Economics
The Open-Access, Open-Assessment E-Journal

Discussion Paper
No. 2009-45 | October 16, 2009 | http://www.economics-ejournal.org/economics/discussionpapers/2009-45

Statistical Theories of Income and Wealth Distribution

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Please cite the corresponding journal article:

http://www.economics-ejournal.org/economics/journalarticles/2010-4

Abstract

The distributions of income and wealth in countries across the world are found to possess some robust and stable features independent of the specific economic, social and political conditions of the countries. We discuss a few physics-inspired multi-agent dynamic models along with their microeconomic counterparts, that can produce the statistical features of the distributions observed in reality. A number of exact analytical methods and solutions are also provided.

JEL: C31, D51, C00

Keywords: Income distribution; wealth distribution; inequality; exchange economies; money

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1 Introduction

Even from everyday experience, one can understand that almost without any exception income and wealth in a society are unequally distributed among its people and from time immemorial, this inequality has been a constant source of irritation in all societies. There are several non-trivial issues and questions related to this obseravtion. In fact, the issue of inequality in terms of income and wealth has been perhaps the most fiercely debated one in economics. Economists and philosophers have spent much time on the normative aspects of this problem (Sen, 1999; Foucault, 2003; Scruton, 1985; Rawls, 1971). The direct and indirect effects of inequality on the society have also been studied extensively. In particular, the effects of inequality on the growth of the economy (Aghion *et al*, 1999; Barrow, 1994; Benabou, 1994; Forbes, 2000) or on the political-economic scenario (Alesina and Rodrik, 1992; Benabou, 2000; Alesina and Perotti, 1993; Blau and Blau, 1982) have attracted major attention. Relatively less emphasis had been put on the sources of the problem itself. But there remain

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very important questions that beg to be answered. How are income and wealth distributed? What are the forms of the distributions? Are they universal or do they depend upon the specific conditions in the individual country? And the most important question is, if inequality is universal, as some of its gross features obviously are, then what is the reason for such universality? More than a hundred years back, this problem caught attention of Pareto and he found that wealth distribution follows a power law decay for the richer section of the society (Pareto, 1897). Much later Champernowne also considered this problem systematically and he came up with a probabilistic theory to justify Pareto's claim (Champernowne, 1953). Separately, Gibrat worked on the same problem and he proposed a law of proportionate development (Gibrat, 1931). It was subsequently found in numerous studies that the distributions of income and wealth indeed possess some globally stable and robust features (see e.g., Yakovenko and Rosser, 2009 for a review). In general, the bulk of the distribution of both income and wealth seems to fit both the log-normal and the gamma distributions reasonably well. Economists usually prefer the log-normal distribution (Montroll and Shlesinger, 1982; Gini, 1921) whereas statisticians (Hogg *et al*, 2007) and more recently physicists (Yakovenko and Rosser, 2009; Chatterjee *et al*, 2005; Chatterjee and Chakrabarti, 2007), tend to rely more on the gamma distribution. And the upper end of the distribution, that is the tail of the distribution, is agreed to be described well by a power law as was found by Pareto. Although the exact nature of such distributions are yet to be finalized, there is a general agreement on the observation that income and wealth distributions show regularities independent of the country-specific conditions and these observed regularities in patterns may be indicative of a *natural law* of economics.

Here, we survey some multi-agent dynamic models inspired by the physics of energy distribution in many-body thermodynamic systems. Specifically, we intend to discuss a very simple microeconomic model with a large number of agents and consider the asset transfer equations among the agents due to trading in such an economy. It will be shown that this type of asset transfer among the agents in an economy closely resembles the process of energy transfer due to collisions among particles in a thermodynamic system like an ideal gas. The steady state distribution for such a system is an exponential one, as was found by Gibbs a hundred years back (see e.g., Yakovenko and Rosser, 2009). We then see that several modified versions of the same model produce gamma function like behavior for the distribution of income among the agents in the economy. A further modification of the model produces a power law for the upper or tail end of the distribution of income/wealth, as has been found empirically. Next, we discuss the analytical aspects of the models and provide some exact results and derivations of the same. So far this is the only known class of models which, starting from microeconomics of utility maximization and solving for the resultant dynamical equations in the line of rigorously established statistical physics, can reproduce quite reliably the major features of both of the income and wealth distributions in economies.

This paper is organized as follows. In section 2, we review the data gathered on income and wealth distributions. In section 3, we consider a simple microeconomic framework as our basic model. In the next section, we discuss a number of different modifications of the model focusing on different economic behaviorial assumptions that lead to a number of intriguing results. In section 5, we discuss a method to find out the moments of the distributions upto any order and also review some earlier analytical results of the models considered.

2 A short review of data

The distributions of income and wealth have long been subject to detailed empirical analysis and tests. To put the result briefly, these studies (Yakovenko and Rosser, 2009; Chatterjee *et al*, 2005; Chakrabarti *et al*, 2006) so far indicate that

$$P(m) \sim \begin{cases} m^{\alpha}\exp(-m/T) & \text{for } m < m_c, \\ m^{-(1+\nu)} & \text{for } m \geq m_c, \end{cases} \qquad (1)$$

where P denotes the number density of people with income or wealth m and α, ν denote exponents and T denotes a scaling factor. The power law in income and wealth distribution (for $m \geq m_c$) is named after Pareto and the exponent ν is called the Pareto exponent. The crossover point (m_c) is extracted from the crossover of the Gamma (or log-normal) distribution to the power law tail. The existence of both features in the same distribution was possibly first demonstrated by Montroll and Shlesinger (1982) who observed that while the top 2-3 % of the population (in terms of income) followed a power law with Pareto exponent $\nu \simeq 1.63$, the rest followed a lognormal distribution. That study led economists to fit the region below m_c to a log-normal form, $\log P(m) \propto -(\log m)^2$. This form has indeed been seen in several studies (see e.g., Souma, 2000; Di Matteo *et al*, 2004; Clementi and Gallegati, 2005). But there are strong empirical evidences that the Gamma distribution form Eqn. (1) fits better with the data, (see e.g., the remarkable fit with the Gibbs distribution in Silva and Yakovenko, 2005 and also Drăgulescu and Yakovenko, 2001, Drăgulescu and Yakovenko, 2001a). There are many studies concluding that the tail is described well by a power law (see e.g., Souma, 2000; Drăgulescu and Yakovenko, 2001; Drăgulescu and Yakovenko, 2001a; Aoyama *et al*, 2000). Interestingly, the tail of the distribution of income of companies also follows a power law (see e.g., Okuyama *et al*, 1999; Axtell, 2001).

