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A Tale of Two Debt Crises: A Stochastic Optimal Control Analysis

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Abstract
Banks should evaluate whether a borrower is likely to default. The author applies several techniques in the extensive mathematical literature of stochastic optimal control/dynamic programming to derive an optimal debt in an environment where there are risks on both the asset and liabilities sides. The vulnerability of the borrowing firm to shocks from either the return to capital, the interest rate or capital gain, increases in proportion to the difference between the Actual and Optimal debt ratio, called the excess debt. As the debt ratio exceeds the optimum, default *becomes ever more likely*. This paper is "A Tale of Two Crises" because the analysis is applied to the agricultural debt crisis of the 1980s and to the sub-prime mortgage crisis of 2007. A measure of excess debt is derived, and the author shows that it is an early warning signal of a crisis.

JEL: C61, D81, D91, D92

Keywords: Optimization; banking; stochastic optimal control; agriculture debt crisis; subprime mortgage crisis

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A TALE OF TWO DEBT CRISES: A STOCHASTIC OPTIMAL CONTROL ANALYSIS

Jerome L. Stein

1. Introduction

Bubbles are based upon anticipated but non-sustainable capital gains that are not closely related to the net productivity of capital. As a consequence, the rising debt payments/net income makes the system more vulnerable to shocks either from the capital gains, productivity of capital or the interest rate. A crisis then occurs with bankruptcies and defaults. This paper addresses the question: *How should creditors, banks and bank regulators evaluate and monitor risk of an excessive debt that significantly increases the probability of default?*

This paper may be called: A Tale of Two Crises. The agricultural debt crisis of the 1980s is emblematic of the bubble-crisis phenomenon. I use this as a specific example of the usefulness of the stochastic optimal control analysis, because data are readily available[1] that correspond to the theoretical variables. One can just copy/paste the agriculture story in understanding the sub-prime mortgage crisis of 2007.

Agriculture flourished in the 1970s. Farm exports grew rapidly and along with the domestic inflation farm incomes reached all-time highs. These factors produced capital gains on farm assets. Equity rose significantly. Credit was readily available. Real interest rates were low and farmers used the rising value of farm assets as collateral for loans. Farmers would purchase farm real estate with moderate down payments and, after the value of the newly purchased land increased, would use the increased equity to buy additional farmland with minimal down payments. Higher levels of real estate debt were supplemented by debt to finance machinery and equipment. The speculation in land produced capital gains and raised the market value of equity (EQUITY). The ratio of interest service on the debt/value added (INTVA), the debt burden, rose significantly. See Figure 1.

In the fall of 1979, the Federal Reserve undertook a restrictive monetary policy in order to reduce inflation and interest rates rose drastically. The resulting appreciation of the US dollar

[1] I draw upon the study of the Federal Deposit Insurance Corporation and use data from the Economic Research Service USDA.

reduced foreign demand for US agricultural products. The decline in foreign demand was exacerbated by the debt crisis in the less developed countries. Farm exports declined by 40% from 1981 to 1986, at a time when productive capacity had increased. The result was an accumulation of huge surpluses of farm commodities in the early 1980s. When the bubble collapsed in 1980, asset values and equity fell drastically. The resulting rise in the debt burden was devastating, and the delinquency rate on loans (DELINQRATEFCS) rose drastically.

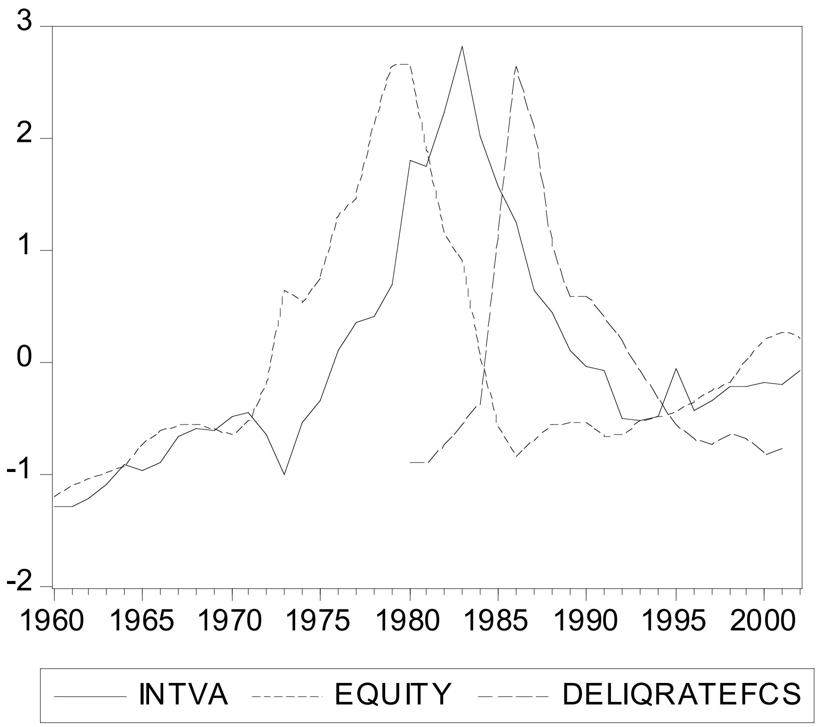

Figure 1. Agricultural Bubble. Normalized variables. INTVA = interest payments/value added = debt burden. DELIQRATEFCS = delinquency rate, Farms Security Administration as a percent of loans. EQUITY = assets – liabilities. Source: USDA, Economic Research Service, Agriculture Income and Finance. Economic Research Service, USDA, Farm Income and Balance Sheet Indicators.

The sub-prime mortgage crisis of 2006-2007 is similar. Demyanyk and Van Hemert utilized a data-base containing information about one half of all subprime mortgages originated between 2001 and 2006. They explored to what extent the probability of delinquency/default can be attributed to different loan and borrower characteristics and housing price appreciation. I use data from the FRED and OFHEO, cited under Figure 2. In part 6 below, I relate the statistical results to the Stochastic Optimal Control/Dynamic Programming (SOC/P) analysis.

