expression is the derivative of the revenue function, so a seller will strictly increase revenues by setting $r > \underline{v}$.

Suppose $m = \Sigma_i^N t_i(\underline{v})(J_i(\underline{v}) - x_0) < 0$. Then, by definition of the limit, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in (\underline{v}, \underline{v} + \delta)$, $m - \varepsilon < \Sigma_i^N H_i(x) f_i(x)(J_i(x) - x_0) < m + \varepsilon$. Since $m < 0$, we can pick ε small enough so that $\Sigma_i^N H_i(x) f_i(x)(J_i(x) - x_0) < 0$, and thus $-\Sigma_i^N H_i(x) f_i(x)(J_i(x) - x_0) > 0$ for

[11] We are grateful to an anonymous referee whose suggestions helped us to write this generalized theorem.

all $x \in (\underline{v}, \underline{v}+\delta)$. But this last expression is the derivative of the revenue function, so a seller will strictly increase revenues by setting $r > \underline{v}$.

Part b) Assume that J_i is increasing for each i and $\Sigma_i^N t_i(\underline{v})(J_i(\underline{v}) - x_0) > 0$ and an effective reserve price $r > \underline{v}$ exists. By Lemma 1, this r must satisfy $\Sigma_i^N H_i(r) f_i(r)(J_i(r) - x_0) = 0$. But then for all $x \in (\underline{v}, r)$ we have $\Sigma_i^N H_i(x) f_i(x)(J_i(x) - x_0) < 0$ by the fact that J_i are increasing, and $H(.)$ and $f(.)$ are positive. But then the assumption $\Sigma_i^N t_i(\underline{v})(J_i(\underline{v}) - x_0) > 0$ cannot hold. This is a contradiction; hence, no effective reserve price exists.

Now suppose $\Sigma_i^N t_i(\underline{v})(J_i(\underline{v}) - x_0) = 0$ and an effective reserve price $r > \underline{v}$ exists. Again, by Lemma 1, this r must satisfy $\Sigma_i^N H_i(r) f_i(r)(J_i(r) - x_0) = 0$. But we have $\Sigma_i^N H_i(r) f_i(r)(J_i(r) - x_0) > 0$ by the fact that $\Sigma_i^N t_i(\underline{v})(J_i(\underline{v}) - x_0) = 0$, J_i are increasing, and $H(.)$ and $f(.)$ are positive. Again, this is a contradiction; hence, no effective reserve price exists.

■

Before discussing the implications of the theorem, we give a corollary providing easier sufficient conditions to check.

Corollary 4 *a) If $J_i(\underline{v}) < x_0$ for each i then $r > \underline{v}$.*

b) Assume that J_i are increasing and $J_i(\underline{v}) > x_0$ for each i then $r = \underline{v}$. That is, effectively, there is no reserve price.

Proof Part a) If the seller values the object at x_0, the derivative of the expected revenue function becomes:

$$\begin{aligned} x_0 \Sigma_i^N f_i(r) H_i(r) + \Sigma_i^N H_i(r)(1 - F_i(r) - f_i(r) r) \\ = \Sigma_i^N H_i(r)(1 - F_i(r) - f_i(r)(r - x_0)) = -[\Sigma_i^N H_i(r) f_i(r)(J_i(r) - x_0)] \end{aligned} \tag{4}$$

By the assumption $J_i(\underline{v}) < x_0$ and continuity of revenue function and the fact that derivative of the revenue function is zero exactly at $\underline{v}$, we can find a small $\varepsilon > 0$, $-\Sigma_i^N[(J_i(\underline{v}+\varepsilon) - x_0) f_i(\underline{v}+\varepsilon) H_i(\underline{v}+\varepsilon) > 0$ without making any assumption on the curvature of J_i. But then, by setting the reserve price slightly more than $\underline{v}+\varepsilon$ increases revenue. This completes the proof.

Part b) The first order derivative of expected revenue function, $-[\Sigma_i^N H_i(\underline{v}) f_i(\underline{v})(J_i(\underline{v}) - x_0)]$, is zero. For any $r > \underline{v}$ in the support, the first order derivative is negative since $J(.)$ is increasing and $J_i(\underline{v}) > x_0$, and H and f are positive. Since increasing the reserve price decreases the revenue, the optimal reserve price should be set at $r = \underline{v}$.[12] ■

Corollary 4 implies that when $x_0 > \underline{v}$ (and f_i are positive as the standard assumption), setting $r > 0$ (i.e., using an effective reserve price) is always optimal

[12] The seller may set any reserve price less than or equal to $\underline{v}$ but this is effectively using no reserve price.

when the support is on $[0,\overline{v}]$. Therefore, we generalize this result in the literature (e.g. Krishna (2009)) with our proposition above.