While there is no dearth of empirical analysis on the income distribution, relatively few studies have considered the distribution of wealth due to the lack of an easily available data source. However, Drăgulescu and Yakovenko (2001a), Levy and Solomon (1997), Coelho *et al* (2004), Sinha (2006) have studied wealth distributions extensively. Hegyi *et al* (2007) studied the wealth distribution in Hungarian medieval society. Similar studies are done on the wealth distribution of ancient Egyptian societies (14-th century BC) (Abul-Magd, 2002) as well. The general feature observed in these limited empirical studies of wealth distribution is that of a power law behavior for the wealthiest $5 - 10\%$ of the population, and gamma or log-normal distribution for the rest of the population.

To sum up, numerous investigations during the last ten years revealed that the tail of the income distribution indeed follows a power law with the value of the Pareto exponent ν generally varying between 1 and 3 (Di Matteo *et al*, 2004; Clementi and Gallegati, 2005; Drăgulescu and Yakovenko, 2001a; Levy and Solomon, 1997; Sinha, 2006; Oliveira *et al*, 1999; Aoyama *et al*, 2003, Clementi and gallegati, 2005a). The rest of the low income population, follow a different distribution which is debated to be either gamma (Drăgulescu and Yakovenko, 2001; Levy and Solomon, 1997; Aoyama *et al*, 2003; Chakrabarti and Marjit, 1995; Ispolatov *et al*, 1998; Drăgulescu and Yakovenko, 2000) or log-normal (Di Matteo *et al*, 2004; Clementi and Gallegati, 2005; Clementi and Gallegati, 2005a).

The striking similarities observed in the income distributions for different countries indicate that probably the same process governs the distributions of assets in different economies though these economies are superficially different. There is a huge literature on modelling the economies in analogy with large systems of interacting particles, by physicists. From that perspective, the economy is often viewed as a thermodynamic system in which the distribution of income among the agents is readily identified with the distribution of energy among the particles in a gas. In particular, a class of kinetic exchange models have provided a simple mechanism for understanding the unequal distribution of assets. These models have been successful to capture the key factors in economic interactions that results in different economies with different socio-political structures converging to similar forms of unequal distribution of resources (see Chatterjee *et al*, 2005 and Chakrabarti *et al*, 2006, which consists of a collection of large number of technical papers in this field).

3 The model

We intend to discuss a minimal model to analyze the effects of stochastic trading processes on the asset holding in the steady state of an economy. Chakrabarti and Chakrabarti (2009) considered an N-agent exchange economy. Each of the agents produces a single perishable commodity which is different from all other commodities produced. Money is treated as a non-perishable commodity which facilitates transactions. All commodities alongwith money can enter the utility function of any agent as arguments. These agents care for their future consumptions and hence they care about their savings in the current period as well. Initially, all of these agents are endowed with an equal amount of money which is assumed to be unity. As will be shown, the steady state distribution of money is independent of the initial amount endowment. At each time step, two agents are chosen at random to carry out transactions among themselves following the utility maximization principle. The utility functions are of Cobb-Dauglas type. We also assume that the preference structure of the agents are time-dependent that is the parameters of the utility function vary over time (Lux, 2005; Silver *et al*, 2002). Below, we consider a typical transaction that leads to the dynamics of money among the agents.

Suppose agent 1 produces Q_1 amount of commodity 1 only and agent 2 produces Q_2 amount of commodity 2 only and the amounts of money in their possession at time t are $m_1(t)$ and $m_2(t)$ respectively. Since neither of the two agents possess the commodity produced by the other agent, both of them will be willing to trade with each other and buy the other good by selling a fraction of their own productions as well as with the money that they hold. Hence, at each time step there would be a net transfer of money from one agent to the other due to trade. Our focus is on how the amounts money held by the agents change over time due to the repetition of such a trading process. For notational convenience, we denote $m_i(t+1)$ as m_i and $m_i(t)$ as M_i (for $i = 1, 2$).

Utility functions are defined as follows. For agent 1, $U_1(x_1, x_2, m_1) = x_1^{\alpha_1} x_2^{\alpha_2} m_1^{\alpha_m}$ and for agent 2, $U_2(y_1, y_2, m_2) = y_1^{\alpha_1} y_2^{\alpha_2} m_2^{\alpha_m}$ where the arguments in both of the utility functions are consumption of the first (i.e., x_1 and y_1) and second good (i.e., x_2 and y_2) and amount of money in their possession respectively. For simplicity, we assume that the sum of the powers is normalized to 1 i.e., $\alpha_1 + \alpha_2 + \alpha_m = 1$. Let the commodity prices to be determined in the market be denoted by p_1 and p_2. Now, we can define the budget constraints as follows. For agent 1 the budget constraint is $p_1 x_1 + p_2 x_2 + m_1 \leq M_1 + p_1 Q_1$ and similarly, for agent 2 the constraint is $p_1 y_1 + p_2 y_2 + m_2 \leq M_2 + p_2 Q_2$. In this set-up, we get the market clearing price vector $(\hat{p}_1, \hat{p}_2)$ as $\hat{p}_i = (\alpha_i/\alpha_m)(M_1 + M_2)/Q_i$ for $i = 1$, 2 (see Chakrabarti and Chakrabarti, 2009).

By substituting the demand functions of x_i, y_i and p_i for $i = 1$, 2 in the money demand functions, we get the most important equation of money exchange in this model. We make a restrictive assumption that α_1 in the utility function can vary randomly over time with α_m remaining constant. It readily follows that α_2 also varies randomly over time with the restriction that the sum of α_1 and α_2 is a constant (1-α_m). In the money demand equations derived from the above-mentioned problem, we substitute α_m by λ and $\alpha_1/(\alpha_1+\alpha_2)$ by ϵ to get the money evolution equations as

$$\begin{aligned} m_1(t+1) &= \lambda m_1(t) + \epsilon(1-\lambda)(m_1(t)+m_2(t)) \\ m_2(t+1) &= \lambda m_2(t) + (1-\epsilon)(1-\lambda)(m_1(t)+m_2(t)) \end{aligned} \tag{2}$$

where $m_i(t) \equiv M_i$ and $m_i(t+1) \equiv m_i$. For a fixed value of λ, if α_1 is a random variable with uniform distribution over the domain $[0, 1-\lambda]$, then ϵ is also uniformly distributed over the domain $[0, 1]$. For the limiting value of α_m in the utility function (i.e., $\alpha_m \to 0$ which implies $\lambda \to 0$), we get the money transfer equation describing the random sharing of money without savings (see Chakrabarti and Chakrabarti, 2009 for derivation and a discussion in details).

A noteworthy feature of this model is that the exchange equations are not sensitive to the level of production that is even if for some reason the level of production alters (due to production shock) the form of the transfer equations will remain the same provided the form of the utility function remains the same. Also, the model captures the possibility of coupling in the evolution of assets (money). This set of equations forms the basis of our subsequent analysis.

