A short note on multiplicative processes subordinated to renewal counting processes

Enrico Scalas¹.∗

¹Department of Mathematics, University of Sussex, UK

(Dated: December 14, 2018)

Abstract

This note is an erratum on equation (14) for the paper A Parsimonious Model for Intraday European Option Pricing by Mauro Politi and myself, available as a discussion paper at http://www.economics-ejournal.org/economics/discussionpapers/2012-14 and later published as Scalas, Enrico and Politi, Mauro (2013) A note on intraday option pricing. International Journal of Applied Nonlinear Science, 1 (1). pp. 76-86. ISSN 1752-2862.

PACS numbers: 02.50.-r, 02.50.Ey, 89.65.Gh

∗Electronic address: e.scalas@sussex.ac.uk; URL: http://www.sussex.ac.uk/profiles/330303
I. DEFINITIONS AND BASIC PROPERTIES

Definition 1: Let \( \{U_i\}_{i=1}^{\infty} \) and \( \{J_i\}_{i=1}^{\infty} \) be two sequences of independent and identically distributed positive random variables. Let \( F_U(w) \) and \( F_J(w) \) denote their respective cumulative distribution functions. Let \( \mathbb{E}(U_1) < \infty \). Further assume that the two sequences are mutually independent.

Definition 2: The epochs are defined as the sequence \( \{T_n\}_{n=1}^{\infty} \) of cumulative sums of the random variables \( J_i \)s.

\[
T_n := \sum_{i=1}^{n} J_i. 
\] (1)

Definition 3: The counting process \( N(t) \) is

\[
N(t) := \max\{k : T_k \leq t\}. 
\] (2)

Definition 4: Let \( Y_n \) denote the following product

\[
Y_n := U_0 \prod_{i=1}^{n} U_i, 
\] (3)

with \( U_0 = 1 \).

Definition 5: The multiplicative process \( Y(t) \) is defined as the (càdlàg) random product of \( N(t) \) of the \( U_i \)s

\[
Y(t) := Y_{N(t)} = U_0 \prod_{i=1}^{N(t)} U_i, 
\] (4)

with \( Y(0) = U_0 = 1 \).

Proposition 1. The multiplicative process \( Y(t) \) belongs to the class of semi-Markov processes.

Proof. This is true by construction. However, one can see that, for any Borel set \( A \in \mathbb{R}^+ \)

\[
\mathbb{P}(Y_n \in A, J_n \leq t | Y_1, \ldots, Y_{n-1}, J_1, \ldots, J_{n-1}) = \mathbb{P}(Y_n \in A, J_n \leq t | Y_{n-1}) = P(Y_n \in A | Y_{n-1}) F_J(t). 
\] (5)

Lemma 1. The expected value of \( Y_n \) is given by

\[
\mathbb{E}(Y_n) = [\mathbb{E}(U_1)]^n. 
\] (6)
Proof. This is an immediate consequence of the fact that the $U_i$s are i.i.d.

**Definition 6:** The *natural filtration* $\mathcal{F}_t$ is the $\sigma$-field generated by the process $Y(t)$. We can write

$$\mathcal{F}_t := \sigma(Y(s), s \leq t).$$

**Definition 7:** Let $\mathcal{G}_n := \sigma(J_1, \ldots, J_n, U_1, \ldots, U_n)$.

**Lemma 2.** We have

$$\mathcal{F}_t := \sigma(Y(s), s \leq t) = \sigma(J_1, \ldots, J_k, U_1, \ldots, U_k : k \leq N(t)) = \mathcal{G}_{N(t)}.$$  \hfill (8)

**Proof.** By definition

$$\sigma(Y(s), s \leq t) = \sigma(Y_{N(s)}, s \leq t) = \sigma \left( U_0 \prod_{i=1}^{N(s)} U_i, s \leq t \right) \subseteq \sigma(J_1, \ldots, J_k, U_1, \ldots, U_k : k \leq N(t)).$$ \hfill (9)

In the other direction, if we know all the values of $Y(s)$ for $s$ up to time $t$, we can find the corresponding sequences $J_1, \ldots, J_k$ and $U_1, \ldots, U_k$ with $k \leq N(t)$. In other words, the sequences are measurable with respect to $\sigma(Y(s), s \leq t)$ and $\sigma(Y(s), s \leq t) \supseteq \sigma(J_1, \ldots, J_k, U_1, \ldots, U_k : k \leq N(t))$. \hfill \qed

**Lemma 3.** Consider the ratio $Y(t)/Y(s)$ with $s < t$, this is given by

$$\frac{Y(t)}{Y(s)} = 1$$ \hfill (10)

if $N(t) < N(s) + 1$ (no events between $s$ and $t$) and

$$\frac{Y(t)}{Y(s)} = \prod_{i=1}^{N(t)} U_i$$ \hfill (11)

if $N(t) \geq N(s) + 1$ (at least one event between $s$ and $t$). The conditional expectation $\mathbb{E}(Y(t)/Y(s)|\mathcal{F}_s)$ is given by

$$\mathbb{E}(Y(t)/Y(s)|\mathcal{F}_s) = \sum_{n=0}^{\infty} \mathbb{P}(N(t) - N(s) = n|\mathcal{F}_s)[\mathbb{E}(U_1)]^n = \sum_{n=0}^{\infty} \mathbb{P}(N(t) - N(s) = n|\mathcal{F}_s)\mathbb{E}(Y_n).$$ \hfill (12)
Proof. As a consequence of Lemma 2, the $U_i$s are independent of $F_s$ for $i > N(s)$. Incidentally, you can convince yourself that this is the case, by noting that, if you are sitting at any time $s$, the size of the next $U_i$s does not depend on the previous history of the process and hence on $F_s$. Then remember that the the $U_i$s and $J_i$s are mutually independent (see Definition 1). The expected value of the product of $n$ $U_i$s is given by Lemma 1. Between $s$ and $t$, there can be $n$ events with any non-negative value of $n$. Then, the expectations of the product of the $U_i$ must be multiplied by the probability of the event $P(N(t) - N(s) = n)$. This probability does depend on $F_s$ because it depends on the residual lifetime (a.k.a. forward recurrence time) $T_{N(s)+1} - s$ which, in turn, depends on the previous history as discussed in [1]. Then, the sum over all the possible values of $n$ gives the desired expectation.

**Corollary 1.** If $E(U_1) = 1$ then $E(Y(t)/Y(s)|F_s) = 1$.

Proof. This is an immediate consequence of Lemma 3.

II. MARTINGALE PROPERTY

**Theorem** Let $Y(t)$ be a multiplicative process as defined in Definition 5 and further assume that $E(U_1) = 1$, then $Y(t)$ is a martingale with respect to its natural filtration $F_t$.

Proof. Considering Definition 4, one has that $|Y(t)| = Y(t)$ and one finds that $|Y(t)|$ is integrable, namely

$$E(|Y(t)|) = E(Y(t)) = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n)E(Y_n) = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n) = 1 < \infty. \quad (13)$$

Using the fact that it is possible to take out what is known in conditional expectations, it is possible to write the following chain of equalities

$$E(Y(t)|F_s) = E(Y(s)Y(t)/Y(s)|F_s) = Y(s)E(Y(t)/Y(s)|F_s). \quad (14)$$

Now, since $E(U_1) = 1$, using Corollary 1, we can conclude that

$$E(Y(t)|F_s) = Y(s). \quad (15)$$