When will the seller not use the reserve price? $J_i(\underline{v}) > x_0$ implies that $1 < f_i(\underline{v})(\underline{v} - x_0)$ which in turn implies that $\underline{v} > x_0$. Hence, a necessary condition not to use reserve price is $\underline{v} > x_0$. Also, $f_i(\underline{v})$ should be big enough. This means, there has to be an enough mass of consumers with valuations close to $\underline{v}$. Note that Myerson's optimal auction will not use effective reserve price under these conditions. Our contribution is to explicitly state the sufficient conditions for not using the effective reserve price.

We believe that there are certain cases in which x_0 can be negative. For example, there may be a storage cost of keeping the object to the seller or a firm that is making a loss under the current management may have negative values to the seller. In these cases, the seller may not use the reserve price (or equivalently set the reserve price equal to $\underline{v}$).

Example (No reserve price): Assume two bidders. Their valuation is drawn from the support $[4,5]$. The first one has a uniform density function $f_1 = 1$. The second one has a density function $f_2 = -3.5 + x$. The seller values the object at $x_0 = 1$. Then, the seller would not set an effective reserve price.

Englebrecht-Wiggans (1987), by assuming that bidders have a cost to participate in the auction, and Dastidar (2010), by assuming a model with non-quasi linear utility functions, show that the optimal reserve price will be equal to zero (effectively no reserve price). Also, McAfee and McMillan (1987) and Lu (2010) show that sellers may not use reserve price but the "endogenous number" of bidders have to buy information/pay entry fee. Unlike all these papers, we use asymmetric bidders and find conditions when the seller will use the reserve price and when it will not use it. If the reserve price is not used (like in the example above), this implies that the auction is efficient since the object will always be sold. When using a reserve price is not optimal in a second price auction, we do not have to be concerned with the assumption of whether the seller can commit not to sell the object in a second auction.[13]

2.1 Stochastic Dominance with N types and a Reserve Price

We will introduce N types that we will order from "strongest" to "weakest" in the sense of reverse hazard rate and hazard rate dominance.[14] Specifically, we assume

[13] With the exception of McAfee and Vincent (1997) who assume seller will re-auction the object if it is not sold, the literature generally assumes that seller will make this "not re-auction" commitment credibly (e.g. Myerson (1981).

[14] If one assumes likelihood ratio dominance, it would imply both reverse hazard rate and hazard rate dominance (Krishna (2009:60,289)).

that $F_1/f_1 < F_2/f_2 < ... < F_N/f_N$ in the interior of the support.[15] In addition, we assume f_i nondecreasing which in turn implies that J_i are increasing. The proof of the proposition shows that the derivative of the revenue function is single-crossing zero from above –for the interior values–. This implies that the revenue function is quasi-concave, and hence, the reserve price is unique.

Proposition 5 *Suppose that all f_i are non-decreasing and $J_i(\underline{v}) < x_0$ for each i (i.e., $r > \underline{v}$). There are N types of bidders ranked from strongest to weakest in the sense of both hazard rate dominance, $\frac{1-F_1(x)}{f_1(x)} > \frac{1-F_2(x)}{f_2(x)} > ... > \frac{1-F_N(x)}{f_N(x)}$ and reverse hazard rate dominance, $F_1(x)/f_1(x) < F_2(x)/f_2(x) < ... < F_N(x)/f_N(x)$ for $x \in (\underline{v}, \overline{v})$ and M_i bidders for each type. Then, there is a **unique** reserve price r that maximizes revenue which can be calculated from equation 3.*

Proof of Proposition 5: From equation 3 in Lemma 1, the optimal reserve price satisfies:

$\Sigma_{i=1}^{N} M_i H_i(r)(1 - F_i(r) - f_i(r)(r - x_0)) = 0.$

First note that likelihood ratio dominance implies hazard rate dominance (i.e. $\lambda_i(r) = (f_i(r)/(1-F_i(r)) < \lambda_j(r))$ and first order stochastic dominance (i.e., $F_i(r) < F_j(r)$) in the interior of the support. This implies that $(1-\lambda_i(r)(r-x_0))(1-F_i(r)) > (1-\lambda_j(r)(r-x_0))(1-F_j(r)) \Leftrightarrow (1-F_i(r)-f_i(r)(r-x_0)) > (1-F_j(r)-f_j(r)(r-x_0))$ for $i < j$. Therefore, there cannot be a unique r that makes $1-F_i(r)-f_i(r)(r-x_0) = 0$ for all i.

It is clearly not the case that all terms in the first order condition sum, $\Sigma_{i=1}^{N} M_i H_i(r)(1-F_i(r)-f_i(r)(r-x_0))$, are positive or negative. So for some $j \in \{2, ..., N-1\}$, $(1-F_k(r)-f_k(r)(r-x_0)) > 0$ for all $1 \leq k \leq j-1$ and $1-F_k(r)-f_k(r)(r-x_0) < 0$ for $j \leq k \leq N$. From now on, j will denote that cutoff agent.

