

Alan Greenspan, the Quants and Stochastic Optimal Control

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Abstract Alan Greenspan's paper (March 2010) presents his retrospective view of the crisis. His theme has several parts. First, the housing price bubble, its subsequent collapse and the financial crisis were not predicted either by the market, the FED, the IMF or the regulators in the years leading to the current crisis. Second, financial intermediation tried to function on too thin layer of capital—high leverage—owing to a misreading of the degree of risk embodied in ever more complex financial products and markets. Third, the breakdown was unpredictable and inevitable, given the “excessive” leverage—or low capital—of the financial intermediaries. The proposed legislation for the “reform” of the financial system requires that the FED “identify, measure, manage and mitigate risks to the financial stability of the United States”. The focus is upon capital requirements or debt ratios. The “Quants” ignored systemic risk and just focused upon risk transfer in very liquid markets.

The FED, IMF, Treasury and the “Quants”/market lacked the appropriate tools of analysis to answer the following questions: what is an optimal leverage or capital requirement that balances the expected growth against risk? What are theoretically founded early warning signals of a crisis? The author explains why the application of stochastic optimal control (SOC)/dynamic risk management is an effective approach to determine the optimal degree of leverage, the optimum and excessive risk and the probability of a debt crisis. The theoretically derived early warning signal of a crisis is the excess debt ratio, equal to the difference between the actual and optimal ratio. The excess debt starting from 2004–05 indicated that a crisis was most likely. This SOC analysis should be used by those charged with surveillance of financial markets.

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ALAN GREENSPAN AND STOCHASTIC OPTIMAL CONTROL

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1. Motivation of Paper

1.1. Alan Greenspan's Retrospective

Prior to the subprime crisis of 2007, there was a false sense of safety in financial markets. Greenspan said (2004) that the rise in home values was “not enough in our judgment to raise major concerns”. Ben Bernanke said (2005) that a housing bubble was a “pretty unlikely possibility”. Moreover he said (2007) that the Fed does “not expect significant spillovers from the subprime market to the rest of the economy”.

Greenspan's paper (March 2010) presents his retrospective view of the crisis. His theme has several parts. First, in the years leading to the current crisis, financial intermediation tried to function on too thin layer of capital – high leverage – owing to a misreading of the degree of risk embodied in ever more complex financial products and markets. Second, the breakdown was unpredictable and inevitable, given the “excessive” leverage – or low capital – of the financial intermediaries.

The market and the FED did not consider these mortgages to be very risky. In February 2004, a few months before the FED formally ended a run of interest rate cuts, Greenspan told the Americans that they would be missing out if they failed to take advantage of cost-saving adjustable rate mortgages. In 2005, he trumpeted the expansion of the subprime mortgage market and argued that “Where once more-marginal applicants would simply have been denied credit lenders are now able to quite efficiently judge the risk posed by individual applicants and to price that risk appropriately”. By 2007, the yield spread (CCC bonds – 10 year US Treasury) fell to a record low.

Greenspan tries to qualify the situation. He wrote that almost all of his contacts in the financial market were aware of the growing risks. “Similarly in 2002, I expressed my concerns before the Federal Open Market Committee that ‘...our extraordinary housing boom ... financed by very large mortgage debt cannot continue indefinitely.’ It lasted until 2006”.

The failure to realize that there was an unsustainable bubble that would damage the world economy was pervasive. As late as April 2007, the IMF noted that “...global

economic risks *declined* since...September 2006...The overall US economy is holding up well ...[and] the signs elsewhere are very encouraging.” The UK Financial Services was unable to anticipate and prevent the bank run that threatened Northern Rock. The venerated credit rating agencies bestowed credit ratings that implied Aaa smooth sailing for many a highly toxic derivative product.

In 2008 Greenspan said “Those of us who have looked to the self-interest of lending institutions to protect stockholders’ equity, myself included, are in a state of disbelief”. In his retrospective he asks: could the breakdown have been prevented? The FED was lulled into complacency about a bursting of the bubble and its aftermath because of recent history. First, they anticipated that the decline in home prices would be gradual. Second, there were only modestly negative effects of the 1987 stock market crash. The injections of FED liquidity apparently helped stabilize the economy.

The failure to anticipate the bubble, its collapse and effect upon the economy stemmed from the absence of a theoretical model, with explanatory power, that measures what is an “excessive” debt or leverage that will raise the probability of a crisis. *Such a model must take into account that the future movements of key variables are stochastic, and optimally balances expected return against risk.* The appropriate technique for the analysis is stochastic optimal control (SOC). On the basis of the SOC analysis, I derive a theoretically founded Early Warning Signal (EWS). This EWS is the “excessive” debt, equal to the difference between the actual and optimal debt ratio, that would have predicted the crisis. Greenspan, Bernanke would have benefitted had their staff had the analytic tool developed here. It is hoped that the FED will not be like the ancien régime: “Ils n’ont rien appris, ni rien oublié”.

1.2. Outline

There are several parts, with subsections, to this paper. Part Two sets the stage for the use of SOC, which is the subject of Part Three. Section 2.1 explains the Greenspan-Rubin view that the FED should not be concerned with “bubbles”. This philosophy was embodied in the Jackson Hole Consensus, and was criticized by Otmar Issing, who was the Chief Economist of the European Central Bank. Section 2.2 explains why the market did not anticipate a housing bubble. Section 2.3 explains why the Quants and Rating

Agencies underestimated risk. The reasons are several. First, they assumed that the distribution of housing price changes based upon data from 1980 to 2007 would be stable. Second, they were not worried that mortgages were taken out to finance consumption without the ability to service the debt from income/ability to pay. They assumed that households could service the debt from capital gains. They ignored the requirement that there be “no free lunch” that the capital gains would continue to exceed the interest rate. Third, the Quants thought that they could tranche the CDO’s in such a way that the senior tranches would have very little risk. They ignored the likelihood of systemic risk. Part 2.3.1 concerns leveraging in the financial sector. It was leveraging that converted the collapse of the housing price crash into a financial crisis. A specific example is given in part 2.3.2 concerning the Atlas Fund. Part 2.4 describes the critics who warned of the collapse but whose warnings were disregarded by the FED and by the market.

The main issue is: what is an “excessive leverage”? In this case one wants to balance risk with the main function of an economic system which is to channel saving to investment to maximize the expected growth. This leads into part 3 that is an exposition of the Stochastic Optimal Control (SOC) approach. I focus upon the following questions: What is an optimal debt ratio or leverage? What are the consequences of having a debt ratio in excess of the optimal ratio? Part 4 concerns the use of the SOC analysis in predicting bubbles and derives Early Warning Signals of the crisis. Conclusions are in part 5.

2.1. The Jackson Hole Consensus

Otmar Issing, former Chief economist of the European Central Bank (ECB), discussed the Lessons to be learned by Central banks from the recent financial crisis. The main thrust of his argument was a criticism of the Jackson Hole Consensus (JHC, 2005) for the relation between asset price bubbles and the conduct of monetary policy.

During the boom years, abundant liquidity and low interest rates led to a situation of excessive risk taking and asset price bubbles. The JHC has been the prevailing regulatory approach taken by the FED. It is based upon three principles. Central banks: (1) should not target asset prices, (2) should not try to prick an asset price bubble, (3)

should follow a “mopping up” strategy after the bubble bursts by injecting enough liquidity to avoid serious effects upon the real economy. A justification for this policy was seen in the period 2000-02 with the collapse of the dot.com bubble. The “mopping up” seemed to work well and there were no serious effects upon the real economy from following the JHC.

Issing objects to the JHC because it constitutes an asymmetric approach. When asset prices rise, without inflationary effects measured by the CPI, this is deemed irrelevant for monetary policy. But when the bubble bursts, central banks must come to the rescue. This, he argues, produces a moral hazard. He notes that although the JHC strategy worked well in the 2000-02 period it should not have justified the assumption that it would work afterwards in other cases. The JHC strategy certainly did not work in the 2007-08 crisis that was precipitated by the bursting of the housing price bubble.

He wrote: Did we really need a crisis that brought the world to the brink of a financial meltdown to learn that the philosophy which was at the time seen as state of the art was in fact dangerously flawed?...we must conduct a thorough discussion as to appropriate strategy of central banks with respect to asset prices. ”

Issing favors giving the central banks a mandate for macro-prudential supervision, the proposal by the Larosière group. The ECB should be responsible for identifying macroeconomic imbalances and for issuing warnings and recommendations addressed to national policy makers. The “solution” proposed is one that monitors closely monetary and credit developments as the potential driving forces for consumer price inflation in the medium to short run. “As long as money and credit remain broadly controlled, the scope for financing unsustainable runs in asset prices should also remain limited.” He notes: ”numerous empirical studies have shown that almost all asset price bubbles have been accompanied, if not preceded by strong growth of credit and or money”.

However, these studies such as reported by the BIS (Borio and Lowe) are vague and inconclusive. Even their authors conclude that the existing literature provides little insight into the key question that is of concern to central banks and supervisory authorities: When should credit growth be judged “too fast”? Moreover, contrary to Issing, it is very difficult to find a relation between recent money growth and the 2007-08 financial crisis. The BIS makes suggestions for further research. (1) Such work should pay

greater attention to conceptual paradigms and be more closely tailored to the needs of policymakers: length of horizons in identifying cumulative processes, the use of ex-ante information, balancing type I/II errors. (2) The definition of financial strains should be examined more carefully. (3) There is a need for analytical research concerning the interaction between financial imbalances and the real economy.

2.2. Market Anticipations of the Housing – Mortgage Debt Crisis

It has been commonly asserted that root of the problem lies with the subprime mortgage market. It is more accurate to say that the subprime market was the trigger for the crisis, although any one link in the highly leveraged financial intermediaries could have precipitated the crisis, as explained in section (2.3) below. I now turn to the market anticipations of housing prices: the methods used and why they were so erroneous.

Gerardi et al (Brookings Papers) explore whether market participants could have or should have anticipated the large increase in foreclosures that occurred in 2007. They decompose the change in foreclosures into two components: the sensitivity of foreclosures to a change in housing prices times the change in housing prices. The authors conclude that investment analysts had a good sense of the sensitivity of foreclosures to a change in housing prices, but missed drastically the expected change in housing prices. The authors do not analyze whether housing was overvalued in 2005-06 or whether the housing price change was to some extent predictable.