From 1998-2005 rising home prices produced above average capital gains (CAPGAIN), which increased owner equity. This induced a supply of mortgages, and the totality of household financial obligations as a percent of disposable personal income (DEBTRATIO) rose (Figure 2). The rises in housing prices and owner equity induced a demand for mortgages by banks and funds. In about 45-55% of the cases, the purpose of the sub-prime mortgage taken out in 2006 was to extract cash by refinancing an existing mortgage loan into a larger mortgage loan. The quality of loans declined. The share of loans with full documentation substantially decreased from 69% in 2001 to 45% in 2006 (Demyanyk and Van Hemert). Funds held packages of mortgage-backed securities either directly as asset-backed securities or indirectly through investment in central funds. The purchases were financed by short-term bank loans. Neither the funds nor the banks worried about the rising debt, because equity was rising due to the rise in home prices.

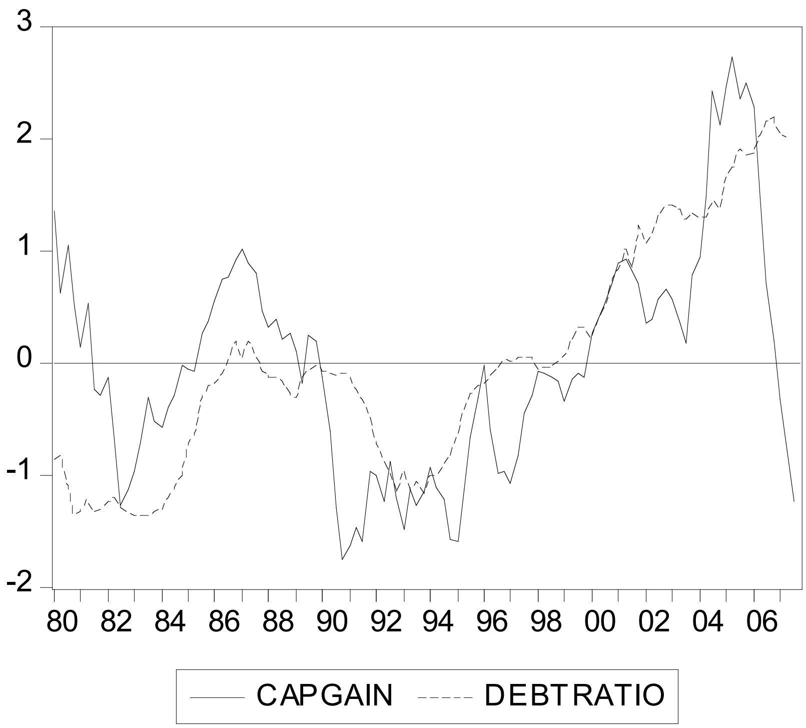

Figure 2. Mortgage Market Bubble. Normalized variables. Appreciation of single-family housing prices, CAPGAIN, 4q appreciation of US Housing prices HPI, Office Federal Housing Enterprise Oversight (OHEO); Household debt ratio DEBTRATIO = household financial obligations as a percent of disposable income. Federal Reserve Bank of St. Louis, FRED, Series FODSP.

The large capital gains from 2003 – 2005 fell drastically from 12.2% pa in 2006q1 to 1.79% pa in 2007q3. The delinquency rates in 2006, for each age of mortgage, were the highest in the previous five years. Figure 2 shows that the level and change in capital gain was the lowest over the period.

Many borrowers had little equity in their homes and found it difficult to sell or to refinance, *because the debt exceeded the market value of the home.* It was cheaper to default and avoid debt service than to rent new housing. Large banks and investors who made sub-prime loans or bought securities backed by them reported billions of dollars of losses. The massive unwinding of positions by highly leveraged investors such as hedge funds pushed the prices of both low and high quality sub-prime securities lower. Equity was further reduced, and the debt/equity ratio of borrowers and financial intermediaries rose. Banks reacted by reducing the supply of credit to the economy, and induced the Federal Reserve to change its monetary policy. *One can just copy/paste the agriculture story in understanding the sub-prime mortgage crisis.*

Banks should evaluate whether the borrower is likely to default. I apply several techniques in the extensive mathematical literature of stochastic optimal control (SOC) to derive an optimal debt in an environment where there are risks on both the asset and liabilities sides. The ratio debt/net worth *per se* is not a significant explanation of defaults. The vulnerability of the firm to shocks, from either the return to capital, the interest rate or capital gain, increases in proportion to the *excess* debt, which is defined as the difference between the Actual and Optimal debt ratio. As the debt ratio exceeds the optimum, risk rises relative to expected return and default *becomes ever more likely.*

There are several parts to the analysis: A criterion function, A structural model, Specification of the stochastic processes, and the solution using the Ito equation and (DP) Dynamic Programming. The basic references for the mathematical techniques used in this paper[2] are Fleming & Soner (2006), Fleming (1999), Fleming & Stein (2004), and Stein (2004, 2005 and 2006 ch. 3).

2. The Criterion Function

[2] These techniques were used in Stein (2007) to analyze the whether the current US external debt should be a cause for serious concern.

The lender evaluates what debt would maximize the expected (E) growth rate of the borrower's net worth over the period of the loan, an horizon of length T from the present t=0. This would be the optimal debt that a prudent lender would want to offer. The bank/lender wants to avoid borrower's bankruptcy (X = 0) by placing an infinite penalty on a debt that would lead to a zero net worth, bankruptcy. The borrower has a net worth X(t) equal to the value of capital K(t) less debt L(t). Initially net worth X = X(0) > 0. Eq. (1) is the criterion function. Equation (1a) is an alternative form of Eq. (1). The lender is very risk averse, since X(T) = 0 implies that W is minus infinity..

(1) $W(X,T) = \max E \ln [X(T)/X(0)], \qquad X = K - L > 0.$

(1a) $E [X(T)] = X(0)\, e^{W(X,T)}$

The next steps are to: explain the stochastic differential equation for net worth, relate it to the debt ratio, and specify what are the sources and characteristics of the risk and uncertainty.

3. Dynamics of Net Worth

In view of equation (1), the bank/lender should focus upon the change in net worth dX(t) of the borrower. It is the equal to the change in capital dK(t) less the change in debt dL(t). Capital K = PQ, the product a physical quantity Q times the relative price P of the capital asset to the price of output, such as the GDP deflator. The change in capital has two components. The first is the change due to the change in relative price of capital, which is the capital gain or loss, K(dP/P) term. The second is investment, which is I = P dQ, the change in the quantity times the relative price. The change in debt dL is the sum of expenditures less income. Expenditures are the debt service r(t)L(t) at real interest rate r(t), plus investment I = P dQ plus either consumption, dividends or distributed profits C(t). Income Y(t) = β(t)K(t) is the product of capital times β(t) its productivity. Combining these effects, the change in net worth is equation (2) and (2a).

