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Modelling Trades-Through in a Limited Order Book Using Hawkes Processes

Ioane Muni Toke
Ecole Centrale Paris and ERIM, University of New Caledonia, Nouméa

Fabrizio Pomponio
Ecole Centrale Paris and BNP Paribas Equity & Derivatives Quantitative Research & Development, Paris

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Abstract We model trades-through, i.e. transactions that reach at least the second level of limit orders in an order book. Using tick-by-tick data on Euronext-traded stocks, we show that a simple bivariate Hawkes process fits nicely our empirical observations of trades-through. We show that the cross-influence of bid and ask trades-through is weak.

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Correspondence Fabrizio Pomponio, Chair of Quantitative Finance, MAS Laboratory, Ecole Centrale Paris, Grande Voie des Vignes, 92295 Châtenay-Malabry Cedex, France; e-mail: fabrizio.pomponio@ecp.fr

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Introduction

Recent contributions have emphasized that Hawkes processes exhibit interesting features for financial modelling. For example, these self- and mutually exciting point processes can model arrival times of orders in an order book model (Large (2007); Muni Toke (2011)), or explain the Epps effect in a microstructure toy model (Bacry et al. (2011)). A comprehensive econometric framework can be derived (Bowsher (2007)).

In this paper, we are interested in modelling trades-through, i.e. transactions that reach at least the second level of limit orders in an order book. Trades-through are very important in price formation and microstructure. Since traders usually minimize their market impact by splitting their orders according to the liquidity available in the order book, trades-through may contain information. They may also reach gaps in orders books, which is crucial in price dynamics.

In a first part, we give basic statistical facts on trades-through, focusing on their arrival times and clustering properties. Our second part is a general introduction to Hawkes processes. In a third part, using tick-by-tick data on Euronext-traded stocks, we show that a simple bi-dimensional Hawkes process fits nicely our empirical data of trades-through. We show that the cross-influence of bid and ask trades-through is weak. Following Bowsher (2007), we improve the statistical performance of our maximum likelihood calibrations by enhancing the stationary model using deterministic time-dependent base intensity.

1 Trades-through

1.1 Orders splitting and trades-through

It has been shown several times that the times series built from trading flows are long-memory processes (see e.g. Bouchaud et al. (2009)). Lillo and Farmer (2004) argues that this is mainly explained by the splitting of large orders. Indeed, let us assume that a trader wants to trade a large order. He does not want to reveal its intentions to the markets, so that the price will not "move against him". If he were to submit one large market order, he would eat the whole liquidity in the order book, trading at the first limit, then the second, then the third, and so on. When "climbing the ladder" this way, the last shares would be bought (resp. sold) at a price much higher (resp. lower) than the first ones. This trader will thus split its large order in several smaller orders that he will submit one at a time, waiting between each submitted order for some limit orders to bring back liquidity in the order book. We say that the trader tries to minimize its market impact.

Limit number considered	Number of trades-through per day (all)	Number of trades-through per day (bid side)	Number of trades-through per day (ask side)
2	829.0	401.8	427.2
3	124.1	59.0	65.1
4	30.5	14.6	15.9

Table 1: Occurrences of trades-through at bid and ask sides for BNP Paribas.

In practice, this mechanism is widely used: traders constantly scan the limit order book and very often, if not always, restrict the size of their orders to the quantity available at the best limit. But sometimes speed of execution is more important than minimizing market impact. In this case, orders larger than the size of the first limit may be submitted: thus, trades-through are precisely the trades that stand outside the usual trading pattern, and as such are worth being thoroughly studied.

Trades-through have already been empirically studied in Pomponio and Abergel (2010): their occurrences, links with big trades, clustering, intraday timestamps distribution, market impact, spread relaxation and use in lead-lag relation. In this paper, we model trades-through with Hawkes processes.

1.2 Definition of trades-through

In general, we call a n-th limit trade-through any trade that consumes at least one share at the n-th limit available in the order book. For example, a second limit trade-through completely consumes the first limit available and begins to consume the second limit of the order book. Our definition is inclusive in the sense that, if p is greater than q, any p-th limit trade-through is also part of the q-th limit trades-through. In this study, we will focus on second limit trades-through, and simply call them trades-through in what follows. Figure 1 shows an example of trade-through.

1.3 Occurrences of trades-through

Here, we look at the occurrences of trades-through on the different sides of the order book. Basic statistics are given in table 1. These statistics are computed using Thomson-Reuters tick-by-tick data of the Euronext-Paris limit order book for the stock BNP Paribas (BNPP.PA) from June 1st 2010 to October 29th 2010. We can see that for second limit trades-through, there are around 400 events per day on each side of the book.