The authors looked at the records of market participants from 2004-06 to understand why the investment community did not anticipate the subprime mortgage crisis. Several themes emerge. The first is that the subprime market was viewed as a great success story in 2005. Second, mortgages were viewed as lower risk because of their more stable prepayment behavior. Third, analysts used sophisticated tools but the sample space did not contain episodes of falling prices. Fourth, pessimistic feelings and predictions were subjective and not based upon quantitative analysis. Fifth, analysts were remarkably optimistic about Housing Price Appreciation (HPA).

Analysts who looked at past data on housing prices, such as the four-quarter appreciation, could construct the histogram below. This is taken from Stein (2010). In the aggregate, housing prices never declined from year to year during the period 1980q1-

2007q4. The mean appreciation was 5.4% pa with a standard deviation of 2.94% pa. The optimism could be understood if one asks: on the basis of this sample of 111 observations, what is the probability that housing prices will decline? Given the mean and standard deviation, there was only a 3% chance that prices would fall.

The best estimates of the analysts were that the rates of housing price appreciation CAPGAIN or HPA in 2005 - 2006 of 10 to 11 % per annum would be unlikely to be repeated but that it would revert to its longer term average. A Citi report in December 2005 stated that "...the risk of a national decline in home prices appears remote. The annual HPA has never been negative in the United States going back at least to 1992." *Therefore no mortgage crisis was anticipated.* There was no economic theory or analysis in this approach. It was simply a VaR value at risk implication from a sample based upon relatively recent data. More fundamentally, no consideration was given to the economic determinants of the probability distribution of capital gains or housing price appreciation.

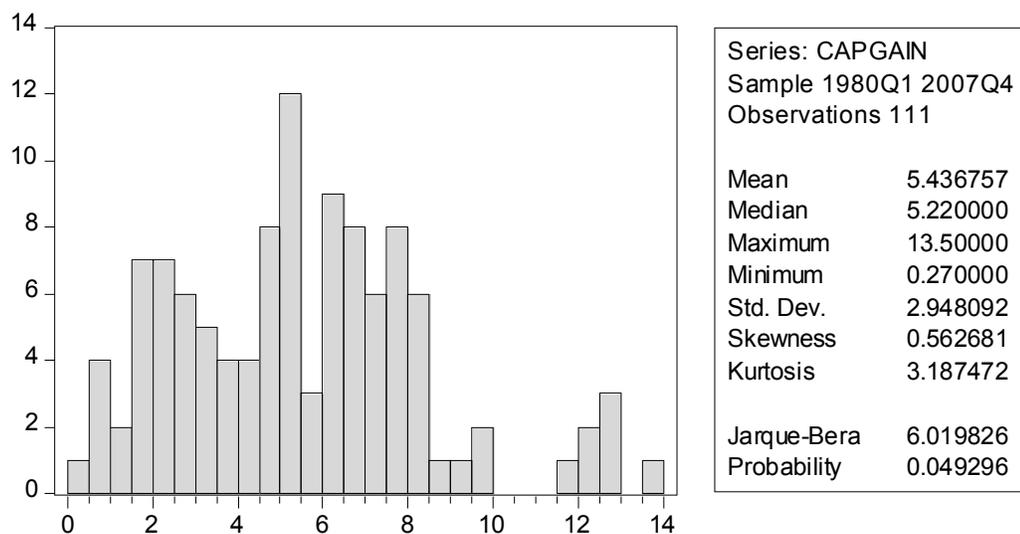


Figure 1. Histogram and statistics of CAPGAINS = Housing Price Appreciation HPA, the change from previous 4-quarter appreciation of US housing prices, percent/year, on horizontal axis. Frequency is on the vertical axis. Source of data: Office of Federal Housing Price Oversight. ADF(trend,intercept) = -2.09, Pr = 0.54.

2.3. The Quants, Securitization and Leverage

The financial innovation underlay Greenspan's (2005) view that "Where once more-marginal applicants would simply have been denied credit lenders are now able to quite efficiently judge the risk posed by individual applicants and to price that risk appropriately". By 2007, the yield spread (CCC bonds – 10 year US Treasury) fell to a record low.

The financial market consisted of several stages. At one end were the mortgagors, the households who borrowed against negligible collateral or ability to service the loans from income. Their loans were packaged or securitized into bundles by financial intermediaries who could not perform due diligence since they had no idea of the quality of the loans. In turn these packages were sold to institutional investors, who relied upon the rating agencies to evaluate risk. This method was used by FNMA and then followed by the private sector.

Since there was no way to evaluate the risk of a package of many mortgages of dubious quality, a prioritization method was used. Tranches of the pool were sold, like bonds, preferred stock and common stock. Losses or defaults would first affect the lowest tranche ("common stock") then when that tranche's assets were exhausted, the losses would affect the medium tranche ("preferred stock"). The top tranche ("bonds") would only lose if the losses exceeded the assets in all the tranches. Say that there were two tranches A(top) and B(bottom). The probability that there would be default of tranche B $\Pr(B)$ is $1 > p > 0$. That is the average probability of default of the entire package. The probability that owners of tranche A would suffer from defaults is if the losses exhausted both tranches. That is $\Pr(AB)$ the assets in both tranches defaulted.

The probability $\Pr(AB) = \Pr(A|B)\Pr(B)$. There are two relevant cases. (a) The defaults in the two tranches are independent. The conditional probability $\Pr(A|B) = \Pr(B) = p$. Then $\Pr(AB)$ that the senior tranche would be exhausted is $p^2 < p$. Risk of the senior tranche is less than that of the entire package. Taken to the extreme, let there be n *independent* tranches. Each has a probability of $1/2$ of defaulting. Then the probability that the senior (n -th) tranche defaults is equal to the probability that all " n " tranches default. This is $(1/2)^n$. The senior tranche is in effect a silk purse made from a sow's ear.

Case (b) is that the value/return of assets in each tranche are perfectly correlated.

Then the conditional probability that assets in tranche A defaults, given that B defaults, is $\Pr(A|B) = 1$. Then the probability that the owners of tranche A loses is $\Pr(AB) = 1 \cdot \Pr(B) = p$. The senior tranche is as risky as is the average of assets in the portfolio.

Tranching/securitization can allow risk reduction if there is no systemic risk, where $\Pr(A|B) = 1$. Since there are many assets in the CDO and no due diligence was performed, it is extremely difficult or impossible to evaluate the correlations or conditional probabilities. This was the impossible challenge for the rating agencies, discussed in part 2.3.2. The errors of the Quants were as follows. First, they assumed that the distribution of housing price changes based upon data from 1980 to 2007 would be stable. Second, they were not worried that mortgages were taken out to finance consumption without the ability to service the debt from income/ability to pay. They assumed that households could service the debt from capital gains. They ignored the economic fact that the entire structure of derivatives was based upon the ability of the mortgagors to service the debt. The financial structure characterized by securitization, absence of due diligence and leveraging would make the system unstable. I now turn to leveraging.

2.3.1. Leveraging

It is now widely believed that “excessive” leveraging, an “excessive” debt ratio, at key financial institutions helped convert the initial subprime turmoil in 2007 into a full blown financial crisis of 2008. Leverage is the ratio of debt $L(t)$ /net worth $X(t)$, alternatively called the debt ratio, and denoted $f(t) = L(t)/X(t)$. Although leverage is a valuable financial tool, “excessive” leverage poses a significant risk to the financial system. For an institution that is highly leveraged, changes in asset values highly magnify changes in net worth. To maintain the same debt ratio when asset values fall either the institution must raise more capital or it must liquidate assets.

The relations are seen through equations (i) – (iv). In (i) net worth $X(t)$ is equal to the value of assets $A(t)$ less debt $L(t)$. Equation (ii) is just a way of expressing the debt ratio. Equation (iii) relates the debt ratio $f(t) = L(t)/X(t)$ to the ratio $A(t)/X(t)$ of assets/net worth. Equation (iv) states that the percent change in net worth $dX(t)/X(t)$ is equal to the leverage $(1+f(t))$ times $dA(t)/A(t)$ the percent change in the value of assets.

- (i) $X(t) = A(t) - L(t)$.
- (ii) $L(t)/X(t) = f(t) = 1/[(A(t)/L(t) - 1]$.
- (iii) $A(t)/X(t) = 1 + f(t)$.
- (iv) $dX(t)/X(t) = (1 + f(t)) dA(t)/A(t)$.

The Congressional Oversight Panel COP reported that, on the basis of recent estimates just prior to the crisis, investment banks and securities firms, hedge funds, depository institutions, and the government sponsored mortgage enterprises – primarily Fanny Mae and Freddie Mac - held assets worth \$23 trillion on a base of \$1.9 trillion in net worth, yielding an overall average leverage of $A/X = 12$. The leverage ratio varied widely as seen below.

Broker-dealers and hedge funds	27
Government sponsored enterprises	17
Commercial banks	9.8
Savings Banks	6.9
Average	12

Consider the average, where $A(t) = \$23$ trillion, $X(t) = \$1.9$ trillion, $L(t) = \$21.1$ trillion, then leverage $f = 11.1$. From equation (iv), a 3% decline in asset values would reduce net worth by $dX(t)/X(t) = (1+11.1)(0.03) = 36\%$. The loss of net worth is equal to $(0.36)(\$1.9 \text{ trillion}) = \0.69 trillion. To maintain the same leverage $f = 11$, the institutions must either raise capital to offset the decline in asset values $dX = dA < 0$, or it must sell off assets to reduce its debt by the same proportion $dL(t)/L(t) = dX(t)/X(t)$, derived from equation (ii). A 3% decline in asset value would require the sale of $(0.03)(21.1 \text{ trillion}) = \630 billion in assets to repay the debt.