4 Stochastic models

As is shown above, the results of the economic activities (production, trade and consumption) is represented by a pair of asset transfer equations (see Eqn. (2)). What we basically do is to study the steady state behavior of some modifications of this pair of equations. In the following models, one considers a closed economic system where total money M and total number of agents N is fixed. It is assumed that the system is conservative and no migration occurs. Each agent i posesses money $m_i(t)$ at time t. In any trading, a pair of agents i and j exchange their money (Chakrabarti and Marjit, 1995; Ispolatov *et al*, 1998; Drăgulescu and Yakovenko, 2000; Chakraborti and Chakrabarti, 2000; Chakrabarti and Chakrabarti, 2009). such that their total money is (locally) conserved and none end up with negative money ($m_i(t) \geq 0$, i.e., debt not allowed):

$$m_i(t+1) = m_i(t) + \Delta m;\; m_j(t+1) = m_j(t) - \Delta m \tag{3}$$

following local conservation:

$$m_i(t) + m_j(t) = m_i(t+1) + m_j(t+1); \tag{4}$$

time (t) changes by one unit after each trading.

4.1 Model A: Random sharing of money

The simplest model considers random sharing of the total money between the trading partners. Assuming $\lambda \to 0$ in Eqn. (2), we get

$$\begin{aligned} m_i(t+1) &= \epsilon[m_i(t) + m_j(t)] \\ m_j(t+1) &= (1-\epsilon)[m_i(t) + m_j(t)] \end{aligned} \tag{5}$$

for the i-th and the j-th agent, where ϵ is a random fraction uniformly distributed between 0 and 1. This is a set of very basic equations of money transfer among the agents. Interestingly, the same set of equations represents transfer of energy among particles due to collisions in an ideal gas except that there all m_i's (money) in Eqn. (5) are substituted by the colliding particles' energies. Using the multiplicative property of probabaility i.e.,

$$P(m_i)P(m_j) = P(m_i + m_j), \tag{6}$$

one can solve for the steady state ($t \to \infty$) distribution of money which is a Gibbs (exponential) distribution:

$$P(m) = (1/T)\exp(-m/T); T = M/N. \tag{7}$$

Using the same technique, we get the same result in our model for money distribution. Hence, no matter how uniform or justified the initial distribution is, the eventual steady state correspond to the exponential distribution where most of the people have got very little money. This steady state result is seen to be very robust. Several variations of the mode of trading, and of the 'trading network' (on which the agents can be put at the nodes and each agent trade with its 'neighbors' only), whether compact, fractal or small-world like (Oliveira *et al*, 1999) leaves the distribution unchanged. There are still other studies where variations like random sharing of an amount $2m_2$ only (not of $m_1 + m_2$) when $m_1 > m_2$ (trading at the level of the relatively poorer agent in the trade), lead even to a drastic situation: all the money in the market drifts to one agent and the rest become truely pauper (Hayes, 2002; Chakraborti, 2002).

4.2 Model B: With constant λ

We now consider Eqn. (2) with constant λ for the i-th and j-th agents:

$$m_i(t+1) = \lambda m_i(t) + \epsilon(1-\lambda)(m_i(t) + m_j(t))$$

$$m_j(t+1) = \lambda m_j(t) + (1-\epsilon)(1-\lambda)(m_i(t) + m_j(t)).$$

Note that λ acts as a savings factor in this model, where each trader at time t saves a fraction λ of its money $m_k(t)$ (for $k = i,\ j$) and trades randomly with the rest (see Chakraborti and Chakrabarti, 2000).

The market (non-interacting at $\lambda = 1$) becomes 'interacting' for any non-vanishing $\lambda(< 1)$. For fixed λ (same for all agents), the steady state distribution $P(m)$ of money is exponentially decaying on both sides of the mode of the distribution i.e., the most-probable amount of money per agent. The mode also shifts away from $m = 0$ (for $\lambda = 0$) to M/N as $\lambda \to 1$ (Fig. 1). This self-organizing feature of the market, induced by sheer self-interest of saving by each agent without any global perspective, is very significant since the fraction of people below a particular poverty line decrease as the saving fraction λ increases and most people end up with some finite, non-zero fraction of the average money in the market (Chakraborti and Chakrabarti, 2000). Although this fixed saving propensity does not give yet the Pareto-like power-law distribution, the Markovian nature of the scattering or trading processes (see Eqn. (6)) is effectively lost. Indirectly through λ, the agents get to know (start interacting with) each other and the system co-operatively self-organises towards a stable form with a non-vanishing most-probable income (see Fig. 1).

Angle (Lux, 2005; Angle, 1986; Angle, 2006) proposed an early version of the above model several years back in sociology journals.

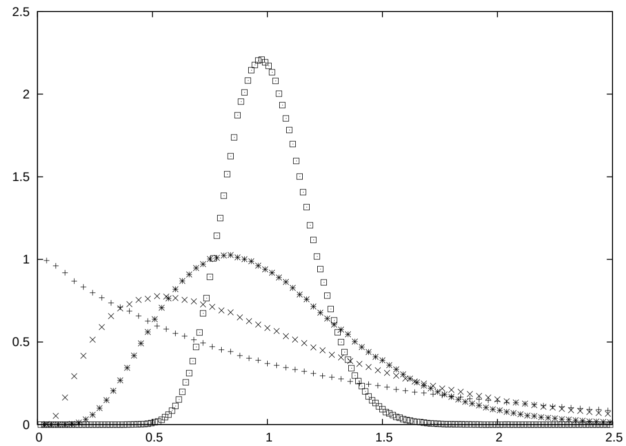

Figure 1:
Steady state money distribution $P(m)$ (y-axis) is plotted against money m (x-axis) for the model with uniform savings. $+$, $\times$, $*$ and $\Box$ denotes distributions with $\lambda = 0$, 0.3, 0.6 and 0.9 respectively. All data sets shown are for average money per agent $M/N = 1$ and $N = 100$.

Angle's 'Inequality Process' is described by the following equations:

$$\begin{aligned} m_i(t+1) &= m_i(t) + D_t w m_j(t) - (1 - D_t) w m_i(t) \\ m_j(t+1) &= m_j(t) + (1 - D_t) w m_i(t) - D_t w m_j(t) \end{aligned} \tag{8}$$

where w is a fixed fraction and D_t takes value 0 or 1 randomly. The numerical simulation results of Angle's model fit well to Gamma distributions.

In the model with uniform savings, the distribution of the monetary assets shows a self organizing feature. A peaked distribution with a most-probable value indicates an economic scale. Empirical findings in homogeneous groups of individuals as in waged income of factory labourers in UK and USA (Willis and Mimkes, 2004) and data from population survey in USA among students of different school and colleges support this observation (Angle, 2006).