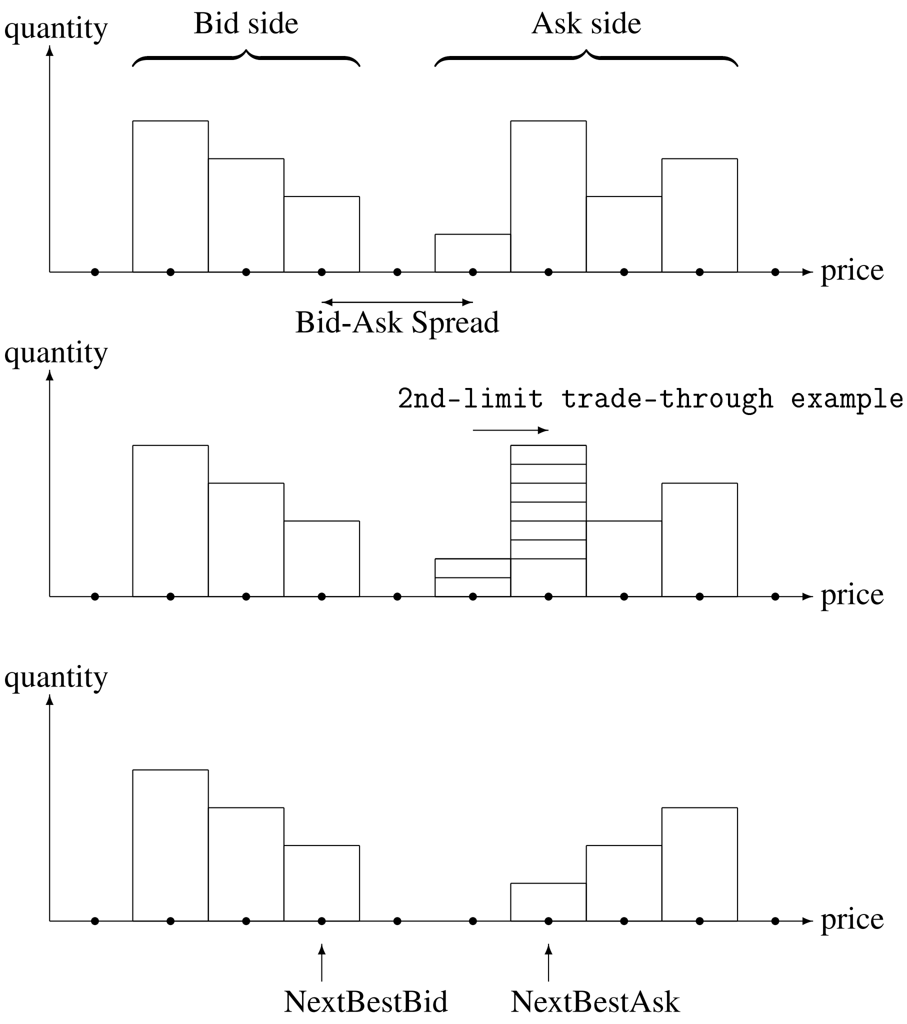

Figure 1: Example of a trade-through: *(up)* Limit order book configuration before the trade-through; *(middle)* Trade-through; *(down)* Limit order book configuration after the trade-through.

Impact studied	Mean waiting time until next trade-through (in seconds)
$(\lambda^+ + \lambda^-) \rightarrow (\lambda^+ + \lambda^-)$	36.9
$(\Lambda^+ + \Lambda^-) \rightarrow (\lambda^+ + \lambda^-)$	51.8
$(\lambda^+) \rightarrow (\lambda^+ + \lambda^-)$	36.3
$(\Lambda^+) \rightarrow (\lambda^+ + \lambda^-)$	51.7
$(\lambda^-) \rightarrow (\lambda^+ + \lambda^-)$	37.5
$(\Lambda^-) \rightarrow (\lambda^+ + \lambda^-)$	51.7
$(\lambda^+) \rightarrow (\lambda^+)$	76.1
$(\Lambda^+) \rightarrow (\lambda^+)$	107.9
$(\lambda^-) \rightarrow (\lambda^-)$	71.6
$(\Lambda^-) \rightarrow (\lambda^-)$	98.1
$(\lambda^+) \rightarrow (\lambda^-)$	80.4
$(\Lambda^+) \rightarrow (\lambda^-)$	101.8
$(\lambda^-) \rightarrow (\lambda^+)$	91.1
$(\Lambda^-) \rightarrow (\lambda^+)$	111.6

Table 2: Clustering of trades-through on bid and ask sides (on BNP Paribas data).

1.4 Clustering

Trades-through are clustered both in physical time and in trade time (see Pomponio and Abergel (2010)). Here we study in detail several aspects of this problem that will be helpful for further modelling: is the global clustering of trades-through still true when looking only at one side of the book? If so, is there an asymmetry in trades-through clustering at the bid and at the ask sides? Is there a cross-side effect for trades-through, in other words will a trade-through on one side of the book be followed more rapidly than usual by a trade-through on the other side of the book? Which is the stronger from those different effects?

In order to grasp the clustering of trades-though, we compute the mean of the distribution of waiting times between two consecutive trades-through, and we compare it with the mean waiting time between one trade (of any kind) and the next trade-through.

Table 2 summarizes our result on BNP Paribas stock in the considered period of study. We use the notation λ when looking at trades-through and Λ when looking at all the trades. When a specific side of the book is under scrutiny we mention it with a $+$ for ask side and a $-$ for bid side. For example, $(\Lambda^+) \rightarrow (\lambda^+ + \lambda^-)$ means that we look at the time interval between a trade at the ask side and the next trade-through, whatever its sign.