Both actions have adverse consequences for the economy. Firms in the financial sector, the financial intermediaries, are interrelated as debtors-creditors. Banks lend short term to hedge funds who invest in longer term assets and who may also buy credit default swaps. Firms that lost \$690 billion in net worth would have difficulty in raising capital to restore net worth, without drastic declines in share prices. Similarly, the attempt by group G_1 to sell \$630 billion in assets to repay loans will have serious repercussions in the financial markets. The prices of these assets will fall, and the leverage story repeats for

other sectors. Institutions G_j who hold these assets will find that the value of their portfolio has declined, reducing their net worth. In some cases, there are triggers. When the net worth of a Fund G_j falls below a certain amount (“break the buck”) the fund must dissolve and sell its assets. These may include AAA assets. In turn the sale of AAA assets affects group G_k . Investors in this group thought they were holding very safe assets, but to their dismay they suffer capital losses. The conclusion is that in a highly interrelated system, “high leverage” can be very dangerous. What seems like a small shock in one market can affect via leverage the whole financial sector. The Fed and the IMF seemed oblivious to this systemic risk phenomenon because of the history of two previous bubbles. In the S&L and agricultural crises of the 1980s, there was not a strong linkage between the specific sector and a highly leveraged interrelated financial sector based upon CDO and CDS. Therefore the collapse of these earlier bubbles only had localized effects.

2.3.2. The Incredible Leverage of Atlas Capital Funding

The story of the Atlas Capital Fund is an excellent example of leveraging discussed above. This is based upon a paper given by Jichuan Yang, one of the principals of Atlas, given at an Applied Mathematics Colloquium at Brown University September 2009. See also the paper by Ren Cheng (former Chief Investment Officer at Fidelity) at the same Colloquium. A group of talented financial engineers: mathematicians, physicists specializing in mathematical finance, decided to establish a Fund in 2003 with \$12 *billion* of assets, and \$10 *million* of capital, - a leverage of 1200. This Fund was called the Atlas Capital Fund, due to its huge size. The fund portfolio would contain thousands of individual bonds, loans and other types of financial securities. These had longer term maturities, such as 8 years. The liabilities were commercial paper and mid-term notes with maturities ranging from 30 days to 5 years. Atlas would borrow short term and lend longer term to the Hedge Funds. The Funds were set up not to hedge risk but to seek maximum return and they were not in fear of taking risk. Atlas would make its profits from the difference between the lending rate charged to the hedge funds and the cost of short term borrowing. The latter could be reduced to a minimum if Atlas received a AAA rating. This was remarkable goal since most global banks are rated no higher than AA.

Since the portfolio had a much longer maturity than the loans, a major risk to Atlas would be the variable short term borrowing rate. When the 30-day loan matured, Atlas would roll over the 30-day loan at the current rate. If there were difficulties in rolling over, Atlas would have to find banks to give Atlas “emergency” loans to pay off the 30-day debt. These standby banks are called “liquidity providers”.

The “financial engineers” built a model to evaluate the risk, which they used to convince the rating agencies to give them the AAA rating. A higher rating lowers the cost of borrowing. The model would simulate the movement of the \$12 billion of individual assets as well as their correlated behavior. These assets ranged from bonds, loans, to more complicated structured securities backed by all kinds of collateral. The mismatch of the timing of cash flows of assets and liabilities, the price movements, the rating changes, the defaults and recovery had to be “accurately” modeled, calculated and simulated. For each potential future price movement, the model would calculate the loss and return. After tens of thousands of such simulations, the financial engineers would get the expected loss and expected return by certain types of averaging the individual outcomes. These simulations did in fact convince the rating agencies to give Atlas a AAA rating and hence a low cost of borrowing.

At the beginning Atlas was extremely profitable. Stock holders received 100% of their money back in the first year of operations. This was due to the leverage of \$12 billion of assets/\$10 million of capital = 1200. The Fed was most accommodating with its low interest policy. Moreover, Chairman Alan Greenspan was the champion of financial innovation and was fighting off regulatory reform on all fronts. About three years after Atlas started its operations, the US financial industry went into one of its worst crises. The cascading effects of leverage discussed above then occurred. Atlas was blamed as being one of the main culprits for causing the crisis. Lowenstein (2010) presents an exciting chronicle of the 2008 financial collapse. Jichuan Yang, one of the principals of Atlas, wrote in 2009: “Today, if someone tells me that all these things can be simulated by an elegant mathematical model with any realistic accuracy, I would be tempted to say that he’s probably an overconfident idiot”.

2.4. The Disregarded Warnings

Greenspan, Bernanke and the IMF were insouciant, but there were Cassandras who warned of the housing price bubble and likelihood of a collapse. Shiller (1997) looked at a broad array of evidence concerning the recent boom in home prices, and concluded that it does not appear possible to explain the boom in terms of fundamentals such as rent and construction costs. Instead he proposed a psychological theory or social epidemic. This “explanation” is not convincing theoretically, and was not able to overcome the Jackson Hole Consensus. One can do much better than invoke vague phrases such as “epidemic”, “contagion”, “irrationality”.

From 1998-2005 rising home prices produced above average capital gains, which increased owner equity. This induced a supply of mortgages, and the totality of household financial obligations as a percent of disposable personal income rose. Figure 2 graphs the ratio of housing prices/disposable income PRICEINC and the debt service DEBTSERVICE, which is interest payments/disposable income. In figure 2, both variables are normalized, with a mean of zero and standard deviation of one.

The rises in housing prices and owner equity induced a demand for mortgages by banks and funds. In about 45-55% of the cases, the purpose of the subprime mortgage taken out in 2006 was to extract cash by refinancing an existing mortgage loan into a larger mortgage loan. The quality of loans declined. The share of loans with full documentation substantially decreased from 69% in 2001 to 45% in 2006 (Demyanyk and Van Hemert). The ratio of debt/income rose drastically. The only way to service or refinance the debt was for the capital gain to exceed the interest rate. This is an unsustainable situation since it implies that there is a “free lunch” or that the present value of the asset diverges to infinity.

The fatal error was to ignore the fact that the quality of mortgages declined and that it was ever less likely that the mortgagors could service their debt from current income. Sooner or later the defaults would affect housing prices and turn capital gains into capital losses. The market gave little to no consideration of what would happen if the probability distribution/histogram would change.

Both the supporters and the critics of the Jackson Hole Consensus agree that asset price bubbles are a source of danger to the real economy if the financial structure is fragile

and not properly capitalized. The danger from “overvaluation” of housing prices is that the debt used to finance the purchase is excessive.

It is seen in Figure 2 that the ratio $PRICEINC = P(t)/Y(t)$ and the $DEBTSERVICE$ ratio were stable, almost constant from 1980 almost to 2000. Then there was a housing bubble, the price/income shot up from 2000 to 2006. As a result of the rise in homeowner’s equity the debt ratio rose – to finance consumption. The debt service ratio rose to two standard deviations above the longer term mean. The great deviation of the price/income ratio from its long term mean would suggest that there was a housing price “bubble” and that housing prices were greatly overvalued. A housing crisis would be predicted, when the ratio $P(t)/Y(t)$ would return to the long term mean, which is the zero line. Households would then default on their mortgages and leverage would transmit the shock to the financial sector. The market – as well as the FED – discounted that apprehension. There was no theory that could identify an asset price bubble and its subsequent effect upon the economy. The Jackson Hole Consensus ignored the microeconomy.



Figure 2. PRICEINC = Ratio of housing prices/disposable income. DEBTSERVICE = Debt service/disposable income. Both variables are normalized. FRED data set of the Federal Reserve Bank of St. Louis, Office of Federal Housing Enterprise Oversight.

There were financial firms who may have had qualms about the sustainability of the housing price appreciation, but they assumed that they would be able to anticipate the onset of a crisis in time to retrench. Charles Prince's remark is emblematic: "When the music stops, in terms of liquidity, things will be complicated. But as long as the music is playing, you've got to get up and dance". They certainly were mistaken, because they ignored systemic risk that the negative shock could be pervasive, and liquidity and capital would disappear in the wake of a mass exodus from the markets for derivatives.

There were a few hedge firms such as Scion Capital (SC) that anticipated the crash and took appropriate actions. Michael Burry of SC realized in 2005 that the bubble would burst and acted upon that view. He purchased credit default swaps (CDS) on

billions of dollars worth both of subprime mortgage backed securities and bonds of many financial corporations that would be devastated when the real estate bubble burst. Then as the value of the bonds fell, the value of CDS would rise. The investors in this hedge fund still “wanted to dance” and profit from the rising house prices. Despite pressure from the investors, Burry liquidated the CDS at a substantial profit. But since he was operating in face of strong opposition from both his investors and from the Wall Street community, he shut down SC in 2008. Lowenstein ch.7 describes the divergent opinions in the market where the pessimists were in the minority.

3. Stochastic Optimal Control (SOC)/Dynamic risk management

Shojai and Feiger, in their article “Economists’ Hubris – The Case for Risk Management” – write that “...the tools that are currently at the disposal of the world’s major global financial institutions are not adequate to help them prevent such crises in the future and that the current structure of these institutions makes it literally impossible to avoid the kind of failures that we have witnessed.” The FED, IMF, Treasury and the market have lacked the appropriate tools of analysis. To his credit, Greenspan stressed the importance of the financial sector and debt to “optimally” allocate saving to investment. The social objective should be the maximization of the expectation of the logarithm of net worth over a given horizon. The current proposals for “financial reform” go to the other extreme. They focus almost exclusively upon risk reduction rather than upon the balance between risk and growth.

The approach that I now discuss concerns my recent work, which applies the techniques of stochastic optimal control (SOC) to derive an optimal debt ratio, optimal leverage or capital requirement that “optimally” balances risk and expected growth - where the future is unpredictable. I explain: What are the consequences of a debt ratio that deviates in either direction from the derived optimal ratio. Why did the observed leverage deviate from the optimal? What are early warning signals (EWS) of a debt crisis? How successful were the EWS of the housing crisis? The theoretically derived early warning signal of a crisis is the excess debt ratio, equal to the difference between the actual and optimal ratio. The excess debt starting from 2004-05 indicated that a crisis

was most likely. This SOC analysis should be used by those charged with surveillance of financial markets.