4.3 Model C: With distributed λ

In reality, people face different constraints resulting in different patterns of saving or at a more basic level, their attitudes towards savings may not be the same i.e., the parameters of their utility functions may differ from one person to another. This in turn implies that the saving

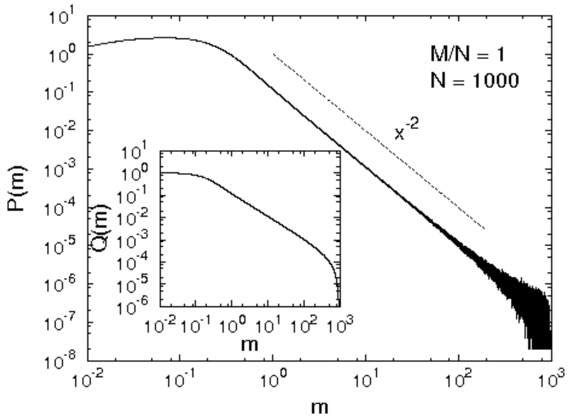

Figure 2:
Steady state money distribution $P(m)$ for the distributed λ model with $0 \leq \lambda < 1$ for a system of $N = 1000$ agents with the average money per agent $M/N = 1$. A power-law is observed with $1 + \nu = 2$.

parameter λ is very heterogeneous. To imitate this situation, we allow λ to be widely distributed within the population (Chatterjee *et al*, 2003; Chakrabarti and Chatterjee, 2004; Chatterjee *et al*, 2004). The evolution of money in such a trading can be written as:

$$\begin{aligned} m_i(t+1) &= \lambda_i m_i(t) + \epsilon\left[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)\right] \\ m_j(t+1) &= \lambda_j m_j(t) + (1-\epsilon)\left[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)\right]. \end{aligned} \tag{9}$$

The trading rules are same as before, except that

$$\Delta m = \epsilon(1-\lambda_j)m_j(t) - (1-\lambda_i)(1-\epsilon)m_i(t) \tag{10}$$

here; where λ_i and λ_j are the saving propensities of agents i and j. The agents have fixed (over time) saving propensities, distributed independently, randomly and uniformly (white) within an interval 0 to 1: agent i saves a random fraction λ_i $(0 \leq \lambda_i < 1)$ and this λ_i value is quenched for each agent (λ_i are independent of trading or t). Studies show that for uniformly distributed saving propensities, $\rho(\lambda) = 1$ for $0 \leq \lambda < 1$, one gets eventually $P(m) \sim m^{(1+\nu)}$, with $\nu = 1$ (see Fig. 2). The eventual deviation from the power law in $Q(m)$ in the inset of Fig. 2 is due to the exponential cutoff contributed by the rare statistics for high m value.

An analytical derivation of the pareto law found above, is provided in section 5.3. It is found that the variation in ϵ plays no role in it. The key factor is the distribution of the savings propensity λ. Refering to section 3, we can define the utility functions as follows. For agent 1, $U_1(x_1, x_2, m_1) = x_1^{\alpha_1} x_2^{\alpha_2} m_1^{\alpha_m}$ and for agent 2, $U_2(y_1, y_2, m_2) = y_1^{\beta_1} y_2^{\beta_2} m_2^{\beta_m}$. Again for simplicity, we assume that the sums of the powers are normalized to 1 i.e., $\alpha_1 + \alpha_2 + \alpha_m = 1$ and $\beta_1 + \beta_2 + \beta_m = 1$. We make a crucial assumption that $\alpha_m \simeq \beta_m$ and $\alpha_m, \beta_m \to 1$. Also, we assume that $\alpha_1 = \alpha_2 \simeq \beta_1 = \beta_2$.Then by doing the same exercise as in section 3 and denoting α_m and β_m by λ_1 and λ_2 respectively, we can approximate the money evolution equations in the following form,

$$\begin{aligned} m_1(t+1) &= \lambda_1 m_1(t) + \frac{1}{2}\left((1-\lambda_1)m_1(t) + (1-\lambda_2)m_2(t)\right) \\ m_2(t+1) &= \lambda_2 m_2(t) + \frac{1}{2}\left((1-\lambda_1)m_1(t) + (1-\lambda_2)m_2(t)\right). \end{aligned} \tag{11}$$

Note that ϵ is constant here (equals to 1/2) and λ is the variable. The above set of equations also produce the Pareto distribution in the steady state (see sections 5.2 and 5.3 for analytical derivations of the same).

4.4 Model D: Taxation and redistribution

This model was studied by Drăgulescu and Yakovenko (2001) and Guala (2007). We return to Eqn. (5) which captures the process of random sharing of money. In this model trading process takes place in two steps. In the first step, we assume that prior to trade a fixed fraction τ of money is taxed from both of them. Random sharing occurs with the rest of the money. In the second step, the total amount of money taxed is distributed equally among all the agents in the economy.

$$\begin{aligned} m_i(t+1/2) &= \epsilon(1-\tau)(m_i(t) + m_j(t)) \\ m_j(t+1/2) &= (1-\epsilon)(1-\tau)(m_i(t) + m_j(t)) \end{aligned} \tag{12}$$

For all k,

$$m_k(t+1) = m_k(t+1/2) + \tau\frac{(m_i(t) + mj(t))}{N}. \tag{13}$$

This model also gives rise to gamma-like features in the steady state distribution. But it has a pecularity in that it shows transition from exponential to gamma function as τ goes up and then after a threshold, it returns to an exponential for higher values τ. Guala (2007) shows that the optimal tax rate is about 0.325 where optimality refers to equality among the agents.

4.5 Model E: Risk aversion and insurance

Chakrabarti and Chakrabarti (2009) proposed this model with a different interpretation. In this model, the money transfer process takes place in two steps. The process is again governed by Eqn. (5). We assume that the agents are risk-averse. Hence, prior to trade they reach an agreement that whoever will be the winner, shall transfer a fraction f of his *excess* of money to the loser. This is akin to an insurance where the agents sacrifice higher gains to avoid losses. In the first step, the agents trade in an absolutely random fashion. This step follows from Eqn. (2) above if we consider that $\lambda \to 0$. Hence,

$$m_i(t+1/2) = \epsilon[m_i(t) + m_j(t)]$$
$$m_j(t+1/2) = (1-\epsilon)[m_i(t) + m_j(t)].$$

The agents agree to split the *excess* income. Hence the agent with more money, transfers a fraction f of the excess income to the agent with less money. It is reasonable to assume that $0 \le f \le 0.5$. If $m_i(t+1/2) \ge m_j(t+1/2)$, excess income $\delta = m_i(t+1/2) - m_j(t+1/2)$. Hence,

$$m_i(t+1) = m_i(t+1/2) - (f\delta)$$
$$m_j(t+1) = m_j(t+1/2) + (f\delta).$$

This process is repeated at each time step until the system reaches a steady state and the distribution $p(m)$ of income among the agents in the steady state are studied. Substituting for δ, $m_i(t+1/2)$ and

$m_j(t+1/2)$ in the above equation, we get the reduced equations

$$\begin{aligned} m_i(t+1) &= g[m_i(t)+m_j(t)] \\ m_j(t+1) &= (1-g)[m_i(t)+m_j(t)]. \end{aligned} \tag{14}$$

The expression of g in the above equations is $g = f+(1-2f)\epsilon$. It may be noted that g is a linear transformation of an uniformly distributed variable ϵ. Hence, g is also uniformly distributed and its domain is $[f, 1-f]$. With rising values of f, this model shows a transition from pure exponential (for $f = 0$) to a Δ distribution (for $f = 0.5$) in money holding. Gamma like distributions emerge for values of f between the two extremes.