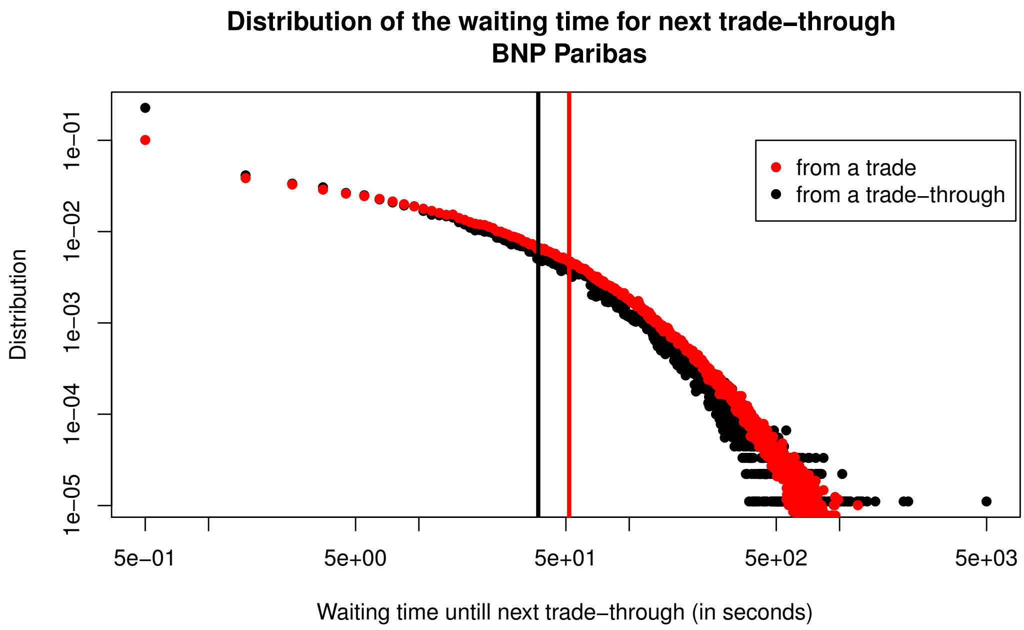

Figure 2: Global trades-through clustering for BNP Paribas.

Analysing the first group of statistics ($(\lambda^+ + \lambda^-) \to (\lambda^+ + \lambda^-)$ and $(\Lambda^+ + \Lambda^-) \to (\lambda^+ + \lambda^-)$), we see that previous result on global clustering of trades-through is confirmed: you wait less the next trade-through when you already are on a trade-through, compared to when you are on a trade. Moreover, when looking at the second group of statistics, we see there is no asymmetry in this effect: both trades-through at the ask and at the bid are more closely followed in time by trades-through (whatever their sign), than trades at the bid and trades at the ask are.

The third group of statistics indicates that if you restrict the study to only one side of the book, the clustering is still valid. Finally, the fourth group of statistics shows that there seems to be a cross-side effect of clustering of trades-through: a trade-through at one side of the book will be more closely followed in time by a trade-through on the other side of the book. But comparing the relative difference between mean waiting times of $(\lambda^+) \to (\lambda^+)$ and $(\Lambda^+) \to (\lambda^+)$, we have approximately a 30% decrease on the same side of the book. Whereas there is only a 20% decrease of mean waiting time between $(\lambda^+) \to (\lambda^-)$ and $(\Lambda^+) \to (\lambda^-)$, which reflects that cross-side clustering effect is weaker than same side clustering for trades-through.

Figure 2 plots the distributions of waiting times $(\lambda^+ + \lambda^-) \to (\lambda^+ + \lambda^-)$ and $(\Lambda^+ + \Lambda^-) \to (\Lambda^+ + \Lambda^-)$ studied in this paragraph.

In brief, looking at these distributions of durations gives us global tendencies on clustering and relative comparisons of the influences of trades-through with respect to limit order book sides. A more quantitative measurement of those effects will be done in the following part using the analysis of calibrated parameters of an adapted stochastic model, namely Hawkes processes.

1.5 Intraday timestamp distribution

We also look at the intraday distribution of timestamps for second-limit trades-through on BNP Paribas stock. We can see that the distribution is globally the sum of two parts: a U-shape curve (linked to the global U-shape trading activity curve) and two peaks at very precise hours (2:30 pm and 4:00 pm - Paris time) reflecting the impact of major macro-economic news released at that moment of the day.

What is important for further modelling is to notice that it seems very difficult to find a pure stochastic model able to capture both the local behaviour and fluctuations of trades-through arrival times and the two big peaks at very precise hours of the day. A first attempt may be to simply remove those peaks in the distribution. In the remaining of the paper, we will restrict ourselves to a two-hour interval, thus removing major seasonality effects.

2 Hawkes processes

Let us first recall standard definitions and properties of Hawkes processes. These processes were introduced by Hawkes (1971) as a special case of linear self-exciting processes with an exponentially decaying kernel.