(2) $dX(t) = K(t)\mathbf{[(dP/P) + \boldsymbol{\beta}(t)\, dt]} - \mathbf{r(t)}L(t)\, dt - C(t)\, dt$

$\qquad = K(t)\, \mathbf{b(t)\, dt} - \mathbf{r(t)}L(t)\, dt - C(t)\, dt.$

(2a) $\mathbf{b(t)} = \mathbf{(dP/P) + \boldsymbol{\beta}(t)\, dt}$

Stochastic variables in bold are the real capital gain or loss (dP/P), the productivity of capital β(t) and r(t) the real interest rate. Term b(t) in (2a) subsumes the two sources of risk on capital: the capital gain or loss and the productivity of capital. The agricultural debt crisis and the sub-prime mortgage crisis can be understood in terms of equations (1) - (2). In one case, capital is land and equipment, and in the other it is residential housing.

4. The Stochastic Processes

Figure 3 graphs the time series of two stochastic variables in the agricultural sector: the productivity of capital $\beta(t) = Y(t)/K(t)$ and the interest rate $r(t)$. The productivity of capital is measured as GVACAP = $\beta(t)$ = gross value added/value of farm assets. The second is INTDEBT = r = total interest payments/debt. The capital gain term dP/P (not graphed here) is not significantly different from zero, but has a very high variance. It is stationary, so that it is mean reverting to zero. For the housing market, the productivity of capital $\beta(t)$ is the imputed rental value of the housing and dP/P is the capital gain CAPGAIN in figure 2.

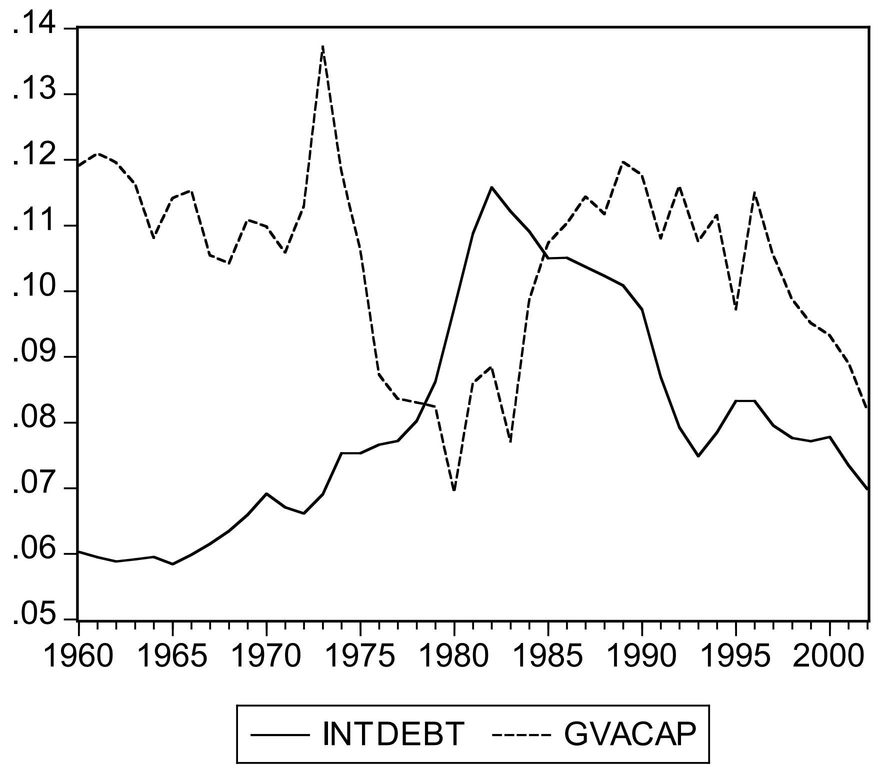

Figure 3. Agriculture. GVACAP = gross value added/capital = productivity of capital = $\beta(t)$, INTDEBT = total interest payments/debt = $r(t)$.

A crucial assumption motivating the use of SOC/DP is that *the future is unpredictable*[3]. The uncertainty may have different forms. Since there is some ambiguity about describing the specific form of the stochastic processes in Figure 3, I consider several cases. *Case A* assumes that the return on capital $b(t) = (dP/P) + \beta(t)$ follows an Ornstein-Uhlenbeck process, Erogodic Mean Reversion (EMR), equation (3a). The solution of this equation implies that the return b(t) converges to a distribution with a mean of b, and a variance is $\sigma^2_b /2\alpha$, where α is the speed of response. The interest rate r(t) is equation (4a), a Brownian Motion with Drift (BMD) process. The mean is r and the variance is $\sigma^2_r dt$. *Case B* assumes that both the return to capital (Eq. 3b), and the interest rate (Eq. 4a) are described by Brownian Motion with Drift[4]. *Case C*, equations (3b) and (4b), is the reverse of case A. The return is BMD but the interest rate is EMR.

BOX 1. STOCHASTIC PROCESSES, UNCERTAINTY

(3a) $db(t) = \alpha_1(b - b(t))\, dt + \sigma_b\, dw_b$	Ornstein-Uhlenbeck (EMR)
(3b) $b(t) = b\, dt + \sigma_b\, dw_b$	Brownian Motion-Drift (BMD)
(4a) $r(t) = r\, dt + \sigma_r\, dw_r$	Brownian Motion-Drift (BMD)
(4b) $dr(t) = \alpha_2\, (r - r(t))dt + \sigma_r\, dw_r$	Ornstein-Uhlenbeck (EMR)

The mean (b, r) is written without any time index. Case A is eq. (3a), (4a); Case B is eq. (3b), (4a), Case C is eq. (3b), (4b).

5. Solution and interpretation of the optimal debt/net worth

The solution for the optimal debt ratio concerns the maximization of the expected growth rate of net worth, eq. (1) subject to eq. (2) and the appropriate stochastic process in BOX 1. There are other reasonable criteria functions and stochastic processes. The mathematical techniques for their solution are discussed in the Fleming and Stein references above. Here, I simply state the results and provide a graphic interpretation that relates to the economics literature. Then I use the results in discussing the two crises.

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[3] The popular concept of "the inter-temporal budget constraint" is meaningless in such a context. See Stein (2006, pp. 7, 32-33, 63 and 228) for a detailed explanation.