5 Analytical studies

There have been a number of attempts to study the uniform savings model (Model B, Sec. 4.2) analytically (see e.g., Das and yarlagadda, 2003), but no closed form expression for the steady state distribution $P(m)$ has yet been arrived at. The exact distributions for the model with taxation (Model D) and the model with risk averse agents (Model E) is also unknown whereas the model with distributed savings (Model C) has been solved in several ways (Chatterjee *et al*, 2005a; Chatterjee *et al*, 2005b; Mohanty, 2006; Repetowicz *et al*, 2005; Richmond *et al*, 2005). Here, we discuss a very simple method of obtaining the moments of the distributions upto any order without knowing the actual distribution.

5.1 Moments of the distribution

We denote *expectation* or average of a variable x by $E(x)$ and the central moment of order $n > 1$ (μ_n) of a variable x as

$$E(x-E(x)^n) = E(\sum_{l=0}^{n} \begin{pmatrix} n \\ l \end{pmatrix} x^l E(-x)^{(n-l)}).$$

For $n = 2$, $E(x-E(x)^n)$ corresponds to the variance of x and is denoted by $V(x)$. Since the systems are conservative and the initial endowments were unity for all agents, it is obvious that $E(m_i)$ would be unity. So we can write the n-th moment of the distribution of money without subscript as

$$E((m-1)^n) = E(\sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) (-m)^l). \tag{15}$$

We assume that m_i and m_j are independent variables. Using the money-transfer equation in Eqn. (15), one can find out μ_n iteratively for any n (i.e., if the moments upto $(n-1)$-th order are known then it is possible to find the n-th moment by the above equation). For example, we find out the second moment of the steady state distribution of model B in section 4.2 by assuming that the first moment is set to unity i.e., the average money $M/N = 1$. Note that for the i-th agent, the time evolution of money is

$$m_i(t+1) = \lambda m_i(t) + \epsilon(1-\lambda)[m_i(t) + m_j(t)].$$

By taking expectations on both sides we get $E(m) = 1$. Also, in the steady state, $V(m_i) = E(x^2) - [E(x)]^2$ where $x = \lambda m_i + \epsilon(1-\lambda)(m_i + m_j)$. Using the fact that $E(x) = 1$, we get

$$V(m_i) = \lambda^2 E(m_i^2) + (1-\lambda)^2 E(\epsilon^2[m_i+m_j]^2) + 2\lambda(1-\lambda)E(\epsilon)E(m_i[m_i+m_j]) - 1.$$

Using the symmetry in agent indices i and j one gets $E(\epsilon^2[m_i+m_j]^2) = [V(\epsilon) + 1/4][2V(m) + 4]$. Since $V(\epsilon) = 1/12$ (as ϵ is uniformly distributed), we get the following equation after rearranging terms

$$V(m) = \lambda^2[V(m)+1] + \frac{2}{3}(1-\lambda)^2[V(m)+2] + \lambda(1-\lambda)[V(m)+2] - 1.$$

Simplifying the above expression we get the result for $\lambda \neq 1$,

$$V(m) = \frac{(1-\lambda)}{(1+2\lambda)}.$$

Chakrabarti and Chakrabarti (2009) also discusses the second moment of the distribution of the model E.

Patriarca *et al* (2004) claimed through heuristic arguments (based on numerical results) that the distribution of model B is a close approximate form of the Gamma distribution

$$P(m) = Cm^{\alpha} \exp[-m/T] \tag{16}$$

where $T = 1/(\alpha+1)$ and $C = (\alpha+1)^{\alpha+1}/\Gamma(\alpha+1)$, Γ being the Gamma function whose argument α is related to the savings factor λ as:

$$\alpha = \frac{3\lambda}{1-\lambda} \tag{17}$$

which implies $T = (1-\lambda)/(1+2\lambda)$ and it may be noted that in the case considered here (with $M/N = 1$), T happens to be equal to the variance of the distribution itself. The same value of the parameter is obtained through moment calculation above (Chakrabarti and chakrabarti, 2009). Also, when compared with Eqn. (1), $m_c \to \infty$. The qualitative argument forwarded here Patriarca *et al* (2004) is that, as λ increases, effectively the agents (particles) retain more of its money (energy) in any trading (scattering). This can be taken as implying that with increasing λ, the effective dimensionality increases and temperature of the scattering process changes. This result has also been supported by numerical results in Bhattacharya *et al* (2005). However, Repetowicz *et al* (2005) and Richmond *et al* (2005) analyzed the moments, and found that moments upto the third order agree with those obtained from the form of the Eqn. (16), and discrepancies start from fourth order onwards. Hence, the actual form of the distribution for this model still remains to be found out.

It is seen that the values of the parameters of the distribution derived by the above-mentioned technique (without any distributional assumption), are identical to those found by Patriarca *et al* (2004). Although this technique enables one to derive the exact values of the moments upto any order, it has an obvoius drawback that it can not be applied to the models where one does not get any *representative* equation. For example, this technique does not apply to model D (sec. 4.4).

We review now some of the analytical results on the steady state distribution $P(m)$ of money resulting from the equations Eqn. (9) representing the trading and money dynamics (Model C, Sec. 4.3) in the distributed savings case.