2.1 Definition

Let $M \in \mathbb{N}^*$. Let $\{(t_i^m)_i\}_{m=1,\dots,M}$ be a M-dimensional point process. We will denote $\mathbf{N}_t = (N_t^1, \dots, N_t^M)$ the associated counting process. A multidimensional Hawkes process is defined with intensities $\lambda^m, m = 1, \dots, M$ given by:

$$\begin{aligned} \lambda^m(t) &= \lambda_0^m(t) + \sum_{n=1}^{M} \int_0^t \sum_{j=1}^{P} \alpha_j^{mn} e^{-\beta_j^{mn}(t-s)} dN_s^n, \\ &= \lambda_0(t) + \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_i^n < t} \alpha_j^{mn} e^{-\beta_j^{mn}(t-t_i^n)}, \end{aligned} \tag{1}$$

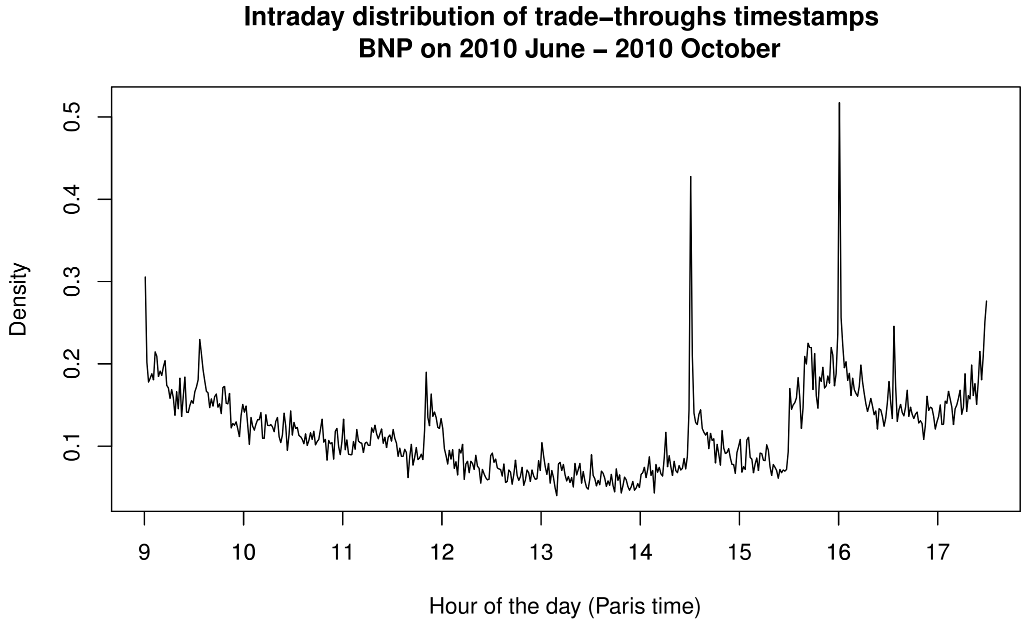

Figure 3: Intraday distribution of timestamps of trades-through for the stock BNP Paribas, using one-minute bins.

where the number P of exponential kernels is a fixed integer, and t_i^n is the i-th jumping time of the n-th variate. In its simplest version with $P = 1$ and $\lambda_0^m(t)$ constant, the definition becomes:

$$\begin{aligned} \lambda^m(t) &= \lambda_0^m + \sum_{n=1}^{M} \int_0^t \alpha^{mn} e^{-\beta^{mn}(t-s)} dN_s^n, \\ &= \lambda_0 + \sum_{n=1}^{M} \sum_{t_i^n < t} \alpha^{mn} e^{-\beta^{mn}(t-t_i^n)}. \end{aligned} \tag{2}$$

Parameters α^{mn} and β^{mn} express the influence (scale and decay) of the past events t_i^n of type n on the m-th coordinate of the process. It follows from this definition that two phenomena are present: self-excitation ($m = n$) and mutual excitation ($m \neq n$).

2.2 Stationarity condition

Taking here $P = 1$ and rewriting equation (2) using vectors to simplify the notations, we have:

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}_0 + \int_0^t \mathbf{G}(t-s) d\mathbf{N}_s, \tag{3}$$

where

$$\mathbf{G}(t) = \left(\alpha^{mn} e^{-\beta^{mn}(t-s)}\right)_{m,n=1,\dots,M}. \tag{4}$$

Assuming stationarity gives $\mathrm{E}[\boldsymbol{\lambda}(t)] = \boldsymbol{\mu}$ constant vector, so that stationary intensities must satisfy:

$$\boldsymbol{\mu} = \left(\mathbf{I} - \int_0^\infty \mathbf{G}(u)du\right)^{-1} \boldsymbol{\lambda}_0 \tag{5}$$

Therefore, a sufficient condition for the process to be linear is that the spectral radius of the matrix

$$\boldsymbol{\Gamma} = \int_0^\infty \mathbf{G}(u)du = \left(\frac{\alpha^{mn}}{\beta^{mn}}\right)_{m,n=1,\dots,M} \tag{6}$$

is strictly smaller than 1. We recall that the spectral radius of the matrix $\mathbf{G}$ is defined as:

$$\rho(\mathbf{G}) = \max_{a \in \mathscr{P}(\mathbf{G})} |a|, \tag{7}$$

where $\mathscr{P}(\mathbf{G})$ denotes the set of all eigenvalues of $\mathbf{G}$.