[4] Brownian Motion results from continuous independent increments with a zero expectation. See Øksendal.

The optimal debt/net worth ratio $f^*(t) = L(t)/X(t)$ in BOX 2 varies according to the three cases (A), (B) and (C) respectively. The asterisk denotes the optimal value. In each case, the numerator is a return less an interest rate, and the denominator is a variance.

BOX 2. OPTIMAL DEBT/NET WORTH RATIO

	Return/interest rate	
(5A) $f^*(t) = [b(t) - r]/\sigma^2_r$	EMR/BMD,	σ^2_r = variance r(t)
(5B) $f^* = (b - r)/\sigma^2 + f(0)$	BMD/BMD,	σ^2 = variance (b-r)
(5C) $f^*(t) = [b - r(t)]/\sigma^2_b$.	BMD/EMR	σ^2_b = variance b(t)

In *case (A*), equation (5A), the optimal ratio of debt/net worth f*(t) varies with time. It is equal to the *current* value of the return to capital b(t) less the *mean* rate of interest r, divided by the variance of the interest rate var (r(t)). In *case (B),* equation (5B), the optimal debt/net worth f* is constant. It is equal to the *mean* return on capital less the *mean* interest rate, (b-r), divided by the variance var [b(t) – r(t)], which contains the covariances. The constant term f(0) is the debt ratio where the total risk is minimal. Case (C), equation (5C) is the reverse of case (A). The debt/net worth ratio f*(t) varies with time. It is equal to the *mean* return less the *current* interest rate divided by the variance of the return var b(t).

Figure 4 is an interpretation of the expected growth rate in equation (6) in terms of a Mean-Variance diagram. There are two terms in the expected growth rate. The first term Mean is the straight line in figure 4/Eq. (6a). It is the expected growth rate for any debt ratio f(t), *if there were no risk.* The intercept is the appropriate return less the ratio C/X of (consumption-dividends-distributed profits)/net worth. The slope is the appropriate return less the interest rate. The appropriate measure depends upon the relevant case, (A), (B) or (C). The second term in equation (6) is graphed as the parabola Risk in figure 4/Eq. (6b). It is the variance of the change in net worth.

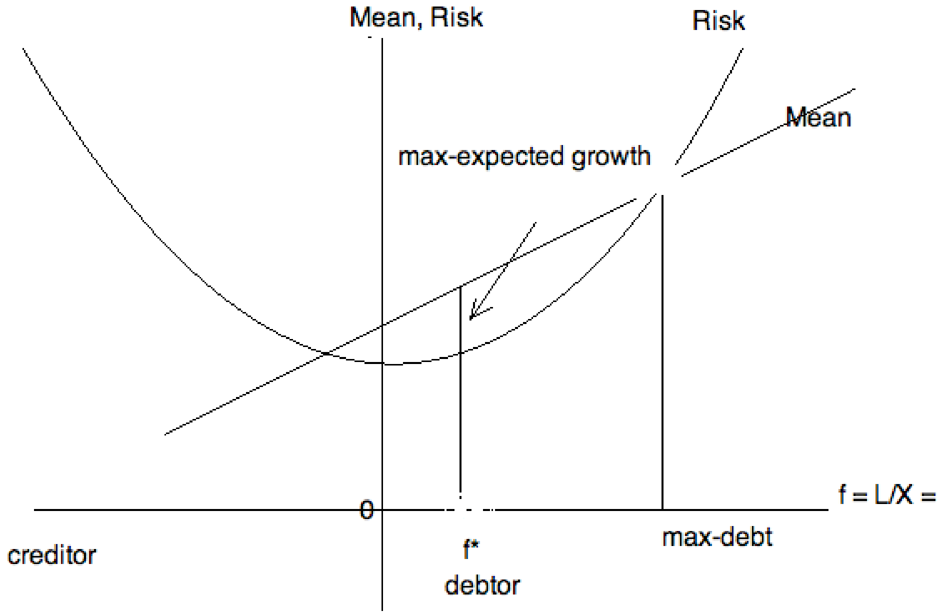

Figure 4 describes the solution of the H-J-B differential equation or Ito equation. The expected growth rate of net worth is the difference between the Mean line and the Risk parabola. Optimal debt ratio f* is found where the slope of the Mean Line equals the slope of the Risk parabola. When the debt ratio exceeds max-debt, the expected growth rate of net worth is negative.

The expected growth rate of net worth in equation (6) corresponds to the difference between the straight line *Mean* and the parabola *Risk* in figure 4. In case (B), Mean and Risk are described by equations (6a) and (6b) respectively. The derivations are in Fleming & Stein (2004). In the other cases, see Stein (2005) for technical details.

(6) $(1/T)\ E\ [\ln X(T)/X(0)] = \text{Mean} - \text{Risk} = \text{expected growth rate.}$

(6a) $\text{Mean} = (b - c) + (b - r)f$

(6b) $\text{Risk} = (1/2)\ \sigma_b^2[f^2\theta^2 + (1+f)^2 - 2f(1+f)\ \theta\rho]$

$\theta = \sigma_r/\sigma_b$ $\quad\quad \rho$ = correlation between (b,r)

The optimal debt ratio f* maximizes the distance between Mean and Risk. It is the value in equations (5A), (5B) or (5C), depending upon the stochastic process. At this ratio, the expected growth rate is maximized for any given ratio C/X of consumption-dividends/net worth.

As the debt ratio f = L/X rises above the optimum f*(t), the Risk rises relative to the Mean, and the expected growth rate of net worth declines. At a debt ratio equal to max-debt, the expected growth rate is zero. When the debt ratio rises above max-debt, the expected growth rate is negative and risk is very high. The liklihood of bankruptcy increases continuously as the debt ratio rises above the optimum. The difference $\Psi(t) = [f(t) - f^*(t)]$ is the "excess debt" ratio. From figure 4, one sees that the likelihood of a serious decline in net worth that threatens bankruptcy is a continuous function of $\Psi(t)$ the excess debt ratio.

Figure 4 can be viewed in terms of the Value at Risk VaR concept[5]. The latter is based upon a probability distribution of the profits or loss. The Value at Risk at the 99% level is VaR(99%) = 2.33 σ, where σ is the standard deviation of the distribution of the net income. This means that the probability of a greater loss is 1%. In figure 4, the Risk rises, and expected return declines, with the excess debt. Hence σ rises with the excess debt, $\Psi(t)$. This means that *the VaR rises with the excess debt,* $\Psi(t) = f(t) - f^*(t)$. An Early Warning Signal EWS of a crisis should be the excess debt, appropriate to the stochastic process[6].