5.2 Distribution of money difference

In the process defined by Eqn. (9), the total money $(m_i + m_j)$ of the pair of agents i and j remains constant, while the difference Δm_{ij}

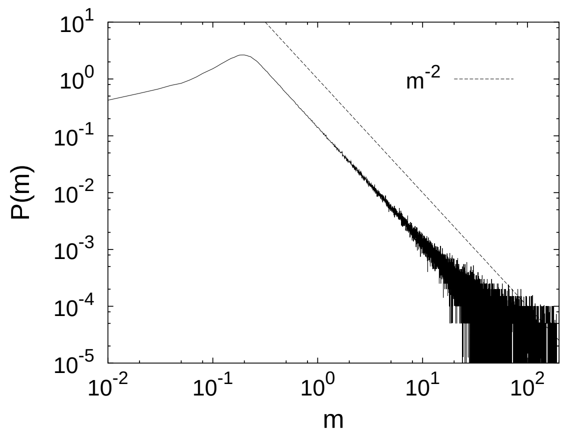

Figure 3:
Steady state money distribution $P(m)$ against m in a numerical simulation of a market with $N = 200$, following equations Eqn. (9) with $\epsilon = 1/2$. The dotted line corresponds to $m^{-(1+\nu)}$; $\nu = 1$. The average money per agent $M/N = 1$.

evolves as

$$\begin{aligned}(\Delta m_{ij})_{t+1} \equiv & \; (m_i - m_j)_{t+1} = \left(\frac{\lambda_i + \lambda_j}{2}\right)(\Delta m_{ij})_t \\ + & \left(\frac{\lambda_i - \lambda_j}{2}\right)(m_i + m_j)_t \\ & +(2\epsilon - 1)[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)].\end{aligned} \tag{18}$$

Numerically, as shown in Fig. 2, we observe that the steady state money distribution in the market becomes a power law, following such tradings when the saving factor λ_i of the agents remain constant over time but varies from agent to agent widely. As shown in the numerical simulation results for $P(m)$ in Fig. 3, the law, as well as the exponent, remains unchanged even when $\epsilon = 1/2$ for every trading (Chatterjee *et al*, 2004). Clearly, the third term in Eqn. (18) is zero for $\epsilon = 1/2$. Even in the case where $\epsilon \to 1$, the third term in the above equation becomes unimportant for the critical behavior. For simplicity, we concentrate on this case, where the above evolution equation for Δm_{ij} can be written in a more simplified form as

$$(\Delta m_{ij})_{t+1} = \bar{\lambda}_{ij}(\Delta m_{ij})_t + \tilde{\lambda}_{ij}(m_i + m_j)_t, \tag{19}$$

where $\bar{\lambda}_{ij} = \frac{1}{2}(\lambda_i + \lambda_j)$ and $\tilde{\lambda}_{ij} = \frac{1}{2}(\lambda_i - \lambda_j)$. As such, $0 \leq \bar{\lambda} < 1$ and $-\frac{1}{2} < \tilde{\lambda} < \frac{1}{2}$.

The steady state probability distribution D for the modulus $\Delta = |\Delta m|$ of the mutual money difference between any two agents in the market can be obtained from Eqn. (19) in the following way provided Δ is very much larger than the average money per agent $= M/N$. This is because, using Eqn. (19), large Δ can appear at $t+1$, say, from 'scattering' from any situation at t for which the right hand side of Eqn. (19) is large. The possibilities are (at t) m_i large (rare) and m_j not large, where the right hand side of Eqn. (19) becomes $\simeq (\bar{\lambda}_{ij} + \tilde{\lambda}_{ij})(\Delta_{ij})_t$; or m_j large (rare) and m_i not large (making the right hand side of Eqn. (19) becomes $\simeq (\bar{\lambda}_{ij} - \tilde{\lambda}_{ij})(\Delta_{ij})_t$); or when m_i and m_j are both large, which is a much rarer situation than the first two and hence is negligible. Consequently for large Δ, the distribution $D(\Delta)$ satisfies

$$\begin{aligned} D(\Delta) &= \int d\Delta' \, D(\Delta') \, \langle \delta(\Delta - (\bar{\lambda} + \tilde{\lambda})\Delta') + \delta(\Delta - (\bar{\lambda} - \tilde{\lambda})\Delta') \rangle \\ &= 2 \left\langle \left(\frac{1}{\lambda}\right) \, D\left(\frac{\Delta}{\lambda}\right) \right\rangle, \end{aligned} \tag{20}$$

where the δ functions take care of the Δ values permitted by Eqn. (19) and we have used the symmetry of the $\tilde{\lambda}$ distribution and the relation $\bar{\lambda}_{ij} + \tilde{\lambda}_{ij} = \lambda_i$, and have suppressed labels i, j. Here $\langle \ldots \rangle$ denote average over λ distribution in the market, and δ denotes the δ-function. Taking now a uniform random distribution of the saving factor λ, $\rho(\lambda) = 1$ for $0 \leq \lambda < 1$, and assuming $D(\Delta) \sim \Delta^{-(1+\nu)}$ for large Δ, we get

$$1 = 2 \int_0^1 d\lambda \, \lambda^{\nu} = 2(1+\nu)^{-1}, \tag{21}$$

giving an unique value of $\nu = 1$. This also indicates that the money distribution $P(m)$ in the market also follows a similar power law variation, $P(m) \sim m^{-(1+\nu)}$ and $\nu = 1$. Distribution of Δ from numerical simulations also agree with this result.

Chatterjee *et al* (2005) and Chatterjee *et al* (2005a) analysed the master equation for the kinetic exchange process and found its solution for a special case. For a pioneering study of the kinetic equations for the two-body scattering process and a more general solution, see Repetowicz *et al* (2005) and Richmond *et al* (2005).

5.3 Average money at any saving propensity and the distribution

Patriarca *et al* (2005) and Patriarca *et al* (2006) studied the relationship between a particular saving factor λ and the average money held by an agent characterized by that savings factor and these numerical studies revealed that the product of the average money and the unsaved fraction remains constant i.e.,

$$\langle m(\lambda)\rangle(1-\lambda) = C \tag{22}$$

where C is a constant; here $\langle x\rangle$ denotes ensemble average over x for a particular value of λ. Mohanty (2006) justifies this result rigorously using a mean-field type approach. It is assumed that the distribution of money of a single agent over time is stationary, which means that the time averaged value of money of any agent remains unchanged independent of the initial value of money. Assuming that the i-th agent interacts with all agents over time and taking the expectation of Eqn. (9), one can write

$$\langle m_i\rangle = \lambda_i\langle m_i\rangle + \langle\epsilon\rangle\left[(1-\lambda_i)\langle m_i\rangle + \langle\frac{1}{N}\sum_{j=1}^{N}(1-\lambda_j)m_j\rangle\right]. \tag{23}$$

The last term on the right can be replaced by the average over the agents (denoted by a constant C) and since ϵ is assumed to be distributed randomly and uniformly in $[0,1]$, so that $\langle\epsilon\rangle = 1/2$, Eqn. (23) reduces to

$$(1-\lambda_i)\langle m_i\rangle = C.$$

Since the right side is free of any agent index, it suggests that this relation is true for any arbitrary agent, i.e., $\langle m_i\rangle(1-\lambda_i) = \text{constant}$, where λ_i is the saving factor of the ith agent (as in Eqn. (22)) and what follows is:

$$d\lambda \propto \frac{dm}{m^2}. \tag{24}$$

Here, m represents $\langle m_i \rangle$ defined above. An agent with a particular saving propensity factor λ therefore ends up with a characteristic average money m given by Eqn. (22) such that one can in general relate the distributions of the two:

$$P(m)\ dm = \rho(\lambda)\ d\lambda. \tag{25}$$

This, together with Eqn. (22) and Eqn. (23) gives (Mohanty, 2006)

$$P(m) = \rho(\lambda)\frac{d\lambda}{dm} \propto \frac{\rho(1-\frac{c}{m})}{m^2}, \tag{26}$$

giving $P(m) \sim m^{-2}$ for large m for uniform distribution of savings factor λ, i.e, $\nu = 1$; and $\nu = 1 + \delta$ for $\rho(\lambda) = (1-\lambda)^{\delta}$. This study therefore explains the origin of the universal ($\nu = 1$) as well as the non-universal ($\nu = 1 + \delta$) Pareto exponent values in the distributed savings model.