A sketch of the SOC approach will facilitate understanding the mathematical analysis below. The object is to select a leverage – ratio of debt/net worth – that will maximize the expected logarithm of net worth at a future date. This is a risk averse strategy because the logarithm is a concave function. Declines in net worth are weighted more heavily than increases in net worth. In fact very severe penalties are placed upon bankruptcy – a zero net worth.

The growth of net worth is affected by leverage. An increase in debt to finance capital (assets) increases net worth by the return on investment, but decreases the growth of net worth by the associated interest payments. The *return on investment* has two components. The first is the *productivity of capital* and the second is the *capital gain* on the assets. An increase in leverage will increase expected growth if the return on investment exceeds the interest rate. The productivity of capital and the interest rate are observed, but the capital gain is unknown and is not observable when the investment decision is made. Figure 1 is the histogram of the capital gain in housing. One must specify the stochastic process on the capital gain if one wants to select the optimal leverage– to maximize the expected logarithm of future net worth.

The true stochastic process is unknown. Three stochastic processes of the capital gain are considered in this paper. In Model I, the logarithm of the price of the asset has a trend and a deviation from the trend. The trend is observable. We constrain the trend to be equal to or less than the rate of interest, to exclude the “free lunch” described above. The deviation from the trend is assumed to be ergodic mean reverting. The deviation tends to a mean of zero but is stochastic. Given the stochastic process, an optimal leverage is derived. It depends upon the productivity of capital, the trend less the interest rate, the deviation of the price from trend, and the variance of the deviation.

Model Ia is a special case of model I, where the deviation from trend is stochastic but not ergodic mean reverting. Then the optimal leverage depends upon the productivity of capital, the trend less the rate of interest and the variance of the diffusion.

In each case, the expected growth of net worth is a concave function of the leverage, figure 3. It is maximal when the optimal leverage is chosen. As the leverage

exceeds the derived optimal, the expected growth declines and the variance/risk rises. Define the excess debt as the actual leverage less the optimal leverage. For a sufficiently high excess debt, the expected growth is zero or negative. *The probability of a decline in net worth or a debt crisis is directly related to the excess debt ratio.*

Model II describes the market behavior. It assumes that the drift of the capital gain is estimated in a Bayesian manner. There is a prior and the sample is the recent realized capital gain. Then the posterior estimate of the drift converges the recent capital gains. The posterior estimate of the drift increased greatly after 2002, and in terms of figure 1 was above 10% pa, far in excess of the rate of interest. Therefore the implied optimal leverage was very high and unsustainable. There was a considerable excess debt.

Some Quants probably realized this, but Charles Prince's remark is emblematic: "When the music stops, in terms of liquidity, things will be complicated. But as long as the music is playing, you've got to get up and dance". This was a delusion because the Quants had the same model, were equally well trained and would all try to get out at the same time. This produces a crash – with the associated fall out from leverage. Other Quants kept searching for what is the best way to model the distribution function, but ignored the fact that it is determined by economics and not by nature. They ignored the "no free lunch constraint".

The FED, IMF, Treasury and the "Quants"/market lacked the appropriate tools of analysis to answer the following questions: what is an optimal leverage or capital requirement that balances the expected growth against risk? In mathematical part Three, I explicitly derive the excess debt from Models I or Ia. The excess debt starting from 2004-05 indicated that a crisis was most likely. Part Four derives the Early Warning Signals of the Crisis, figure 5. This SOC analysis should be used by those charged with surveillance of financial markets.

3. A General Intertemporal Optimization Model

3.1. Performance Criterion

The financial crisis was precipitated by the mortgage crisis and spread through the financial sector. At one end of the financial chain are the mortgagors/debtors who borrow from financial intermediaries – banks, hedge funds, government sponsored enterprises.

The latter are creditors of the mortgagors, but who ultimately are debtors to institutional investors at the other end. For example FNMA borrows in the world bond market and uses the funds to purchase packages of mortgages. If the mortgagors fail to meet their debt payments, the effects are felt all along the line. The stability of the financial intermediaries and the value of the traded derivatives CDO, CDS, ultimately depend upon the ability of the mortgagors to service their debts. For this reason I focus upon the mortgagors.

One must have a performance criterion to answer the question: what is an optimal leverage in a stochastic environment. The techniques of analysis are drawn from the mathematical literature Stochastic Optimal Control, which is just optimal dynamic risk management. The techniques of analysis are developed in Fleming and Stein (2006). The exposition in the text below is a development of Stein (2010). My exposition here attempts to be more intuitive, and focuses upon the ideas and results relevant for economics.

As my criterion of performance, I could either consider maximizing the expected logarithm of net worth of the mortgagors or of the consolidated industry of mortgagors and financial intermediaries. Net worth of a sector X is equal to assets (capital) K less debt L , equation (1). The only difference is that in the first case, debt $L(t)$ is that of the mortgagors and in the second case it is of both the mortgagors and of the financial intermediaries.

The mathematics will be the same in both cases. Let X_1 be the net worth of the mortgagors who have capital K and debt L_1 . Thus $X_1(t) = K(t) - L_1(t)$. Let X_2 be the net worth of the financial intermediaries. Their net worth is $X_2(t) = L_1(t) - L_2(t)$, since their assets are the liabilities of the mortgagors. The net worth of the consolidated mortgagors and financial intermediaries is $X(t) = X_1(t) + X_2(t) = K(t) - L_2(t)$.

The stochastic optimal control problem is to select debt ratios $f(t) = L(t)/X(t)$ during the period $(0, T)$ that will maximize $W(T)$ in equation (1). This ratio is precisely the optimal leverage, and will vary over time. The solution of the stochastic optimal control/dynamic risk management problem tells us what is an optimal and what is an “excessive” leverage.

$$(1) W^*(X, T) = \max_f E \ln [X(T)/X(0)], \quad f = L/X = \text{debt/net worth}$$

(1a) $[X(T)] = X(0) e^{W(X,T)}$ The asterisk denotes the maximum value.

Since $W(T)$ is a positively sloped concave function, both expected return and risk are taken into account. Low values of net worth close to zero may not be likely, but they have large negative utility weights. Hence the criterion function reflects strong risk aversion. Bankruptcy $X = 0$ is severely penalized because, in the deterministic case equation (1a), if net worth $X(T) = 0$, the value of $W(T)$ is minus infinity.

My aim is to model optimum leverage or ratio of debt/net worth.

3.2 . Dynamics of Net Worth

The financial industry has a net worth $X(t)$ equal to the value of assets or capital $K(t)$ less debt $L(t)$, equation (2). Capital $K(t) = P(t)Q(t)$ is the product of a deterministic physical quantity $Q(t)$ times the stochastic price $P(t)$ of the capital asset.

(2) $X(t) = K(t) - L(t) = P(t)Q(t) - L(t)$. The control variable is the debt ratio. The next steps are to explain the stochastic differential equation for net worth, relate it to the debt ratio, and specify what are the sources and characteristics of the risk and uncertainty.

In view of equations (1), (2), focus upon the change in net worth $dX(t)$ of the financial industry. It is the equal to the change in capital $dK(t)$ less the change in debt $dL(t)$. The change in capital $dK(t) = d(P(t)Q(t))$ equation (3) has two components. The first is the change due to the change in price of capital asset, which is the capital gain or loss term, $K(t)(dP(t)/P(t))$. The second is investment $I(t) = P(t) dQ(t)$ the change in the quantity times the price.

$$(3) dK(t) = d(P(t)Q(t)) = Q(t)dP(t) + P(t)dQ(t) = K(t)dP(t)/P(t) + I(t)$$

The change in debt $dL(t)$, equation (4), is the sum of expenditures less income.

Expenditures are the debt service $i(t)L(t)$ at interest rate $i(t)$, plus investment

$I(t) = P(t) dQ(t)$ plus $C(t)$ the sum of consumption, dividends and distributed profits.

Income $Y(t) = \beta(t)K(t)$ is the product of capital $K(t)$ times $\beta(t)$ its productivity.

$$(4) dL(t) = i(t)L(t) + P(t)dQ(t) + C(t) - \beta(t)K(t).$$

Combining these effects, the change in net worth $dX(t) = dK(t) - dL(t)$ is equation (5).

$$(5) dX(t) = dK(t) - dL(t) = K(t)[dP(t)/P(t) + \beta(t) dt] - i(t)L(t) dt - C(t) dt.$$

Since net worth is capital less debt, equation (6) describes the dynamics of net worth equation (5) in terms of the ratio $f(t) = L(t)/X(t)$ of debt net worth and consumption

ratio $c(t) = C(t)/X(t) \geq 0$. Since capital/net worth $k(t) = K(t)/X(t) = (1+f(t))$, the control variable could be either $f(t)$ the debt ratio – leverage - or $k(t)$ the capital ratio.

$$(6) dX(t) = X(t) \{ (1 + f(t)) [dP(t)/P(t) + \beta(t) dt] - i(t) f(t) dt - c(t) dt \}.$$

A simplifying assumption is that the outflow consumption, distributed profits and dividends are paid from the current productivity of capital, equation (7). If the latter is negative, then $c(t) = 0$.

$$(7) c(t) = \beta(t) \geq 0.$$

In that case, the stochastic differential equation (6) becomes (8).

$$(8) dX(t) = X(t)[(1+f(t))(dP(t) /P(t) + (\beta(t) - i(t))f(t)dt]$$

The optimization of (1) subject to (8) depends upon the stochastic processes underlying the $dP(t)/P(t)$, $\beta(t)$ and $i(t)$ variables. The productivity of capital $\beta(t)$ and interest rate $i(t)$ are always observable but change over time. However the change in price $dP(t)$ from t to $t+dt$ is unpredictable, given all the information through present time t . Histogram figure 1 above shows that the annual change of housing prices from 1992 to 2007 was never negative. This led to an underestimation of risk. The mean capital gain over the entire sample period was 5.4% p.a. and the standard deviation was 2.9% p.a. The distribution is not normal but skewed to the right over that sample period. The mean capital gain 5.4% p.a. is close to the rate of interest.

A very important consideration is the model for the prices $P(t)$. In order to take into account the ups and downs of the market, the model should allow periods in which prices have both increasing and decreasing trends.