6. A Tale of Two Debt Crises: Agriculture and Subprime Mortgage

Both the agriculture and subprime mortgage crisis can be understood in terms of the SOC/DP analysis and Early Warning Signals are thereby derived. A bubble is a situation described by equation (7). The capital gain exceeds the interest on the debt, which in turn exceeds the productivity of capital. The only way that the borrower can pay the interest is by cashing in on the capital-gain.

(7) $dP/P > r > \beta$. BUBBLE P = price asset/GDP deflator

(7a) $dP/P < r$. BURST

The basic proposition is that the *relative* price of an asset to the GDP deflator cannot continue to rise. When P stabilizes at a value related to the productivity of capital the capital gain dP/P disappears. A *sufficient* condition for the bubble to burst is that the *relative* price P stabilizes, $dP/P = 0$. The capital gain dP/P is less than r(t) the interest on he debt. When the bubble bursts, Eq. (7a), the borrowers are not able to refinance, cash in the capital gain, at the low old interest

[5] See Crouhy et al Chapter 7 for a discussion of the usefulness of the VaR for risk management.
[6] As the debt ratio exceeds max-debt, the expected growth rate becomes more negative and the variance rises. That is, the probability distribution of the growth rate shifts to the left.

rate. Then $b(t) – r(t)$ becomes negative. The cash flow plus the (zero or negative) capital gain is insufficient to service the debt.

The "market" used improper estimates of the variables that determine the optimal ratio in BOX 2. For the net return $b(t) – r(t)$, the borrowers/lenders used $[\beta(t) + dP(t)/P(t)] – r(t)]$, where dP/P is the capital gain based upon the recent past values, which were not linked to the productivity of capital. The error was to assume that the mean capital gain dP/P in Eq. (5B) could be based upon the values in the recent past[7]. Since their estimate of $b(t)$ was high, say b_1, and they assumed that the risk $\sigma^2 = \sigma_1^2$ was low they incurred a high debt ratio $f_1 = (b_1 – r)/\sigma_1^2$. When the capital gain disappeared – the bubble burst - the optimal debt ratio was considerably below the ratio $f_1 = (b_1 – r)/\sigma_1^2$ they incurred. For example, the new optimal ratio was f* in figure 4, whereas they were holding what is now a value f_1 above max-debt. In figure 1, the fall in agricultural equity – the collapse of the bubble - is seen in EQUITY. In figure 2, the bursting of the mortgage market bubble is seen in the decline in the capital gain CAPGAIN, which became capital losses.

If they optimized on the basis of Eq. (5A), the assumption was that they could quickly and at negligible cost reduce their debt to a new level, based upon a lower $b(t) – r(t)$. The error was to ignore the fact that when the equity bubble burst, they would have great difficulty in selling their assets at the pre-existing prices, because many other borrowers are also trying to sell the asset to pay off the debt. This liquidation generates bankruptcies and defaults.

What are Early Warning Signals of a debt crisis? How should the borrowers and lenders have optimized? Consider each case in turn: agriculture, subprime mortgage market.

6.1. Agriculture

The appropriate measure of the return in Equations (5A) – (5C) depends upon the stochastic process. The capital gain term dP/P has a mean that is not significantly different from zero. Therefore in Eq. (5B), the mean return $b = \beta$, the productivity of capital, is the mean ratio of value added/capital (GVACAP). The numerator of f* should be $(\beta – r)$ = the productivity of

[7] There are many articles in the mathematical finance literature concerning the best way to estimate if the drift term has changed. Some use the Kalman filter. Others use estimates of conditional probability. See for example Blanchet-Scalliet et al. (2007). In the context of the Two Debt Crises, I use the basic proposition that relative price P cannot grow steadily. Optimization should not be based upon capital gains that are unrelated to the productivity of capital.

capital (GVACAP) less interest rate (INTDEBT), which are graphed in figure 3 above. Their difference GVACAP – INTDEBT is the numerator of the optimal debt ratio.

If the stochastic process implies Eq. (5A) or (5C), then the return should be based upon the current value of one variable and the mean value of the other variable. However, in each case, the expected capital gain terms should be set at zero. A *relative price cannot have a long term mean positive growth rate.*

One cannot be sure what is the appropriate stochastic process and hence optimal debt ratio. Therefore I take several approaches. In Case B, the optimal debt ratio is Eq. (5b). Using the mean values for $(\beta - r)$ and its variance σ^2, the optimal debt ratio is

$(L^*/X) = f^* = 23.5$

The actual debt/net worth ratio rose from 21.2 in 1970 to 29.8 in 1985 – a 41% rise. The delinquency rate on the debt (DELIQRATE in fig. 1) rose by more than two standard deviations in the mid 1980s.

A general approach in evaluating debt and obtaining Early Warning Signal is that the optimal debt ratio should follow the net return $(b(t) - r(t))$. In Eq. (5A) the appropriate net return is $[\beta(t) - r]$, in Eq. (5b) it is $(\beta\text{-}r)$ and in Eq. (5c) it is $[\beta - r(t)]$. In figure 5, the curve labeled RETVAINTD is the normalized value of $[\beta(t) - r(t)]$. It is:

(8) $\text{RETVAINTD} = [(\beta(t) - r(t)) - (\beta - r)]/ \sigma,$

σ = standard deviation of $(b(t)\text{-}r(t))$, $(\beta\text{-}r)$ = mean net return

The debt ratio in the optimization is $f = L/X$ = debt/net worth. However, there is a bias in using this as an empirical measure of an Early Warning Signal (EWS). The reason is that as net worth EQUITY collapses, this ratio jumps up violently. For this reason, in empirical work I prefer to use the ratio $h = L/Y$ of debt (L) to (Y) to net income. Call h the debt ratio. In figure 5, the normalized value of the debt ratio is:

(9) $\text{DEBTNINC} = [L(t)/Y(t) - (L/Y)]/\sigma$

σ = standard deviation of $[L(t)/Y(t)]$, L/Y = mean $(L(t)/Y(t))$

The optimal debt ratio should either follow RETVAINTD, (Eq. (5A), (5C)) or be constant (Eq. (5B). My measure of an excess debt $\Psi(t)$ is the difference between the normalized curves in figure 5. Non-optimal debt would occur if the debt ratio were rising relative to its long term mean when the net return was declining relative to its mean.