In the one-dimensional case, we obtain the sufficient stationarity condition $\frac{\alpha_1}{\beta_1} < 1$. In the two-dimensional case, this stability condition can be written:

$$\frac{1}{2}\left(\frac{\alpha^{11}}{\beta^{11}} + \frac{\alpha^{22}}{\beta^{22}} + \sqrt{\left(\frac{\alpha^{11}}{\beta^{11}} - \frac{\alpha^{22}}{\beta^{22}}\right)^2 + 4\frac{\alpha^{12}}{\beta^{12}}\frac{\alpha^{21}}{\beta^{21}}}\right) < 1. \tag{8}$$

Note that this result can also be seen as a particular (linear) case of (Brémaud, 1996, Theorem 7) which deals with general non-linear Hawkes processes.

2.3 Maximum-likelihood estimation

Let $\{t_i\}_{i=1,\dots,N}$ be the ordered pool of all events $\left\{\{t_i^m\}_{m=1,\dots,M}\right\}$. The log-likelihood of a multidimensional Hawkes process can be computed as the sum of the likelihood of each coordinate, i.e. is written:

$$\ln \mathscr{L}(\{t_i\}_{i=1,\dots,N}) = \sum_{m=1}^{M} \ln \mathscr{L}^m(\{t_i\}), \tag{9}$$

where each term is defined by:

$$\ln \mathscr{L}^m(\{t_i\}) = \int_0^T (1 - \lambda^m(s))\, ds + \int_0^T \ln \lambda^m(s) dN^m(s). \tag{10}$$

This partial log-likelihood can be computed as:

$$\begin{aligned} \ln \mathscr{L}^m(\{t_i\}) &= T - \Lambda^m(0,T) \\ &+ \sum_{i=1}^{N} z_i^m \ln \left[\lambda_0^m(t_i) + \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_k^n < t_i} \alpha_j^{mn} e^{-\beta_j^{mn}(t_i - t_k^n)} \right], \end{aligned} \tag{11}$$

where $\Lambda^m(0,T) = \int_0^T \lambda^m(s)ds$ is the integrated intensity, and z_i^m is equal to 1 if the event t_i is of type m, 0 otherwise. Following Ozaki (1979), we compute this in a recursive way by observing that:

$$\begin{aligned} R_j^{mn}(l) &= \sum_{t_k^n < t_l^m} e^{-\beta_j^{mn}(t_l^m - t_k^n)} \\ &= \begin{cases} e^{-\beta_j^{mn}(t_l^m - t_{l-1}^m)} R_j^{mn}(l-1) + \sum_{t_{l-1}^m \le t_k^n < t_l^m} e^{-\beta_j^{mn}(t_l^m - t_k^n)} & \text{if } m \neq n, \\ e^{-\beta_j^{mn}(t_l^m - t_{l-1}^m)} \left(1 + R_j^{mn}(l-1)\right) & \text{if } m = n. \end{cases} \end{aligned} \tag{12}$$

The final expression of the partial log-likelihood may thus be written:

$$\begin{aligned} \ln \mathscr{L}^m(\{t_i\}) &= T - \Lambda^m(0,T) - \sum_{i=1}^{N} \sum_{n=1}^{M} \sum_{j=1}^{P} \frac{\alpha_j^{mn}}{\beta_j^{mn}} \left(1 - e^{-\beta_j^{mn}(T - t_i)}\right) \\ &+ \sum_{t_l^m} \ln \left[\lambda_0^m(t_l^m) + \sum_{n=1}^{M} \sum_{j=1}^{P} \alpha_j^{mn} R_j^{mn}(l) \right], \end{aligned} \tag{13}$$

where $R_j^{mn}(l)$ is defined with equation (12) and $R_j^{mn}(0) = 0$.

2.4 Testing the calibration

A general result on point processes states that it can be transformed into a homogeneous Poisson process by a stochastic time change. More precisely, let N be a point process on $\mathbb{R}_+$ such that $\int_0^\infty \lambda(s)ds = \infty$, and let t_τ be the stopping time defined by

$$\int_0^{t_\tau} \lambda(s)ds = \tau. \tag{14}$$

Then the process $\tilde{N}(\tau) = N(t_\tau)$ is a homogeneous Poisson process with constant intensity $\lambda = 1$. A general proof of such a result can be found in (Bremaud, 1981, Theorem 16) for example.

Bowsher (2007) has shown that this can be generalized for Hawkes processes in a multidimensional settings. Let us compute the integrated intensity of the m-th coordinate of a multidimensional Hawkes process between two consecutive events t^m_{i-1} and t^m_i of type m:

$$\begin{aligned}
\Lambda^m(t^m_{i-1}, t^m_i) &= \int_{t^m_{i-1}}^{t^m_i} \lambda^m(s)ds \\
&= \int_{t^m_{i-1}}^{t^m_i} \lambda^m_0(s)ds + \sum_{n=1}^{M}\sum_{j=1}^{P}\sum_{t^n_k < t^m_{i-1}} \frac{\alpha^{mn}_j}{\beta^{mn}_j}\left[e^{-\beta^{mn}_j(t^m_{i-1}-t^n_k)} - e^{-\beta^{mn}_j(t^m_i - t^n_k)}\right] \\
&\quad + \sum_{n=1}^{M}\sum_{j=1}^{P}\sum_{t^m_{i-1}\leq t^n_k < t^m_i} \frac{\alpha^{mn}_j}{\beta^{mn}_j}\left[1 - e^{-\beta^{mn}_j(t^m_i - t^n_k)}\right]. \qquad (15)
\end{aligned}$$