For each of these models considered in this paper the differential of the prices has the form

$$(9) dP(t) = P(t)(a(t) dt + \sigma dw(t)),$$

where $w(t)$ is a Brownian motion and the drift process $a(t)$ is a slowly varying random process which can have both positive and negative values. The positive periods of $a(t)$ are periods of relative growth of prices, the negative periods of $a(t)$ are periods of relative decline of prices. The Brownian motion $w(t)$ gives the rapid short term variation of the prices.

In the general model I derive the optimal debt ratio, without specifying what is the explicit value $a(t)$ of the time varying drift. In Model I, the price equation (9) is specified to be an ergodic mean reverting process and thereby an explicit value for the drift $a(t)$ is

obtained. Model Ia is also a sensible optimization model. It assumes that the return on capital has a constant drift plus a Brownian Motion term. On the other hand, market Model III based upon a Bayesian analysis assumed an unsustainable value for $a(t)$, the drift, which led to the bubble and subsequent collapse.

3.3. Optimization in the general Model

Consider the problem of choosing the debt ratio $f(t)$ as a function of the past values of the price $P(t)$, the interest rate $i(t)$, and the productivity of capital $\beta(t)$ to maximize the expected value of the logarithm of net worth at a fixed final time T , criterion (1) above

$\max_{f(t)} E[\ln X(T)]$, $X(0) = 1$. This choice is to be made subject to $X(t)$ being a solution of equation (10) based upon equations (8) and (9).

$$(10) dX(t) = X(t)[(1+f(t))a(t) + (\beta(t) - i(t))f(t)]dt + X(t)(1+f(t))\sigma dw(t).$$

When $a(t)$ can be determined from past values of the prices $P(t)$, the choice of $f(t)$ can depend on the information set $\mathfrak{S}(t) = \{P(t), i(t), \beta(t), a(t)\}$.

To carry out this optimization notice that (10) and Ito's differential rule imply

$$(11) d \ln X(t) = [(1+f(t))a(t) + (\beta(t) - i(t))f(t)]dt - (1/2)(1+f(t))^2 \sigma^2 dt + (1+f(t))\sigma dw(t).$$

Or in integrated form

$$(12) \ln X(T) = \ln X(0) + \int_0^T [(1+f(t))a(t) + (\beta(t) - i(t))f(t) - (1/2)(1+f(t))^2 \sigma^2] dt + \int_0^T (1+f(t)) \sigma dw(t).$$

Thus taking expected values,

$$(13) E[\ln X(T)] = \ln X(0) + E[\int_0^T \{(1+f(t))a(t) + (\beta(t) - i(t))f(t) - (1/2)(1+f(t))^2 \sigma^2\} dt]$$

The integrand in (13) is maximized when

$$(14) a(t) + \beta(t) - i(t) - (1+f(t)) \sigma^2 = 0.$$

In the *general case* $f^*(t)$ in equation (15) is the optimum (denoted by an asterisk) debt ratio.

$$(15) f^*(t) = [a(t) + \beta(t) - i(t) - \sigma^2] / \sigma^2$$

It depends upon information set $\mathfrak{S}(t)$, the current observations of productivity of capital $\beta(t)$ interest rate $i(t)$, and the past history of prices. *The specification of the price equation (9) for the alternative models determines what is the appropriate value of $a(t)$ the drift. The value of the drift $a(t)$ thereby obtained is substituted in eqn. (15) to obtain the optimal debt ratio for the alternative models.*

3.4 Model Uncertainty, Excess Debt and Vulnerability to Shocks: General Case

Optimal risk management should focus upon the debt ratio $f(t) = L(t)/X(t)$ that maximizes the expected growth of the logarithm of net worth. This is a risk averse strategy for reasons explained in part (3.1). Risk management should avoid having debt ratios where the expected growth of the logarithm of net worth is very low or negative. Define the excess debt $\Psi(t) = (f(t) - f^*(t))$ as the difference between the observed actual debt ratio $f(t)$ and the optimal debt ratio $f^*(t)$. *In this part, we derive an equation for the excess debt in the general case and show that the greater the excess debt the lower is the expected growth rate of net worth and the greater is the likelihood of a debt crisis.*

One can never be sure of what is the correct model for optimization. There will always be an error of measurement of what is the “true” optimal debt ratio. I therefore derive a relation between errors of measurement of the excess debt resulting from model uncertainty and the loss of expected growth of net worth. I then illustrate the relation by using figure 2.

The growth of net worth over a short period, equation (11) for the general case, is written as eqn. (16). The corresponding expected growth is eqn. (16a). The variance of $d[\ln X(t)]$ is eq. (16b). BOX 1 summarizes the results.

BOX 1

Growth over a short period

$$(16) d[\ln X(t)] = W(f(t)) + (1+f(t))\sigma dw$$

Expected Growth

$$(16a) W(f(t)) = E[d \ln(X(t))] = (1+f(t)) a(t) + (\beta(t) - i(t)) f(t) - (1/2)(1+f(t))^2 \sigma^2$$

Risk

$$(16b) \text{var } d[\ln X(t)] = (1+f(t))^2 \sigma^2 dt$$

The relation between excess debt $\Psi(t) = (f(t) - f^*(t))$ and the difference between maximal and actual expected growth $[W^*(t) - W(t)]$ is derived as follows. The expected growth of net worth $W(f(t))$ is a quadratic function of the debt ratio $f(t)$, which is graphed in figure 3. Since $W(f(t))$ is quadratic in $f(t)$, it has a three term Taylor expansion, eqn. (17), about $f^*(t)$, the optimum value.

$$(17) W(f(t)) = W(f^*(t)) + W'(f^*(t))(f(t) - f^*(t)) + (1/2)W''(f^*(t)) (f(t) - f^*(t))^2$$

It attains a maximum $W^*(t) = W(f^*(t))$ when the debt ratio is optimal at $f^*(t)$, as defined in eqn. (15) for the general model. Hence $W'(f^*(t)) = 0$ and $W''(f) = -\sigma^2$.

It follows that the difference between the maximum expected growth $W(f^*)$ and the actual growth $W(f)$ is equation (18). It is proportional to $\Psi(t)^2$ the square of the excess debt.

$$(18) [W^*(t) - W(t)] = (1/2) \sigma^2 (f(t) - f^*(t))^2 = (1/2) \sigma^2 \Psi(t)^2.$$

The term $W^*(t)$ with the asterisk refers to the case where the debt ratio is always optimal $f^*(t)$ but *time varying*, and the term $W(t)$ refers to the actual debt ratio $f(t)$ during the period. *The loss in expected growth, equation (18), results from model misspecification.* In figure 3, if one thought that f_1 was the optimum debt ratio at time t when it is really f^* , the loss of expected growth of net worth during the period of length dt is $(W^* - W_1)$.

There are several important implications of eqn. (18) or figure 3. First: as $f(t)$ rises above $f^*(t)$, the expected growth $W(t) = E[d \ln X(t)]$ declines and the variance $\text{var } d \ln [X(t)]$, risk equation (16b), rises. Second: this implies that the economy's growth of net worth is ever more vulnerable to shocks, from the capital gains, productivity of capital and the interest rate. When there is an excess debt, bankruptcy and a crisis become ever more probable. Third, one can never know what is the "true" model and hence what is the "true" optimal debt ratio $f^*(t)$. One chooses what seems to be the optimal debt ratio based upon what seems to be the correct model. There will always be a specification error at any time t measured as the excess debt $\Psi(t) = (f(t) - f^*(t))$. Eqn. (19), which is the integral of eq. (16), states precisely what is the loss of expected growth of net worth as a result of using a non-optimal control, debt ratio $f(t)$.

$$(19) E[\ln X^*(T) - \ln X(T)] = \int^T [W^*(t) - W(t)] dt = (1/2) \int^T \sigma^2 \Psi(t)^2 dt.$$

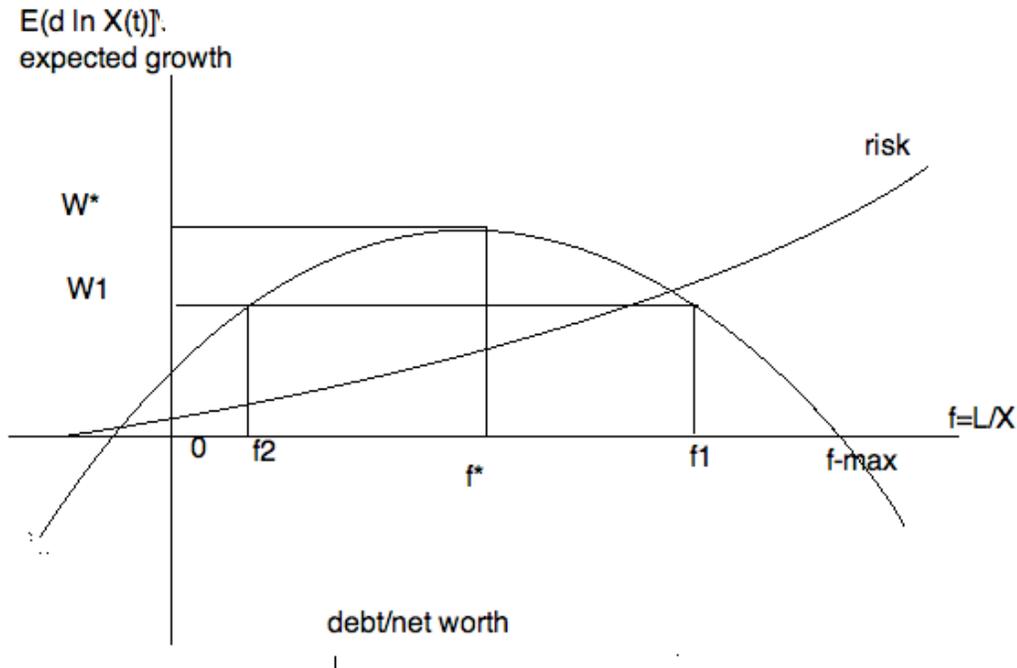


Figure 3. Expected growth in net worth $W(f(t)) = E[d \ln X(t)]$ is a concave function of the debt ratio $f(t) = L(t)/X(t)$. The optimum ratio is $f^*(t)$, and maximum expected growth rate is W^* . Debt ratio in period $t = 1$ is f_1 and in period $t = 2$ it is f_2 . The average is f^* . Excess debt $\Psi^2(t) = (f_1 - f^*)^2 = (f_2 - f^*)^2$ corresponds to the sacrifice of expected growth $(W^* - W_1) = (1/2)(\sigma^2 \Psi(t)^2)$ in each period. Risk is $\text{var } d[\ln X(t)] = (1+f(t))^2 \sigma^2 dt$.