6 Summary and discussion

Income and wealth distributions across the population in many countries are found to possess some robust characteristics. As has been discussed in section 2, it is empirically found that the bulk (about 90%) of the population fits a gamma like distribution: after an initial steep rise in probability with income/wealth, an exponential decay is seen in the number of persons with income/wealth. There are considerable deviations from exponential decay in the high income/wealth range and the income and wealth data in that range (for the top 5-10% of the population in any country) fit well to Pareto distribution (power law) with the value of the exponent ranging between 1 and 3.

As has been discussed in section 3, the simple exercise of utility maximization (with a well-known utility function) in a bilateral trading framework gives rise to a pair of money exchange equations. The system depicted by this set of equations is conserved and the coupling behaviour is captured by the same set of equations. This has led to a completely new, statistical formulation of the models of market economies. The dynamics of money in such a model, reveals interesting features about the steady state distribution of money among the interacting agents. Self-organisation is a key emerging feature of these kinetic exchange models when saving factors are introduced. In the model with uniform savings (see Sec. 4.2), the Gamma-like distribution of money shows stable distribution with a most-probable value indicative of an economic scale dependent on the saving propensity or factor λ. Empirical observations in homogeneous groups of individuals supports this theoretical prediction (see e.g., Willis and Mimkes, 2004; Angle, 2006). The moments of the distribution can be found (see Sec. 5) very easily.

Next, the saving propensity is assumed to vary from agent to agent (see Sec. 4.3). The emergence of a power law tail in money-holding is apparent in cases where the saving factor does not change with tradings or time t for the same agent (i.e., where each trader has a different characteristic saving propensity). The money exchange equations can be cast into a master equation, and the solution to the steady state money distribution giving the Pareto law with $\nu = 1$ have been derived using several approaches (see Sec. 5). Then we discuss two different models focusing on different economic institutions that can also give rise to the same gamma function-like behavior for the distribution of assets in the steady state. The first one considers taxation (see Sec. 4.4) whereas the second one considers insurance against losses (see Sec. 4.5). The moments of the resulting distributions of the last model can be found upto any order (see Sec. 5). The possibilities of emergence of self organizations in markets, evolution of the steady state distributions, emergence of Gamma-like distribution for the bulk and the power law tail, are seen to be captured well by this class of market models.

Though the models considered above (Sec. 4) follow from established principles of the utility maximization paradigm (Sec. 3) and the analysis of their kinetics (Sec. 5) have a rigorous foundation based on hundred years' old statistical physics, they are not matured enough yet to be put to use in practice directly. Nevertheless, they present a workable and tractable approach for analysing a statistically large economy. They illuminate the statistical effects of a number of mechanisms and institutions of the economy and reproduce the distributions of assets seen in reality quite reliably; as such they may provide a new foundation of macroeconomics (Lux and Westerhoff, 2009). In future, policy making may also benefit from such detailed understanding of the mechanisms by which distributions of income and wealth emerge out of collective exchanges (Hogan, 2005).

Acknowledgement: We are greatful to Anirban Chakraborti and Arnab Chatterjee for collaboration at various stages of this development. We are also greatful to Banasri Basu, Satya R. Chakravarty, Arnab Das, Kaushik Gangopadhyay, Jun-ichi Inoue, Subhrangshu S. Manna, Manipushpak Mitra, Pradip K. Mohanty, Srutarshi Pradhan, Abhirup Sarkar, Arunava Sen, Sitabhra Sinha, Robin Stinchcombe, Victor M. Yakovenko and Sudhakar Yarlagadda for useful discussions.

References

Abul-Magd A. Y. (2002). Wealth distribution in an ancient Egyptian society. Phys. Rev. E **66**: 057104

Aghion P., E. Caroli, and C. Garcia-Penalosa (1999). Inequality and economic growth: The perspective of the new growth theories. J. Econ. Lit. **37**: 1615

Alesina A., and D. Rodrik (1992). Distribution, political conflict, and economic growth in *Political Economy, Growth and Business Cycles*, ed. A. Cukierman, Cambridge, MA: MIT Press

Alesina A., and R. Perotti (1993). Income distribution, political instability and investment. Eur. Econ. Rev. **40**: 1203

Angle J. (1986). The surplus theory of social stratification and the size distribution of personal wealth. Social Forces **65**: 293

Angle J. (2006). The inequality process as an wealth maximizing process. Physica A **367**: 388

Aoyama H., W. Souma, Y. Nagahara, M. P. Okazaki, H. Takayasu, and M. Takayasu (2000). Pareto's law for income of individuals and debt of bankrupt companies. Fractals **8**: 293

Aoyama H., W. Souma, and Y. Fujiwara (2003). Growth and fluctuations of personal and company's income. Physica A **324**: 352

Axtell R. L. (2001). Zipf distribution of U.S. firm sizes. Science **293**: 1818

Barrow R. (1994). Population growth and economic growth: Some more correlations . Population and Development Review **20**: 503

Benabou R. (1994). Human capital, inequaltiy and growth: A local perspective. Eur. Econ. Rev. **38**: 817

Benabou R. (2000). Unequal societies: Income distribution and the social contract. Am. Econ. Rev. **90**: 96

Bhattacharya K., G. Mukherjee, and S. S. Manna (2005), in Chatterjee *et al* (2005)

Blau J., and P. Blau (1982). The cost of inequality: Metropolitan structure and violent crime. Am. Soc. Rev. **47**: 114

Chakrabarti A. S., and B. K. Chakrabarti (2009). *Microeconomics of the ideal gas like market models*. Physica A **388**: 4151

Chakrabarti B. K., A. Chakraborti, and A. Chatterjee (Eds.), *Econophysics and Sociophysics*, Wiley-VCH, Berlin, 2006

Chakrabarti B. K., and A. Chatterjee, in *Application of Econophysics*, Ed. H. Takayasu, (Springer, Tokyo, 2004) 280