As in the log-likelihood computation, following Ozaki (1979) we observe that:

$$\begin{aligned}
A^{mn}_j(i-1) &= \sum_{t^n_k < t^m_{i-1}} e^{-\beta^{mn}_j(t^m_{i-1}-t^n_k)} \\
&= e^{-\beta^{mn}_j(t^m_{i-1}-t^m_{i-2})} A^{mn}_j(i-2) + \sum_{t^m_{i-2}\leq t^n_k < t^m_{i-1}} e^{-\beta^{mn}_j(t^m_{i-1}-t^n_k)}, \qquad (16)
\end{aligned}$$

so that the integrated density can be written $\forall i \in \mathbb{N}^*$:

$$\begin{aligned}
\Lambda^m(t^m_{i-1}, t^m_i) &= \int_{t^m_{i-1}}^{t^m_i} \lambda^m_0(s)ds + \sum_{n=1}^{M}\sum_{j=1}^{P} \frac{\alpha^{mn}_j}{\beta^{mn}_j}\Bigg[A^{mn}_j(i-1)\left(1 - e^{-\beta^{mn}_j(t^m_i - t^m_{i-1})}\right) \\
&\quad + \sum_{t^m_{i-1}\leq t^n_k < t^m_i}\left(1 - e^{-\beta^{mn}_j(t^m_i - t^n_k)}\right)\Bigg], \qquad (17)
\end{aligned}$$

where A is defined as in equation (16) with $\forall j, A^{mn}_j(0) = 0$.

Hence following Bowsher (2007), we can easily define tests to check the goodness-of-fit of a Hawkes model to our empirical data. Since the integrated intensity $\Lambda^m(t^m_{i-1}, t^m_i)$ is a time interval of a homogeneous Poisson Process, we can test for each $m = 1, \ldots, M$:

1. whether the variables $\left(\Lambda^m(t^m_{i-1}, t^m_i)\right)_{i\geq 0}$ are exponentially distributed ;
2. whether the variables $\left((\Lambda^m(t^m_{i-1}, t^m_i)\right)_{i\geq 0}$ are independent.

In section 3.2, the independence test will be carried out with a Ljung-Box test up to the twentieth term, and we will use a standard Kolmogorov-Smirnov test the empirical data against the exponential distribution.

Having all these results at hand, we can now turn to the modelling of trades-through in an order book model.

3 A simple Hawkes model for trades-through

3.1 Model

Since empirical evidence shows that trades-through obviously occur in a clustered way, it makes sense to try to model them with self-exciting Hawkes processes. We thus define our basic model as follows. Let $\left(t_i^A\right)_{i\geq 1}$ be the point process of trades-through occurring on the ask side of the limit order book, and $\left(t_i^B\right)_{i\geq 1}$ be the point process of trades-through occurring on the bid side. Let N^A and N^B denote the associated counting processes. These two processes are assumed to form a two-dimensional Hawkes process with intensities λ^A and λ^B defined with parameters $\left(\alpha_{ij}, \beta_{ij}\right)_{(i,j)\in\{A,B\}^2}$ as follows:

$$\begin{aligned}
\lambda^A(t) &= \lambda_0^A(t) + \int_0^t \alpha_{AA} e^{-\beta_{AA}(t-s)} dN_s^A + \int_0^t \alpha_{AB} e^{-\beta_{AB}(t-s)} dN_s^B, \\
\lambda^B(t) &= \lambda_0^B(t) + \int_0^t \alpha_{BA} e^{-\beta_{BA}(t-s)} dN_s^A + \int_0^t \alpha_{BB} e^{-\beta_{BB}(t-s)} dN_s^B. \qquad (18)
\end{aligned}$$

This is a standard bivariate Hawkes model of section 2 with $P = 1$.

3.2 Calibration

Empirical data

We use Thomson-Reuters tick-by-tick data of the Euronext-Paris limit order book for the stock BNP Paribas (BNPP.PA) from June 1st, 2010 to October 29th, 2010, i.e. 109 trading days. This data gives us trades (timestamp to the millisecond, volume and price) and quotes (volume, price, side of the order book) for the stock, from the opening to the close of the market. For each trading day, we extract the series of timestamps $\left(t_i^A\right)_{i\geq 1}$ and $\left(t_i^B\right)_{i\geq 1}$ of the trades-through. In what follows, we will restrict the study to a two-hour time interval, from 9:30 am to 11:30 am. During these five months of trading, we count in average each day during this time interval 2737 trades, 206 of which are trades-through (100 on the ask side and 106 on the bid side). Thus roughly 8% of the recorded transactions are trades-through.