Fourth, errors of using non-optimal debt ratios do not average out. Suppose that the optimal debt ratio is f^* in figure 3 for two periods. In period $t = 1$ the actual ratio is f_1 , in period $t = 2$ it is f_2 , and the average is $(1/2)(f_1 + f_2) = f^*$. The square of the errors $\Psi(t)^2$ imply a loss of $(W^* - W_1)$ in each period so that the errors of using an incorrect debt ratio do not average out to zero. *The conclusion is that as the excess debt rises, the expected return declines and the risk rises.*

3.3. Stochastic Processes

The basic stochastic variable in equation (8) is $dP(t)$ the change in the housing price. Two stochastic processes are considered: Model I, special case Model Ia and Model II. Models I/Ia are sensible candidates for optimality. Model II describes the market behavior that led to the crisis.

3.3.1 Models I and Ia

Equations (20) – (22) concern the price $P(t)$ of the asset. They contain two ideas, inspired by Bielecki and Pliska and Platen-Rebolledo, and discussed in Fleming (1999).

The first, in equation (20)/(20a), is that ρ is a price trend. The initial value of the price is P , which can be normalized at one. Variable $y(t)$ in (20)/(20a) is a deviation from the trend. The second idea, expressed in equations (21)-(22), is that deviation $y(t)$ is an ergodic mean reversion term whereby the price converges towards the trend. The speed of convergence of the deviation $y(t)$ towards the trend is described by finite coefficient $\alpha > 0$. The stochastic term is $\sigma dw(t)$. The solution of stochastic differential equation (21) is (22). The deviation from trend converges to a distribution with a mean of zero and a variance of $\sigma^2/2\alpha$.

$$(20) P(t) = P \exp(\rho t + y(t)), \quad P = 1, \quad (20a) y(t) = \ln P(t) - \ln P - \rho t.$$

$$(21) dy(t) = -\alpha y(t)dt + \sigma dw(t). \quad \infty > \alpha > 0, \quad E(dw) = 0, \quad E(dw)^2 = dt.$$

$$(22) \lim y(t) \sim N(0, \sigma^2/2\alpha).$$

The choice of price trend ρ is very important in determining the optimal leverage. I impose a constraint that the assumed price trend must not exceed the rate of interest. If this constraint is violated, as occurred during the housing price bubble, debtors were offered a “free lunch” as described above. Borrow/Refinance the house and incur a debt that grows at the rate of interest. Spend the money in any way that one chooses. Insofar as the house appreciates at a rate greater than the rate of interest, at the terminal date T the house is worth more than the value of the loan, $P(T) > L(T)$. The debt $L(T)$ is easily repaid by selling the house at $P(T)$ or refinancing. One has had a free lunch. In the optimization, one must constrain the trend ρ not to exceed the rate of interest $i(t)$. This constraint is equation (23).

$$(23) \rho \leq i(t). \quad \text{No free lunch constraint}$$

An alternative justification for equation (23) is as follows. The present value of the asset

$$(23a) PV(T) = P(0) \exp[(\rho - i)t],$$

where trend ρ is the rate of appreciation or capital gain and i is the interest rate. If $(\rho - i) > 0$, the present value diverges to plus infinity. An infinite present value is not sustainable.

The Market estimated the price trend from recent experience, described by histogram figure 1. From 2000 to 2004, the capital gain greatly exceeded the interest rate. This assumption violates the “no free lunch” constraint, equation (23). That is why the rates of appreciation of 10 – 14 % p.a. were unsustainable. This assumption was to have dire consequences, as discussed below.

3.3.2 Models I/Ia Optimal debt ratio - leverage

The expected growth of net worth is equation (24), graphed as figure 3. It corresponds to equation (16)/(16a), in the general case. It is a concave quadratic function of the control variable, the leverage or debt ratio $f(t) = L(t)/X(t)$. The debt ratio that maximizes the expected growth of net worth is $f^*(t)$, equation (27)/(27a). It corresponds to equation (15) in the general case. This is the time varying ratio that maximizes equation (1) subject to the stochastic processes (8), (20) - (22). At the optimum debt ratio the expected growth of net worth is maximal at W^* . The variance of the growth of net worth $\text{var } d[\ln X(t)]$ is equation (26). It is a quadratic function of the leverage times the variance. As the debt ratio deviates from the optimum, the expected growth of net worth declines - as seen in figure 3.

The optimum debt ratio, leverage $f^*(t)$ in equation (27) is positively related to the productivity of capital $\beta(t)$ less the real rate of interest $r(t) = i(t) - \rho$, equal to the nominal rate of interest $i(t)$ less ρ the trend of prices. The “no free lunch” constraint is that the real rate of interest must be non-negative.

Expected Growth $W(f(t))$ and Risk

$$(24) W(f(t)) = E[d \ln(X(t))]$$

$$= [(1+f(t))(\rho + (1/2)\sigma^2 - \alpha y(t)) + (\beta(t) - i(t)) f(t) - (1/2)(1+f(t))^2 \sigma^2]$$

$$(25) \text{Real rate of interest constraint}$$

Risk

$$(26) \text{var } d[\ln X(t)] = (1+f(t))^2 \sigma^2 dt$$

One can never be certain of what is the correct way to model: (i) the trend ρ of prices, even with the constraint that it is not greater than the rate of interest, (ii) the deviation from trend. For example is the deviation $dy(t)$ ergodic mean reverting? Model Ia assumes that the deviation is equation (21a) not (21). That is $y(t)$ has a unit root.

$$(21a) dy(t) = \sigma dw(t).$$

In that case, the optimum leverage in Model Ia is equation (27a).

The positive value of the real rate of interest $r(t) > 0$ in (27)/(27a) is unknown, and one is not sure exactly what is the value of α , the mean reversion coefficient. Therefore the choice of an optimum leverage $f^*(t)$ at any time is subject to specification error.

BOX 2 Optimum Debt Ratio $f^*(t)$ Models I/Ia

$$(27) f^*(t) = \{[\beta(t) - (i(t) - \rho) - (1/2) \sigma^2] - \alpha y(t)\} / \sigma^2$$

$$(27a) f^*(t) = \{[\beta(t) - (i(t) - \rho) - (1/2) \sigma^2] / \sigma^2$$

$$r(t) = i(t) - \rho \geq 0$$

3.3.3. Model II

There are several methods in the literature of estimate the value of a time varying drift. Prominent are the papers by Blanchet-Scalliet et al, Rishel, Lister and Shirayayev, the Kalman filter, Yin and Zhang. The priors are updated on the basis of recent information based upon the deviation of the actual price from its predicted price. In effect this approach can be understood in a simpler way – model II.

The market seems to have followed *Model II* where the estimate of drift $a(t)$ in the general price equation (9) was Bayesian equation (28). I am drawing upon De Groot chapter 9 for an analysis of Bayesian estimation. The posterior estimate of the drift $a(t+1)$ is a linear combination of the prior $a(t)$ and a sample which is the recent price appreciation $\pi(t) = [P(t+1) - P(t)]/P(t)$. For simplicity assume that the weight $1 > m > 0$ is constant. Solve (28) for $a(t)$ and derive (29), whose limit is (30). The drift $a(t)$ converges to recent capital gains $\pi(s)$, where $(t-s)$ is distance from present time t . Therefore the “optimal” debt ratio selected by the market is eqn. (31), where drift (30) is substituted into the optimal debt ratio eqn. (15) for the general case.

Market/ Model II

$$(28) a(t+1) = ma(t) + (1 - m(t))\pi(t), \quad 1 > m > 0$$

$$(29) a(t+1) = a(0)m^{t+1} + (1-m)\sum \pi(s)m^{t-s}, \quad t \geq s \geq 0$$

$$(30) \lim a(t) = (1-m)\sum \pi(s)m^{t-s}$$

$$(31) f(t) = [(1-m)\Sigma\pi(s)m^{t-s} + \beta(t) - i(t) - \sigma^2]/\sigma^2$$

Figure 1 describes the statistics underlying the capital gains variable CAPGAIN = $\pi(t)$, which is the four- quarter appreciation of US housing prices. The distribution is highly skewed to the right. These extreme observations are the bubble years. During the period 2005-2006, the actual capital gains ranged from 12% to 13% per annum. There were no significant declines in housing prices during the sample period in table 1.

The 30-year mortgage interest rate, during 2004-07, fluctuated between 6 to 6.5% per annum. Hence the market estimated drift in (28)/(29) is dominated by recent $\pi(t)$. This means that $a(t) - i(t) \sim \pi(t) - i(t) \sim 0.12 - 0.06 = 0.06$ per year. This implies the present value of the asset diverges, and violates equation (23). Hence the market assumed that the “optimal debt ratio” was f_1 or f_{max} in figure 2.

There are several grave deficiencies to Model II. First, it ignores the economic constraint that there is “no free lunch/ no money machine”. As was explained in part 3.1 above, one cannot continue to borrow and spend, and pay the interest from the capital gain. A situation where there is a “free lunch” or “money machine” where $[(\pi(t) - i(t)) > 0]$ is not sustainable.

The second deficiency of Model II concerns a positive economic feedback from the prior $a(t)$ to the subsequent “sample” or capital gain $\pi(t+1)$ in eq. (28). The market debt ratio $f(t)$ in the Model II/eq. (31) was highly influenced by $\pi(t)$, the *recent capital gains*. Recent price appreciation induced the market to raise the estimate of future drift. Most of the firms were doing the same thing, based upon the same Model II. The demand for the asset rose, which further raised the price. For a while the strategy seemed to be profitable. This *positive feedback* from the prior $a(t)$ to the subsequent “sample” $\pi(t+1)$ produced the illusion of a “free lunch”. This situation was unsustainable for reasons cited. The optimal debt ratio should have been based upon Model I/Ia , where the drift was constrained by equation (23).