Take the derivative of this first order equation and get:

$\Sigma_{i=1}^{N}[M_i h_i(r)(1-F_i(r)-f_i(r)(r-x_0)) + M_i H_i(r)(-2f_i(r)-f_i'(r)(r-x_0))]$.

Since the density functions are nondecreasing, we have

$\Sigma_{i=1}^{N} M_i H_i(r)(-2f_i(r)-f_i'(r)(r-x_0)) < 0$, so we will ignore this part of the sum and focus on showing that (*) $\Sigma_{i=1}^{N}[M_i h_i(r)(1-F_i(r)-f_i(r)(r-x_0)) < 0$.

We solve for $1-F_j(r)-f_j(r)(r-x_0)$ from the first order condition and substitute this into (*) to get:

$\Sigma_{i\neq j}^{N} M_i h_i(r)(1-F_i(r)-f_i(r)(r-x_0)) + M_j h_j(r)(-\Sigma_{i=2}^{N} M_i H_i(r)(1-F_i(r)-f_i(r)(r-x_0))/M_j H_j(r))$

[15] Note that the way we write the inequalities are such that F_i dominates F_j for $i < j$ in terms of the reverse hazard rate. Also note that $F_i(\underline{v}) = 0$ and hence the whole term will be equal to zero for all i so we define inequalities only in the interior of the support.

We do not have to be concerned with division by zero since our assumptions ensure that $r > \underline{v}$ by proposition 4. After cancelling M_j in the right hand term and combining both terms, we are left with:

$\Sigma_{i \neq j}^{N} M_i H_i(r)(h_i(r)/H_i(r) - h_j(r)/H_j(r))(1 - F_i(r) - f_i(r)(r - x_0))$

By using $h_k(r)/H_k(r) = \Sigma_{l \neq i} f_l(r)/F_l(r)$, we can re-write the term above as:

$\Sigma_{i \neq j}^{N} M_i H_i(r)(f_j(r)/F_j(r) - f_i(r)/F_i(r))(1 - F_i(r) - f_i(r)(r - x_0))$

This is less than 0: The first $j-1$ terms are less than 0 because for each i such that $1 \leq i \leq j-1$, we have $f_j/F_j - f_i/F_i < 0$ (by reverse hazard rate assumption) and $1 - F_i(r) - f_i(r)(r - x_0) > 0$ (since j is the cutoff agent). The $j+1$ to nth terms are less than 0 because for each i such that $j+1 \leq i \leq N$, $f_j(r)/F_j(r) - f_i(r)/F_i(r) > 0$ and $1 - F_i(r) - f_i(r)(r - x_0) < 0$. Therefore, any r that satisfies the first order condition must also satisfy the second order condition (concavity). This implies uniqueness of an optimal reserve price. To see why, suppose there are two distinct values r_1 and r_2 that maximize expected revenue and suppose without loss of generality that $r_1 < r_2$. Both of these points must satisfy the first order condition and hence the second order condition, as we have just shown. But by the differentiability (and hence, continuity) of the expected revenue function, this would imply that there is a point $r_3 \in (r_1, r_2)$ such that there is a local minimum at r_3 and the first order condition is therefore satisfied at r_3.[16] But this is a contradiction because we have just shown that any point that satisfies the first order condition must be a local maximum. Therefore, the optimal reserve price is unique. ■

For example, let $F_N = x$ be uniform distribution function on $[0,1]$, $F_{N-1} = (F_N)^2 = x^2$,...,$F_1 = (F_N)^N = x^N$. This is the power distribution widely used in asymmetric bidders literature. (see Cantillon (2008), for example). This example satisfies all conditions of our proposition 5; hence, the reserve price that satisfies equation 3 is the optimal unique reserve price. More examples can be created by replacing the uniform distribution in the example with any other distribution that has a non-decreasing density function.

Next, we will show that under some conditions, replacing a weaker bidder with a stronger one may lower the optimal reserve price.

Normally, one would expect that if a stronger bidder joins the auction, the seller should increase the reserve price. Next, we will give an example that this is not correct; that is, there is not a monotonic relation between first order stochastic dominance and optimal reserve price. Let us emphasize that in this example, some f_i are decreasing.