(10) $\Psi(t) = \text{DEBTNINC} - \text{RETVAINTD} > 0$ Excess debt

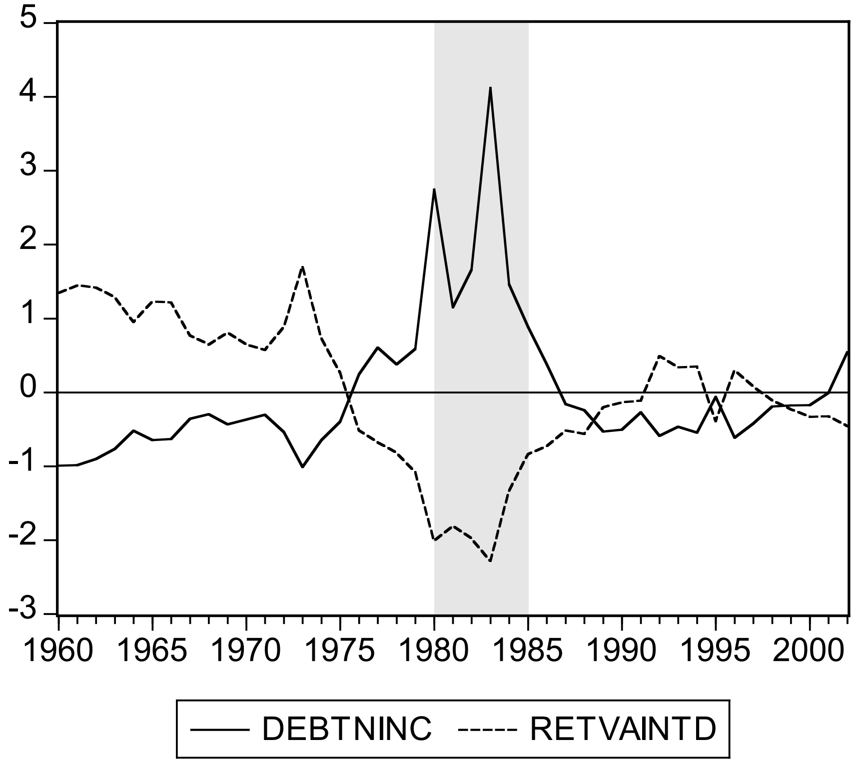

Figure 5. Agriculture. DEBTNINC = L/Y_n = Debt/net income; RETVAINTD = GVACAP - INTDEBT = (gross value added/assets - interest rate). Normalized variable = (variable - mean)/standard deviation.

In figure 5, the normalized net return fell by about 3 standard deviations from 1975 – 1980, but the debt ratio rose by about 3 standard deviations during that period. The excess debt $\Psi(1980)$ was about 4 standard deviations. This corresponds to a large deviation between the actual debt ratio and max-debt in figure 4. A large value of deviation $\Psi(t)$ is an EWS of an impending crisis. This crisis did indeed occur, seen in figure 1, with the bankruptcies and defaults. During the periods when $\Psi(t)$ was small, there were no crises.

6.2.Subprime Mortgage Market

A similar method of analysis can be applied to the subprime mortage market. I interpret the study by Demyanyk and Van Hemert (D-VH) on the basis of the SOC/DP analysis. They had a data base consisting of one half of the US subprime mortgages originated during the period 2001-2006. At every mortgage age, loans originating in 2006 had a higher delinquency rate than in all the other years since 2001. They examined the relation between the probability Π of delinquency/foreclosure/binary variable *z,* denoted as $\Pi = Pr(z)$ and sensible economic variables, vector X. They investigated to what extent a logit regression $\Pi = Pr(z) = \Phi(\beta X)$ can explain the high level of delinquencies of vintage 2006 mortgage loans. Vector β is the estimated regression coefficients.

They estimated vector β based upon a random sample of one million first-lien subprime mortgage loans originated between 2001 and 2006. The first part to their study provides estimates of β, the vector of regression coefficients telling us the importance of the variables in vector X

The second part inquires why the year 2006 was so bad. The approach is based upon the equation (11). The contribution C(i) of component X_i in vector X to why the probability of default in year 2006 was worse than the mean is:

(11) $C(i) = (\delta\Pi/\delta X_i)\, dX_i = \Phi(\beta X_m + \beta_i\, dX_i) - \Phi(\beta X_m)$, m = mean value

The probability of delinquency when the vector X is at its mean value is $\Phi(\beta X_m)$. The added probability resulting from the change in component X_i in 2006 comes from $\beta_i dX_i$ where β_i is the regression coefficient of element X_i whose change was dX_i.

Table 1 below (based upon D-VH, table 3) displays the largest factors that made the delinquencies and foreclosures in year 2006 worse than the mean over the entire period. For year 2006, the largest contribution to delinquency and to foreclosure was the low house price appreciation. It accounted for 1.08% of the greater delinquencies and 0.61% for the greater foreclosures. The debt/income, the balloon dummy and the documentation variables[8] are significantly smaller.

[8] See D-VH table 2 for definitions of variables.

Table 1. Contribution C(i) of factors to probability of delinquency and defaults 2006, relative to mean for the period 2001-2006 (D-VH, table 3)

Variable X(i)	Contribution C(i) to delinquency rate	Contribution C(i) to foreclosure rate
House price appreciation	1.08 %	0.61 %
Balloon	0.18	0.09
Documentation	0.16	0.07
Debt/income	0.15	0.04

Their results can be related to the mathematical analysis above and to the results for agriculture in figure 5. In agriculture or in any other commercial enterprise, the concept of the productivity of capital is explicit. In the home mortgage market, this concept is implicit. One could argue that by owning a home one saves rental payments. Then the productivity of housing capital to households is the implicit net rental income/value of the home plus a convenience yield in owning one's home. This concept would correspond to $\beta = Y/K$ in eq. (2) above. I also assume that the convenience yield in owning a home has been relatively constant. I try to approximate β by using the normalized ratio of rental income/disposable personal income. In figure 7 variable RENTRATIO = [(rental income/disposable personal income) – mean]/standard deviation.

The sub-prime mortgage story is the following. The capital gains in housing CAPGAIN (normalized in figure 2) induced households to take out mortgages in order to extract cash to finance expenditures. Moreover, the rising value of equity served as collateral for home equity loans to finance all sorts of household expenditures.