4. Early Warning Signals of the Crisis

The financial crisis was precipitated by the mortgage crisis for several reasons. First, a whole structure of financial derivatives was based upon the ultimate debtors – the mortgagors. Insofar as the mortgagors were unable to service their debts, the values of the

derivatives fell. Second, the financial intermediaries whose assets and liabilities were based upon the value of derivatives were very highly leveraged. Changes in the values of their net worth were large multiples of changes in asset values. Third, the financial intermediaries were closely linked – the assets of one group were liabilities of another – as described in sections 1.1 and 1.2. A cascade was precipitated by the mortgage defaults. Fourth, Charles Prince’s remark (cited in part 2.4 above) underestimated the fact that, since the “Quants” were following the same rules, the markets would not be liquid.

The “Quants”/financial engineers ignored these points. They have had very microscopic points of view and the unfounded belief that the probability distribution of recent price changes is time invariant. They were focused upon the way to describe the density function: normal, fat tail, Pareto etc. They ignored the fact that other “Quants” were doing the same thing, based upon the same models. They did not consider that their collective behavior would affect the probability distribution and market liquidity. Systemic risk and liquidity considerations were ignored.

For these reasons, I focus upon the excess debt of the mortgagors. The whole structure of derivatives rested upon the mortgagors being able to service their debts. Hence my basic question is: Did the debt ratio of the mortgagors significantly exceed f_{\max} in figure 3?

The application of the optimal dynamic risk management/SOC analysis is done in several steps. First, on the basis of the analysis in part 3, I derive estimates of the excess debt $\Psi(t) = f(t) - f^*(t)$ that lowered the expected return and raised risk. Thereby early warning signals are derived. The bubble was generated by the market view that the trend of prices – the capital gains – exceeded the rate of interest. Second, I show how the collapse occurred when the capital gain fell below the rate of interest. Defaults and bankruptcies occurred.

4.1. The Bubble and Collapse, *Estimates of Excess Debt, Early Warning Signal of a Crisis*

An Early Warning Signal of a debt crisis is a series of excessive debts $\Psi(t) = f(t) - f^*(t) > 0$. As shown in figure 3/eq. (19), the loss of growth from non-optimal debt ratios over a period $(0, T)$ is

$$(19) E[\ln X^*(T) - \ln X(T)] = \int^T [W^*(t) - W(t)] dt = (1/2) \int^T \sigma^2 \Psi(t)^2 dt.$$

When the debt ratio $f(t)$ exceeds $f\text{-max}$ in figure 3, the expected growth is negative and the risk is high. A crisis is likely when $\int^T \sigma^2 \Psi(t)^2 dt$ is large. The next question is: What are the appropriate measures of the actual and the optimal debt ratio to evaluate $\Psi(t)$?

In order to make alternative measures of the debt ratio and key economic variables comparable, I use normalized variables where the normalization (N) of a variable $Z(t)$ called $N(Z) = [Z(t) - \text{mean } Z]/\text{standard deviation}$. *The mean of $N(Z)$ is zero and its standard deviation is unity.*

For the actual debt ratio I use the debt service burden $i(t)L(t)/Y(t)$. There is a great heterogeneity in interest rates charged to the subprime borrowers depending upon the terms of the mortgage, so it is difficult to state exactly what corresponds to interest rate $i(t)$ in the analysis above. I therefore use “Household Debt Service Payments as a Percent of Disposable Personal Income” (This is series TDSP in FRED) as a measure of iL/Y the *debt service burden*. This includes all household debt, not just the mortgage debt, because the capital gains led to a general rise in consumption and debt. The normalized value of the debt service $N(f)$ or debt burden, is equation (33), which is graphed in figure 4 as DEBTSERVICE. This is measured in units of standard deviations from the mean of zero. There is a dramatic deviation above the mean from 1998 to 2006. This sharp rise coincides with the ratio of housing price index $P/\text{disposable income } Y$, $P/Y = \text{PRICEINC}$ in figure 2. During this period, there is more than a two standard deviation rise in P/Y and a two standard deviation rise in iL/Y debt service/disposable income.

$$(32) N(f) = \text{DEBTSERVICE} = [i(t)L(t)/Y(t) - \text{mean}]/\text{standard deviation}.$$

As explained in connection with figure 3 there will always be a specification error in estimating the optimal debt ratio. The main reason is that the price trend ρ cannot be known with certainty, but I require that it not exceed the rate of interest. Therefore a rather flexible approach will be taken to estimate the optimal debt ratio $f^*(t)$.

The *optimum debt ratio* f^* is based upon eqn. (27)/(27a), with the constraint that $r = \rho - i > 0$. From the histogram of the capital gains in figure 1, the mean capital gain was 5.4% per annum with a standard deviation of 2.9%. The 10-year Treasury rate was relatively stable at 5% from 2002-07. It is reasonable to argue that, over a long period, the real appreciation of housing prices was not significantly different from “the mortgage rate of interest”, $(i-\rho) = r = 0$. Equation (27a) for the optimal debt ratio is preferred to

(27) because the deviation from trend $y(t)$ in equation (21) does not seem to be ergodic mean reverting. The evidence is that the ADF statistic for CAPGAIN in figure 1, using a trend and intercept, is not significantly different from zero. Thus the deviation $y(t)$ is more accurately described by (21a) than by (21).

The optimal debt ratio should be (33) below. The normalized optimal debt ratio is $N(f^*)$ in equation (34). Call it the RENTRATIO. It is the (rental income/disposable personal income – mean)/standard deviation.

$$(33) f^*(t) = [(\beta(t) - (1/2)\sigma^2)]/\sigma^2.$$

$$(34) N(f^*(t)) = [(\beta(t) - \beta)]/\sigma(\beta) = \text{RENTRATIO}$$

The main term is $[(\beta(t) - \beta)]$, the deviation of the return on capital from its mean value over the entire period. We must estimate $\beta(t)$, the productivity of capital. The productivity of housing capital is the implicit net rental income/value of the home plus a convenience yield in owning one's home. Assume that the convenience yield in owning a home has been relatively constant. Approximate the return $\beta(t)$ by using the ratio of rental income/disposable personal income. This ratio is not sensitive to the level of housing prices, whereas rents/value of housing is statistically negatively related to the level of housing prices. In figure 4/eqn. (34) variable RENTRATIO is the normalized return, measured in units of standard deviation from the mean β . This ratio was relatively constant from 1994 to 2002 and then fell drastically.

Both the actual (DEBTSERVICE) and optimal (RENTRATIO) are graphed in normalized form in figure 4.

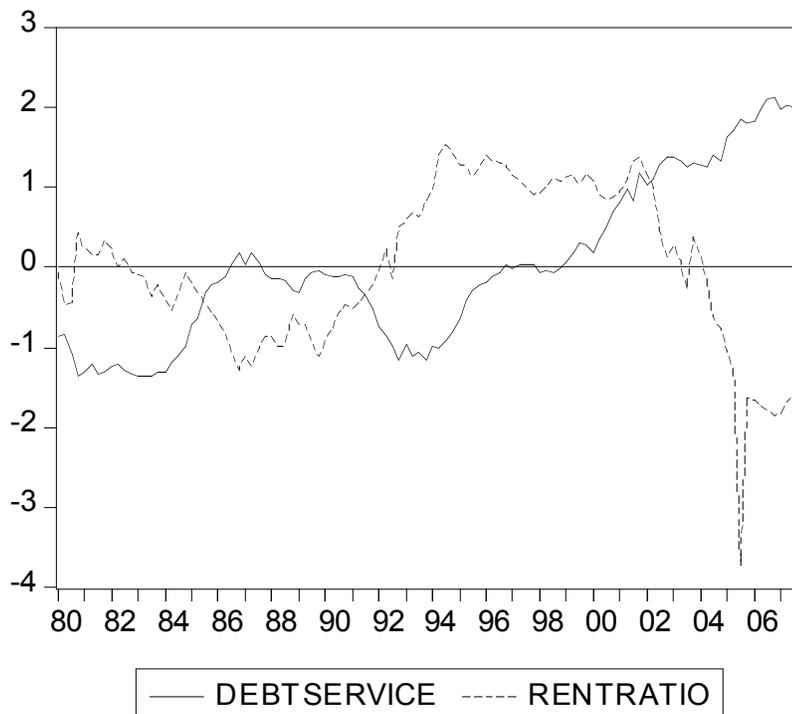


Figure 4. Early Warning Signals: Excess debt $\Psi(t) = N[f(t)] - N[f^*(t)]$.
 $N[f(t)] = \text{DEBTSERVICE} = (\text{household debt service as percent of disposable income} - \text{mean})/\text{standard deviation}$. $N[f^*(t)] = \text{RENTRATIO} = (\text{rental income/disposable personal income} - \text{mean})/\text{standard deviation}$; Sources FRED

The next question is how to estimate the excess debt $\Psi(t)$ that corresponds to eq.19/figure 3, and is consistent with alternative estimates of the optimal debt.