Calibration results

Using computations presented in section 2, we compute the maximum-likelihood estimates for the parameters of our model. Taking into account the huge variations of trading activity during the day, we restrict our empirical observations to a two-hour interval, from 9:30 am to 11:30 am. This may hopefully make the stationarity assumption of section 2.2 with which we work more realistic. In a first step, we also make the assumptions that base intensities λ_0^A and λ_0^B are constants. Tables

	λ_0^A	α_{AA}	α_{AB}	β_{AA}	β_{AB}
Average	1.01E-02	4.13E+00	4.33E-01	3.70E+01	2.48E+01
Median	8.42E-03	6.45E-01	1.05E-01	4.78E+00	1.33E+00
Min	6.62E-06	3.53E-02	1.00E-10	1.84E-01	1.00E-10
Max	3.52E-02	3.09E+01	4.78E+00	2.34E+02	1.48E+03
Stdev	6.27E-03	6.03E+00	8.41E-01	5.21E+01	1.44E+02

Table 3: Statistics summary for the maximum-likelihood estimates of the ask side of model (18).

	λ_0^B	α_{BA}	α_{BB}	β_{BA}	β_{BB}
Average	1.09E-02	3.68E-01	4.81E+00	9.61E+00	3.98E+01
Median	9.08E-03	7.56E-02	3.04E+00	1.46E+00	2.00E+01
Min	2.46E-03	3.83E-13	2.08E-02	1.00E-10	3.71E-02
Max	3.98E-02	4.46E+00	4.62E+01	1.00E+02	3.75E+02
Stdev	6.35E-03	6.77E-01	7.00E+00	1.92E+01	5.52E+01

Table 4: Statistics summary for the maximum-likelihood estimates of the bid side of model (18).

3 and 4 summarize the statistics on the estimated values on the ask and bid sides. These tables shows that the median half-lives associated to the kernels AA, AB, BA and BB are respectively 145, 521, 474 and 35 milliseconds.

It appears that we observe very large variations in the results of the numerical maximization of the likelihood. However, whatever the absolute size of the parameters, it is clear that the cross-excitation effect, i.e. the excitation of trades-through of a given side by the occurrence of trades-through on the opposite side, is much weaker than the self-excitation effect, which translates the clustering of trades-through on a given side. The average value of α_{AB} is 9.5 times smaller that the average value of α_{AA}, while at the same time the associated exponential decay β_{AB} is only 1.5 times smaller than the average β_{AA}. The instantaneous effect is thus much smaller while its half-life is not significantly longer. This observation is also valid for the average α_{BA} which is 13 times smaller than the average α_{BB}, while the average exponentials decays differ only by a factor 4.

In other words, the ratio $\frac{\alpha}{\beta}$, which is equal to the total integrated intensity of an exponential kernel $\int_0^{+\infty} \alpha e^{-\beta u} du$, is much weaker in the cross-excitation cases (taking the average values, $\frac{\alpha_{AB}}{\beta_{AB}} = 0.017$, $\frac{\alpha_{BA}}{\beta_{BA}} = 0.038$) than in the self-excitation cases (still using the average values, $\frac{\alpha_{AA}}{\beta_{AA}} = 0.111$, $\frac{\alpha_{BB}}{\beta_{BB}} = 0.120$). Therefore, we can focus on the calibration and use of a simpler model, where trades-through are

	λ_0^A	α_{AA}	β_{AA}	λ_0^B	α_{BB}	β_{BB}
Average	1.18E-02	6.02E+00	4.76E+01	1.26E-02	8.05E+00	6.64E+01
Median	9.77E-03	4.85E+00	3.63E+01	1.13E-02	5.61E+00	4.72E+01
Min	3.47E-03	1.00E-10	3.92E-02	3.80E-03	1.28E-02	2.64E-02
Max	3.66E-02	3.09E+01	2.34E+02	4.25E-02	4.79E+01	3.91E+02
Stdev	6.53E-03	6.14E+00	4.87E+01	7.00E-03	8.81E+00	7.22E+01

Table 5: Statistics summary for the maximum-likelihood estimates of the simplified model (19) with λ_0^A and λ_0^B constant.

modelled by two one-dimensional Hawkes processes, with no cross-excitation:

$$\begin{aligned}
\lambda^A(t) &= \lambda_0^A(t) + \int_0^t \alpha_{AA} e^{-\beta_{AA}(t-s)} dN_s^A, \\
\lambda^B(t) &= \lambda_0^B(t) + \int_0^t \alpha_{BB} e^{-\beta_{BB}(t-s)} dN_s^B.
\end{aligned} \tag{19}$$

Table 5 summarizes the statistics of the estimated values of this simplified model with the assumption λ_0^A and λ_0^B constant. Values are similar to the previous case, confirming that the cross-effects were negligible. The effect of this simplification will be further discussed with the goodness-of-fit tests.

Finally, in an attempt to grasp small variations of activity independent of the clustering of the trades-through, following ideas presented in Bowsher (2007), we test a third version of the model by getting rid of the assumptions stating that λ_0^A and λ_0^B are constants. In this version of the simplified model (19), base intensities $\lambda_0^A(t)$ and $\lambda_0^B(t)$ are piecewise-linear continuous functions on the subdivision $(9:30 < 10:00 < 10:30 < 11:00 < 11:30)$ of the time interval $[9:30am; 11:30am]$. Note that this assumption implies that the process is not stationary anymore. Tables 6 and 7 summarize the statistics on the estimated values on the ask and bid sides. Let us now discuss the goodness-of-fit of these three calibrations.