Figure 6 describes the statistics underlying the capital gains variable, the four- quarter appreciation of US housing prices dP/P. The distribution is highly skewed to the right. These extreme observations are the bubble years. The median appreciation over the entire sample period is 5.2% p.a. During the bubble period 2004-2007, the 30- year mortgage rate fluctuated between 6 and 6.5% pa. The GDP deflator varied between 2 and 4% pa. It is reasonable to argue that the longer run real appreciation of housing prices was not significantly greater than "the mortgage rate of interest", which is an ambiguous term[9].

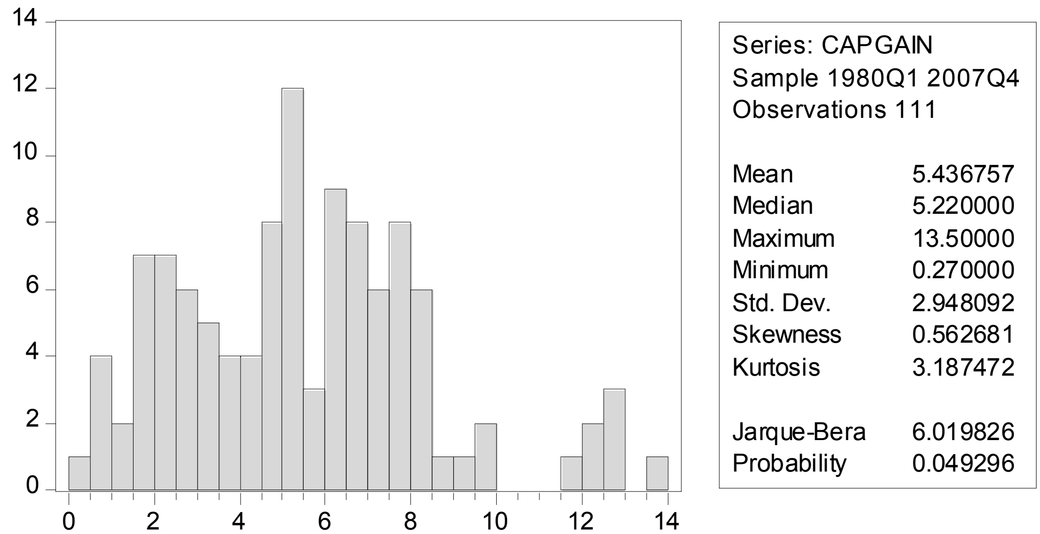

Figure 6. Histogram and statistics of CAPGAIN, the four- quarter appreciation of US housing prices. This is the same variable normalized in figure 2.

The actual debt ratio f(t) was induced by $[\beta(t) + dP/P - r]/\sigma^2$, where dP/P represents the capital gains. The dramatic rise in housing equity induced a drastic rise in *total* household debt (DEBTRATIO, Figure 2). From 1990 the capital gains in housing dP/P rose and the personal saving ratio/disposable income fell. The decline in the household saving ratio is linked to the rise in f(t) the debt ratio, though as Guidolin and La Jeunesse point out there is no simple explanation for the trend decline in the personal saving ratio.

Cash flow is K(t) dP(t)/P(t) and interest payments are r(t)L(t). As long as the quantity net cash flow $Y(t) = [(dP/P(t) - r(t)L(t)/K(t)]$ is positive, more debt is induced to either spend or

[9] Table 1 in (D-VH) contains descriptive statistics for the first lien subprime loans. There are four main mortgage types, each one bearing different "interest rates". They are: Fixed rate mortgages (FRM), Adjustable rate mortgages (ARM), Hybrid and Balloon. The percentage of all the loans in these types varied significantly by period. For example:

	2001	2006
FRM	41.4%	26.1%
ARM	0.9	12.8
Hybrid	52.2	46.2
Balloon	5.5	14.9.

purchase a home. Suppose that it is the latter, $L(t)/K(t) = 1$. The bubble is described by eq. (7) where $dP/P > r > \beta$. The crisis will occur when (7a) $dP/P < r$, the appreciation of housing prices is less than r, the rate of interest. Then net cash flow Y(t) is negative.

Falling growth in housing prices was the most significant variable accounting for the rise in the delinquency and default rates in table 1. This is consistent with the observation (Federal Reserve San Francisco) that there was a negative correlation between the rate of house-price appreciation and level of sub-prime delinquencies among metropolitan statistical areas.

There is a great heterogeneity in interest rates charged to the subprime borrowers, so it is difficult to state exactly what corresponds to r(t) in the analysis above. I therefore use "Household Debt Service Payments as a Percent of Disposable Personal Income" (TDSP in FRED) as a measure of rL/Y the debt burden. This includes all household debt, not just the mortgage debt, because the capital gains led to a general rise in consumption and debt. The normalized value is labeled DEBTSERVICE in figure 7.

DEBTSERVICE = [(Household debt service/disposable personal income) – mean]/st. dev.

Figure 7 plots the values of the two *normalized* variables: DEBTSERVICE and RENTRATIO. The difference between the two *normalized* curves in figure 7 is a measure of excess debt. Variables in figure 7 are measured as standard deviations from their means. Equation (12) for the mortgage market corresponds to eq. (10) in agriculture.

(12) $\Psi(t)$ = DEBTSERVICE - RENTRATIO.

The productivity of capital RENTRATIO was not rising, but L/Y the debt ratio (figure 2) was rising rapidly. The rising debt could only be serviced from capital gains. Assume that over the earlier period 1980 – 1998 the debt ratio was not excessive. From year 2000, the debt service deviated significantly from the rent ratio, because the actual debt ratio f(t) was stimulated by $(dP/P – r)$, the appreciation of housing prices relative to the interest rate.