I estimate excess debt $\Psi(t) = (f(t) - f^*(t))$ by using the difference between two normalized variables $N(f) - N(f^*)$, equation (35). This difference is *measured in standard deviations*.

$$(35) \Psi(t) = \text{Excess Debt} \sim N[f(t)] - N[f^*(t)] = \text{DEBTSERVICE} - \text{RENTRATIO}.$$

Excess Debt graphed in figure 5 corresponds to the difference $\Psi(t) = f^*(t) - f(t)$ on the horizontal axis in figure 3, measured in standard deviations. The probability of a decline in net worth $\Pr(d \ln X(t) < 0)$ in (36) is positively related to $\Psi(t)$ the excess debt. As the excess debt rises, the expected growth declines and the risk increases, equation (36).

$$(36) \Pr(d \ln X(t) < 0) = H(\Psi(t)), H' > 0.$$

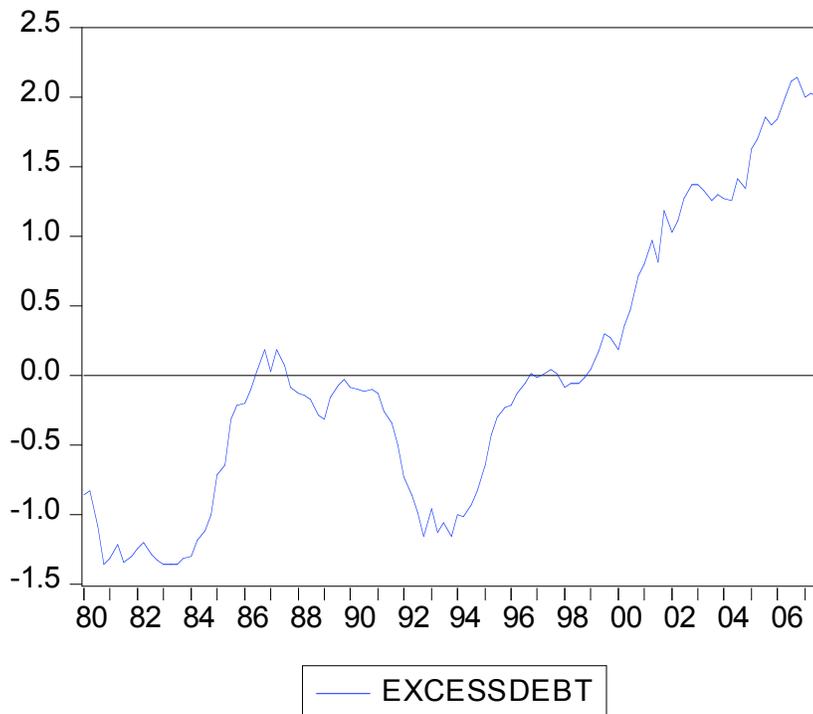


Figure 5. Excess Debt = Debt service – Rent ratio. Normalized.

Assume that over the entire period 1980 – 2007 the debt ratio was not excessive. During the period 2000-2004, the high capital gains and low interest rates induced rises in housing prices relative to disposable income and led to rises in the debt service ratio. By 2005-06 the ratio of housing price/disposable income was about three standard deviations above the long-term mean. See PRICEINC in figure 2. This drastic rise alarmed several economists who believed that the housing market was drastically overvalued. As indicated in part 2 above, they were in a minority. It certainly had a negligible effect upon the market for derivatives and the optimism of the “Quants”.

The advantages of using excess debt $\Psi(t)$ in figure 5 as an Early Warning Signal compared to just the ratio of housing price/disposable income are that $\Psi(t)$ focuses upon the fundamental determinants of the optimal debt ratio as well as upon the actual ratio. The probability of declines in net worth, the inability of the mortgagors to service their debts and the financial collapse due to leverage - a crisis - are directly related to the excess debt. Moreover, the use of normalized variables indicates the magnitude of the excess debt

in terms of standard deviations, and more meaningful estimates can be made of the probability of a crisis.

Based upon figure 5, early warning signals were given as early as 2002. By 2005, the excess debt was two standard deviations above the mean. Hence the debt ratio was in the region of f-max in figure 3. The actual debt was induced by capital gains in excess of the interest rate. The debt could only be serviced from capital gains. This situation is unsustainable. When the capital gains fell below the interest rate, the debts could not be serviced. A crisis was inevitable.

5. Conclusions

Alan Greenspan's paper (March 2010) presented his retrospective view of the crisis. Two main themes emerge. First, the housing price bubble, its subsequent collapse and the financial crisis were not predicted either by the market, the FED, the IMF or the regulators in the years leading to the current crisis. Second, the FED, IMF, Treasury and the "Quants"/market lacked the appropriate tools of analysis to answer the following questions: what is an optimal leverage or capital requirement that balances the expected growth against risk? What are theoretically founded early warning signals of a crisis? What lessons should be learned and what are the contributions of this paper?

Bernanke has studied the Great Depression in depth and the role of bank failures in explaining why it was so severe. He developed the concept of the "financial accelerator." Perhaps one could say that in the subprime crisis he did not draw the lesson he had learned from his study of the Depression regarding the manner in which financial institutions can magnify the severity of an economic downturn. The Fed chairman and staff would have benefited if it had applied the SOC as described in this paper. But not only did they lack the SOC, they apparently lacked other tools which might have indicated that asset values were vastly out of line with fundamentals. They were not searching for such tools because they did not believe that they could or should look for misaligned asset values or excess debt, despite warnings from Shiller and people in the industry. They were blind-sided by the Jackson Hole Consensus which gave them great comfort in adopting a hands off position. So it was not just a lack of appropriate tools which undid the Fed; it was a complete lack of appreciation of what its role should be to

head off an economic catastrophe. Thus there are two separate but related questions: Are identification and containment of a financial bubble legitimate activities of the Fed, and if they are, what are the best tools to carry out this analysis? As the Fed answered "No" to the first question, it saw no need to address the second question.

The Jackson Hole Consensus explains to a considerable extent the Fed's behavior. Greenspan has great knowledge of financial markets and did have some qualms about the housing boom. I think that his behavior can be explained rationally. First he understands that the function of financial markets is to channel saving into investment in the optimal way to promote growth. Second, like most of the economics profession, he accepted the generality of the *First Theorem of Welfare Economics*. This theorem (Koopmans and Bausch) states that a Competitive Equilibrium is a Pareto Optimum. A Competitive Equilibrium is a vector of prices, where (i) supply equals demand, (ii) consumers optimize demand and their supply of labor services, given their preferences and (iii) producers optimize by maximizing their profits, given the technology. A Pareto Optimum is a vector of prices such that (iii) supply equals demand and (iv) it is not possible to select vectors which would make some people better off without making others worse off. The implication is that "market regulation" is superior to regulation by bureaucrats, politicians. Do not try to second guess the markets.

The belief in the generality of the *First Theorem of Welfare* provided a basis for the Jackson Hole Consensus. The Theorem does not hold in financial market because the assumption of atomistic agents operating in perfectly competitive markets with full information and stable preferences is wildly unrealistic. The clearly imperfect markets operating with agents acting without full information concerning the values of the complex derivatives, in fact, almost complete ignorance, implies the conclusion that the situation before the crash cannot be considered to be a Pareto Optimum.

When the crash occurred, Greenspan said (2008) "Those of us who have looked to the self-interest of lending institutions to protect stockholders' equity, myself included, are in a state of disbelief". This statement is ambiguous. The mortgagors optimized by supplying mortgages under the terms offered. The originators optimized by packaging the heterogeneous mortgages, where due diligence was not performed, into CDO's, collecting huge commissions and selling the CDO's to financial intermediaries. The latter

optimized by securitizing them and selling them to investors who were delighted to purchase them at the offered prices. Contrary to Greenspan, the agents believed that they were operating in the “best interests” of the stockholders. The conditions leading to the crash cannot be considered to be a Pareto Optimum. The *First Welfare Theorem* did not seem to be relevant in the financial markets. Greenspan and the academic economics profession were guilty of the failings mentioned by Krugman and described by Shojai/Feiger as “The Economists’ Hubris”. The critique by the Congressional Oversight Panel provides perspective on many of these problems.

The Quants, who were involved in the securitization, focused upon short run profit maximization, but made several egregious errors. First, they ignored the fact that the whole structure rested upon the ability of the mortgagors to service their debts. Second, the Quants assumed that the distribution of capital gains is independent of their actions, and all that they had to do is to find the correct way to describe mathematically the distribution function, whose histogram is graphed in figure 1. The mortgages were supplied and demanded on the assumption that the capital gains would be sufficient to pay the interest. Such a situation is not sustainable. They ignored the economic fact that there can be no free lunch.

Third, they assumed that the markets are liquid so that, at the first sign of trouble, they could unwind their positions. They ignored the fact that all of them were equally well trained and used the same model. Hence they would all want to sell at the same time. They ignored the facts that: their behavior was positively and highly correlated, so that there would be systemic risk and an absence of a liquidity when defaults occurred.

The Stochastic Optimal Control (SOC) analysis in this paper provides another tool of analysis. The housing price bubble was dangerous because it led to a significant rise in the debt ratio. This is systemic risk. Therefore the analysis should focus upon the question: What is an optimal debt or leverage that maximizes the expected growth of net worth and is based upon significant risk aversion? What is optimal risk? These questions were not answered by Shiller and by the other “Cassdandras”. The Stochastic Optimal Control Analysis developed here derives the time varying optimal debt ratio. (1) The optimum debt ratio or leverage maximizes the expected growth of net worth. (2) As the debt ratio rises above the optimum, the expected growth of net worth declines and the

risk rises. (3) The probability of a crisis is positively related to the excess debt, equal to the difference between the actual and optimal debt ratio, measured in standard deviations. (4) An unambiguous early warning signal EWS of a debt crisis would be that the leverage $f(t) = L(t)/X(t)$ exceeds $f\text{-max}$ in Figure 3, so that the expected growth of net worth is negative and the risk is high. (5) The derived EWS would have clearly predicted the crisis, Figure 5.

Peter Clark (2009) wrote: “no measure of underlying or fundamental value will provide consistently accurate predictions of emerging bubbles, but the prior question is whether it is useful to even contemplate the exercise of assessing market values. In light of the huge costs of the housing and credit bubble, the answer must be in the affirmative”. Fed Vice Chairman Kohn indicated that the Fed’s thinking may have changed. He wrote (2009, quoted by Clark): “As researchers, we need to be honest about our very limited ability to assess the ‘fundamental value’ of an asset or to predict its price. But the housing and credit bubbles have had a substantial cost. Research on asset prices...should help to identify risks and inform decisions about the costs and benefits from a possible regulatory or monetary policy decision attempting to deal with a potential asset price bubble.”

Greenspan, Bernanke would have benefitted had their staff added the analytic tool developed here to their array of techniques.. It is hoped that the FED will not be like the ancien régime: “Ils n’ont rien appris, ni rien oublié”.

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