Goodness-of-fit

For each trading days, we have extracted the time series $\left(t_i^A\right)_{i\geq 1}$ and $\left(t_i^B\right)_{i\geq 1}$. For each of the three models discussed above and for each trading day, we can compute the integrated intensities $\left(\Lambda^A(t_i^A, t_{i+1}^A)\right)_{i\geq 1}$ and $\left(\Lambda^A(t_i^B, t_{i+1}^B)\right)_{i\geq 1}$ defined as in (17) and perform the four tests of goodness-of-fit described in section 2. This gives us four tests per model and per trading day. Table 8 shows the results of the tests for a risk of the first kind equal to 1% and 2.5%. These results confirm that the cross-excitation of trades-through of one side of the book on the other side is weak. In the case where λ_0 is constant, the percentage of trading days where the

	$\lambda_0^A(9:30)$	$\lambda_0^A(10:00)$	$\lambda_0^A(10:30)$	$\lambda_0^A(11:00)$	$\lambda_0^A(11:30)$	α_{AA}	β_{AA}
Average	1.94E-02	1.13E-02	1.33E-02	7.67E-03	1.32E-02	6.62E+00	5.64E+01
Median	1.65E-02	9.81E-03	1.25E-02	4.97E-03	1.03E-02	5.10E+00	4.61E+01
Min	1.00E-20	1.00E-20	1.00E-20	1.00E-20	1.00E-20	3.64E-13	1.00E-20
Max	5.40E-02	3.72E-02	5.01E-02	5.45E-02	1.54E-01	3.09E+01	2.34E+02
Stdev	1.29E-02	9.73E-03	9.30E-03	9.93E-03	1.68E-02	6.25E+00	5.14E+01

Table 6: Statistics summary for the maximum-likelihood estimates of the ask side of model (19) with λ_0^A and λ_0^B piecewise-linear continuous functions.

	$\lambda_0^B(9:30)$	$\lambda_0^B(10:00)$	$\lambda_0^B(10:30)$	$\lambda_0^B(11:00)$	$\lambda_0^B(11:30)$	α_{BB}	β_{BB}
Average	1.99E-02	1.25E-02	1.26E-02	9.32E-03	1.33E-02	8.20E+00	6.82E+01
Median	1.67E-02	1.06E-02	1.14E-02	7.77E-03	9.16E-03	5.60E+00	5.15E+01
Min	1.35E-03	1.00E-20	1.00E-20	1.00E-20	7.06E-14	8.70E-04	1.25E-03
Max	6.51E-02	5.38E-02	5.15E-02	5.65E-02	1.26E-01	4.79E+01	3.91E+02
Stdev	1.31E-02	1.00E-02	9.18E-03	9.65E-03	1.53E-02	8.68E+00	6.98E+01

Table 7: Statistics summary for the maximum-likelihood estimates of the bid side of model (19).

Model	Performance	2.5%		1%	
Full Model (18) with λ_0 constant	4 passed	70	(64.2)	83	(76.1)
	3 passed	29	(26.6)	26	(23.9)
	2 or less	10	(9.2)	0	(0.0)
No Cross (19) with λ_0 constant	4 passed	59	(54.1)	77	(70.6)
	3 passed	35	(32.1)	25	(22.9)
	2 or less	15	(13.8)	7	(6.4)
Full Model (18) with λ_0 piecewise-linear	4 passed	84	(77.1)	94	(86.2)
	3 passed	20	(18.3)	13	(11.9)
	2 or less	5	(4.6)	2	(1.8)
No Cross (19) with λ_0 piecewise-linear	4 passed	83	(76.1)	95	(87.2)
	3 passed	20	(18.3)	14	(12.8)
	2 or less	6	(5.5)	0	(0.0)

Table 8: Performance of the calibration of the Hawkes models. For each model, this table gives the number of trading days (out of 109) where 4, 3, or 2 or less tests out of for where successfully passed. The four tests are two independence Ljung-Box tests and two Kolmogorov-Smirnov tests for the exponential distribution. Values in parentheses are percentages.

model passes all 4 statistical tests is 76% in the full specification case, and stays at 71% when cross-excitation is not taken into account. And in the case where λ_0 is allowed to vary as a piecewise-linear continuous function, these two percentages are even equal: in this latter case, we don't have any statistical improvement by including the cross-excitation effect.

Moreover, these results show that adding more flexibility in the modelling of λ_0 using piecewise-linear continuous functions helps the model to grasp the dynamics of trades-through: all tests are passed in more that 87% of the trading days tested in both cases.

Conclusion

We have studied in this paper a model for trades-through based on Hawkes processes. We have shown that the clustering properties of trades-through can be well modelled with such self-exciting processes. Although calibration results may vary a lot from trading day to trading day, general patterns remain, such as the weak cross-excitation effects. Therefore such a model might be used to build indicators on the durations between observed trades-through. We will thus use our model to study trading strategies in a future work.

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