Chakrabarti B. K., and S. Marjit (1995). Self-organization in game of life and economics. Ind. J. Phys. B, **69**: 681

Chakraborti A., and B. K. Chakrabarti(2000). Statistical mechanics of money: How saving propensity affects its distribution. Eur. Phys. J. B, **17**: 167

Chakraborti A. (2002). Distributions of money in model markets of economy. Int. J. Mod. Phys. C, **13**: 1315

Champernowne D. G. (1953). A model of income distribution. The Economic Journal, **63**: 318

Chatterjee A., S. Yarlagadda, and B. K. Chakrabarti (Eds.), *Econophysics of Wealth Distributions*, Springer Verlag, Milan, 2005

Chatterjee A., and B. K. Chakrabarti (2007). Kinetic exchange models for income and wealth distribution. Eur. Phys. J. B **60**: 135

Chatterjee A., B. K. Chakrabarti, and R. B. Stinchcombe (2005a). Master equation for a kinetic model of a trading market and its analytical solution. Phys. Rev. E **72**: 026126

Chatterjee A., B. K. Chakrabarti, and R. B. Stinchcombe, in *Practical Fruits of Econophysics*, Ed. H. Takayasu, (Springer-Verlag, Tokyo, 2005b) 333

Chatterjee A., B. K. Chakrabarti, and S. S. Manna (2003). Money in gas-like markets: Gibbs and Pareto laws. Phys. Scripta T **106**: 36

Chatterjee A., B. K. Chakrabarti, and S. S. Manna (2004). Pareto law in a kinetic model of market with random saving propensity. Physica A, **335**: 155

Clementi F., and M. Gallegati (2005), in Chatterjee *et al* (2005)

Clementi F., and M. Gallegati (2005a). Power law tails in the Italian personal income distribution. Physica A, **350**: 427

Coelho R., Z. Néda, J. J. Ramasco, and M. A. Santos (2005). A family network model for wealth distribution in societies. Physica A **353**: 515

Das A., and S. Yarlagadda (2003). An analytic treatment of the Gibbs-Pareto behavior in wealth distribution. Phys. Scripta T **106**: 39

Di Matteo T., T. Aste, and S. T. Hyde (2004), in *The Physics of Complex Systems (New Advances and Perspectives)*, Eds. F. Mallamace, H. E. Stanley, (Amsterdam, 2004) 435

Drăgulescu A. A., and V. M. Yakovenko (2000). The statistical mechanics of money. Eur. Phys. J. B **17**: 723

Drăgulescu A. A., and V. M. Yakovenko (2001). Evidence for the exponential distribution of income in the USA. Eur. Phys. J. B **20**: 585

Drăgulescu A. A., and V. M. Yakovenko (2001). Exponential and power-law probability distributions of wealth and income in the United Kingdom and the United States. Physica A **299**: 213

Forbes K. J. (2000). A reassessment of the relationship between inequality and growth. Am. Econ. Rev. **90**: 869

Foucault M., *Le Pouvoir psychiatrique: Cors au College de France 1973-74*, Gallimard, Paris, 2003

Gibrat R., *Les Inégalites Économiques*, Libraire du Recueil Sirey, Paris, 1931

Gini C. (1921). Measurement of inequality of incomes. The Economic Journal **31**: 124

Guala S. (2007). Taxes in a simple wealth distribution model by inelastocally scattering particles. arXiv: 0807.4484

Hayes B. (2002). Follow the money. American Scientist **90**, 400

Hegyi G., Z. Néda, and M. A. Santos (2007). Wealth distribution and Pareto's law in the Hungarian medieval society. Physica A **380**: 271

Hogan J. (12 March, 2005). There's only one rule for the rich. New Scientist, 6

Hogg R. V., J. W. Mckean, and A. T. Craig, *Introduction to Mathematical Statistics*, Pearson Education, Delhi, 2007

Ispolatov S., P. L. Krapivsky, and S. Redner (1998). Wealth distributions in asset exchange models. Eur. Phys. J. B **2**: 267

Levy M., and S. Solomon (1997). New evidence for the power law distribution of wealth. Physica A **242**: 90

Lux T. (2005), in Chaterjee *et al* (2005)

Lux T., and F. Westerhoff (2009). Economics crisis. Nature Phys. **5**: 2

Mohanty P. K. (2006). Generic features of the wealth distribution in ideal-gas-like markets. Phys. Rev. E **74**: 011117

Montroll E. W., and M. F. Shlesinger (1982). On 1/f noise and other distributions with long tails. Proc. Natl. Acad. Sci. USA **79**: 3380

Okuyama K., M. Takayasu, and H. Takayasu (1999). Zipf's law in the income distribution of companies. Physica A **269**: 125

Oliveira S. M. D., P. M. C. D. Oliveira, and D. Stauffer, *Evolution, Money, War and Computers*, B. G. Tuebner, Stuttgart, Leipzig, 1999

Pareto V.,*Cours d'economie Politique*,(F. Rouge, Lausanne, 1897)

Patriarca M., A. Chakraborti, and K.Kaski (2004). Statistical model with a standard gamma distribution. Phys. Rev. E **70**: 016104

Patriarca M., A. Chakraborti, K. Kaski, and G. Germano (2005), in Chatterjee *et al* (2005)

Patriarca M., A. Chakraborti, and G. Germano (2006). Influence of the saving propoensity on the power-law tail of the wealth distribution. Physica A **369**: 723

Rawls J., *A Theory of Justice*, Harvard Univ. Press, Harvard, 1971

Repetowicz P., S. Hutzler, and P. Richmond (2005). Dynamics of money and income distribution. Physica A **356**: 641

Richmond P., P. Repetowicz, and S. Hutzler (2005), in Chatterjee *et al* (2005)

Scruton R., *Thinkers of the New Left*, Longman Group Ltd., Harlow, 1985

Sen A., *Poverty and famines*, Oxford Univ. Press, New Delhi, 1999

Silva A. C., and V. M. Yakovenko (2005), in Chatterjee *et al 2005*

Silver J., E. Slud, and K. Takamoto (2002). Statistical equilibrium wealth distribution in an exchange economy with stochastic preference. J. Econ. Theo. **106**: 417

Sinha S. S. (2006). Evidence for power-law tail of the wealth distribution in India. Physica A **359**: 555

Souma W. (2000). Universal structure of the personal income distribution. Fractals **9**: 463

Willis G., and J. Mimkes (2004). Evidence for the independence of waged and unwaged income, evidence for Bolztmann distribution in waged income and the outlines of a coherent theory of income distribution. cond-mat/0406694

Yakovenko V. M., and J. B. Rosser (2009). Colloquium: Statistical mechanics of money, wealth and income. Rev. Mod. Phys. (to be published); also available at arXiv: 0905.1518