Example 6 *Suppose there are N bidders whose valuations are drawn from a uniform distribution F and M bidders whose valuations are drawn from the distribution function $F_j(x) = F^{\alpha}$ with support on $[0,1]$. For any $\alpha > 0$, the optimal*

[16] Since r_1 and r_2 satisfy the second order condition, the expected revenue function cannot be a line between r_1 and r_2.

unique reserve price is determined by the following equation: $(2N+M+M\alpha)r - M(r)^{1-\alpha} - N = 0$. *When* $0 < \alpha < 1$*, the second order condition holds and* M *bidders are weak bidders in the sense of first order stochastic dominance, and as* α *increases, bidders become stronger. Figure 1 shows the optimal reserve price as* α *changes. That is, we keep* N *bidders in the auction but replace* $\alpha = 0.1$ *bidders with* $\alpha = 0.2$ *bidders, then replace* $\alpha = 0.2$ *bidders with* $\alpha = 0.3$ *bidders and so on. Figure 1 shows that the optimal reserve price first decreases, then increases.*

To understand why there is no monotonic relation between α and r, suppose that α is extremely low; that is, there is a very high probability that the weak bidder's valuation is close to zero. Hence, the seller is better off by selecting a reserve price similar to what they would choose if they faced only the strong bidder. As a result, they will either sell the object to the strong bidder at the reserve price or will not sell the object at all. Now, if we increase α slightly, the weak bidder may bid above the reserve price at a time when the strong bidder turns out to have a low valuation that is below r. Lowering r makes sense since the seller is more likely to sell the good to one of the types. The marginal benefit of lowering r is selling to the weak type at the new r. The marginal cost is selling the good to the strong type at the new low reserve price. The first effect must be dominating at low α so that the seller decreases the reserve price. As the weak type becomes stronger (we increase α further), then the risk of not selling the object diminishes considerably; hence, there is no need to decrease the reserve price, rather it is better to increase it.

Hu et. al (2010) shows that the more risk averse the seller is the lower the reserve price (since a more risk averse seller does not want to face the risk of not selling the object). In our paper, we show that the optimal reserve price will be lower when the seller is risk-neutral but the type of bidders change.

3 Conclusion

We characterized the optimal reserve price for asymmetric bidders. First, we showed that, in contrast to the symmetric case, the number of bidders changes the reserve price. When the bidders' types are known, the seller should set a different reserve price for each bidder where the virtual valuations of each bidder is equal to seller's valuation at the optimal reserve price. This may create an inefficient allocation (Myerson 1981) in the sense a bidder that values the object less may win the auction.[17] In our model, we show that the unique reserve price is found by setting a weighted average of bidders' virtual valuations equal to the seller's

[17] We once again acknowledge that the inefficiency in Myerson's paper can be due to the allocation rule; one with the highest virtual valuation gets the object. The mechanism favors the weak.

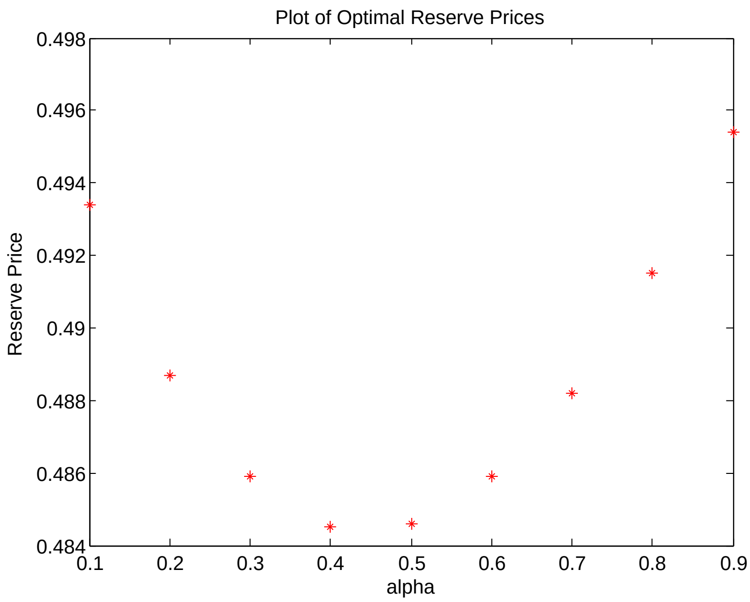

Figure 1: $N = M = 5$. When $0 < \alpha < 1$, optimal reserve price may decrease.

valuation of the object. Hence, the allocation is more efficient; the bidder who values the object most will win the auction as long as his valuation is above the reserve price. Second, we gave sufficient conditions for the uniqueness of the optimal reserve price. Third, with an example, we showed that if a weak bidder is replaced with a strong bidder in an auction, the optimal reserve price may decrease. Hence, there is no monotonic relation between first order stochastic dominance and the reserve price. Finally, we showed when the seller will use a reserve price in an asymmetric auction and when she will not. The conditions that make the seller abstain from using a reserve price are sufficient to make the auction an efficient one, since the object will always be sold to the bidder who values it most.

One potential avenue for future research is finding the optimal reserve price for the first-price auction under these asymmetric bidders assumption. Unfortunately, this is complicated since it is well-known that finding equilibrium explicitly is valid only for limited cases even without reserve price. In addition, Maskin and Riley (2000) and Cantillon (2008) find their results for first price auctions with only two bidders.

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