The excess debt $\Psi(t) = f(t) – f^*(t)$ in 2004 was two standard deviations, which is an EWS of a crisis. The only thing that held off the crisis was the capital gain in excess of the interest rate. Net cash flow Y(t) was positive. But housing prices P cannot continue to grow at a rate above the interest rate[10]. We can be sure that, sooner or later, (7a) will occur. As soon as the

[10] Let the housing price be P(t), the mortgage rate of interest r(t), the rate of inflation of the GDP or CPI is $\pi(t)$ and the real rate of interest is i(t). Thus the mortgage rate of interest is $r(t) = i(t) + \pi(t)$. The difference between the capital gain and the mortgage rate of interest is;

appreciation stopped, dP/P became less than interest rate r. There would be no capital gains that could be converted into cash to pay the interest. When the households lost equity, the choice was between servicing the debt r(t)L(t) or abandoning the property and renting rather than owning housing. When eq. (7) becomes (7a), a crisis occurs with the consequent delinquencies, bankruptcies and defaults. As D-VH found, the most significant variable in explaining why year 2006 was so bad was that housing price appreciation disappeared. In terms of our analysis, debt became excessive, in terms of figure 4.

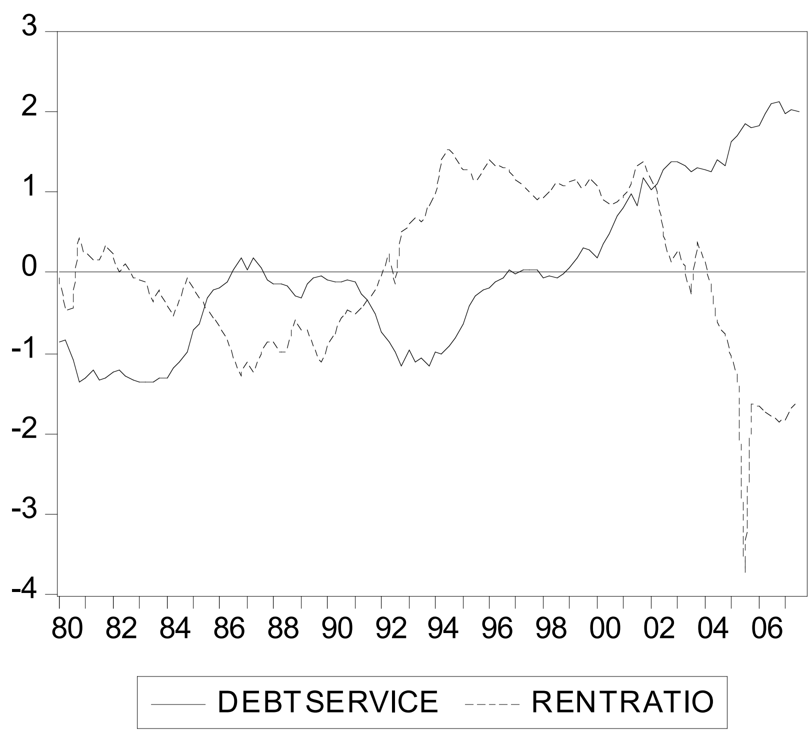

Figure 7. RENTRATIO = normalized rental income/disposable personal income, DEBTSERVICE = normalized household debt service as percent of disposable income. Sources FRED.

Z (t) = [(dP(t)/P(t) – i(t) - π(t)]. This can only be positive if the real appreciation of the housing [dP(t)/P(t) – π(t)] exceeds the real rate of interest i(t). This is not a sustainable situation where the relative price of housing to the general price level is steadily rising.

7. Summary and Conclusions

How should lenders and investors optimally manage risk to avoid losses from the defaults and bankruptcies of the borrowers? The Agricultural debt crisis of the 1980s and the subprime mortgage crisis of 2007 followed similar scenarios. In each case, the growth of the debt was stimulated by capital gains on assets. Capital gains are not sustainable unless they reflect the growth of the productivity of capital. When the capital gains fall below the interest owed, a crisis will occur.

The object of this study is to evaluate if the debt is likely to lead to default and thereby derive *theoretically based* Early Warning Signals EWS of the vulnerability of the debtor to shocks. Given that the future is unpredictable, the optimal debt ratio is derived using the mathematical techniques of stochastic optimal control/dynamic programming (SOC/DP).

There are many sensible criteria of optimization. Since we are looking at the problem from the point of view of the lender/bank, we focus upon the debt/net worth ratio that would maximize the expected growth of the borrower's net worth over a given horizon. This is a risk averse strategy because it corresponds to maximizing the expected logarithm of net worth over a fixed horizon.

The evolution of net worth depends upon three stochastic variables and the selected debt ratio. The stochastic variables are: the productivity of capital, the interest rate and the relative price of assets/price of output. The optimum debt ratio depends upon the stochastic processes. In one case, the sum of the productivity of capital and the capital gain is assumed to be ergodic mean reverting (EMR), and the interest rate is Brownian Motion with drift (BMD). In the second case, the capital gain is assumed to have a zero mean, and both the productivity of capital and the interest rate are BMD. In each case, the optimal ratio debt/net worth is positively related to a measure of the productivity of capital less an interest rate and negatively related to a measure of variance, appropriate to the specific stochastic processes. In neither case should one assume that the capital gain, the growth of a relative price, will continue to exceed the interest rate.

The optimal debt/net worth ratio is derived in these two cases. The vulnerability to shocks from the stochastic variables is not directly related to the actual debt ratio. It is, however, directly related to the *excess* debt, equal to the actual less the optimal debt ratio. As the excess debt rises, the probability of a decline of net worth and the expected loss increase. Thereby our EWS is the magnitude of the excess debt.

The SOC/DP analysis is applied to the two crises. In agriculture, the unit is a commercial firm concerned with profits and there is a clear concept of the productivity of capital. In the home mortgage market, the unit is a household where the implicit rental income/net worth is the closest approximation to the productivity of capital. The story of the subprime mortgage crisis of 2006-2007 is a copy/paste of the agricultural debt crisis of the 1980s. In each case I derives EWS based upon measurable variables of an impending crisis.

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	CAPGAIN %	DEBTRATIO %	DEBT SERVICE %	RENTRATIO
Mean	5.436757	12.06936	12.09000	0.015633
Median	5.220000	11.97000	11.97000	0.015331
Maximum	13.50000	14.48000	14.48000	0.024281
Minimum	0.270000	10.57000	10.57000	-0.005469
Std. Dev.	2.948092	1.099061	1.115338	0.005640
Skewness	0.562681	0.534111	0.528857	-0.530225
Kurtosis	3.187472	2.353906	2.318286	3.294987
Jarque-Bera	6.019826	7.143281	7.323648	5.603520
Probability	0.049296	0.028110	0.025686	0.060703
Sum	603.4800	1327.630	1341.990	1.735219
Sum Sq. Dev.	956.0370	131.6649	136.8376	0.003499
Observations	111	110	111	111