An Idealized View of Financial Intermediation*

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First Draft: March 22, 2006

This Draft: May 30, 2007

Abstract

We consider an environment where the general equilibrium assumption that every agent buys and sells simultaneously is relaxed. We show that fiat money can implement a Pareto optimal allocation only if taxes are type-specific. We then consider intermediated money by assuming that financial intermediaries whose liabilities circulate as money have an important identifying characteristic: they are widely viewed as default-free. The paper demonstrates that default-free intermediaries who issue deposit accounts with credit lines to consumers can resolve the monetary problem and make it possible for the economy to reach a Pareto optimum.

JEL Codes: E5, G2

Keywords: Fiat Money, Cash-in-advance, Financial Intermediation

*Neil Wallace and Chris Waller gave me invaluable comments while developing this paper. I also thank Jim Bullard, Gabriele Camera, Ricardo Cavalcanti, Lee Ohanian, Joe Ostroy, Shouyong Shi, Ross Starr, Ted Temzelides, Aaron Tornell, Randy Wright and seminar participants at the Office of the Comptroller of the Currency for helpful comments. All errors are, of course, my own.

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When a private promise to pay in the future is generally accepted as a means of payment within an economy, we have a single financial asset that fits the definitions of both credit and money. The most common example of such an asset in the modern US economy is a merchant's credit account with a credit card company. The asset is both a privately issued liability and a liability that is almost universally accepted in payment. How can an asset have both of these attributes simultaneously? The essential link between them is this: the private issuer is widely viewed as almost default free. Very little time is spent by merchants worrying about what to do in case Visa, MasterCard or American Express fails to meet its obligations.

This paper uses the assumption that financial intermediaries are default-free to set up a perfect world where intermediation can effortlessly overcome the monetary problem created by the friction in our model. In fact, our perfect world is in many ways a replica of the competitive model – with one important distinction: the role of financial intermediaries and their most important characteristic have been defined. Just as the competitive model posits the existence of an ideal real world in order to articulate the nature of economic relations between agents, we hope that by positing the existence of an ideal financial world we can articulate the role that financial institutions play in the real economy.

The assumption that financial intermediaries are default-free means that their liabilities are accepted as a means of payment, and this is essential to the economy's ability to reach the first-best. This assumption can be motivated by the work of Cavalcanti and Wallace (1999b), which demonstrates that bankers with public histories choose not to default in equilibrium. Clearly, we take this assumption as a starting point and recognize that a full understanding of the nature of financial intermediation will require a careful study of the effects of relaxing our assumptions.

Section 1 of the paper introduces the model. The model is based on a standard infinite horizon general equilibrium endowment economy with one change: the general equilibrium assumption that every agent can buy and sell goods simultaneously is relaxed. In every period of our model each agent is randomly required to either sell first and then make purchases or to buy first and then sell his product. In section 1 we solve for the set of Pareto optimal allocations and the stationary competitive equilibria of the model. In section 2 we assume that debt is not enforceable and introduce fiat money. Now the trading friction implies that each consumer will with probability one half face an endogenous cash-in-advance constraint. We find that implementation of an efficient allocation using fiat money is possible only if the government can collect type-specific

taxes. In section 3 of the paper we consider an alternative monetary regime based on default-free intermediaries and find that a first-best can be attained by a debt contract that is uniform across agents, if our consumers are sufficiently patient. Section 4 concludes.

1 The Model

The time horizon is infinite, and each period is divided into two sub-periods. There are n goods indexed by $j \in \{1, ..., n\}$, and these goods perish in each sub-period. The continuum of infinitely lived consumers has unit mass. In every period each consumer is endowed with a quantity, y, of one good where $y \in \{1, ..., k\}$, and the endowment may arrive in the first or in the second sub-period. Let $i \in I \equiv \{1, ..., n\} \times \{1, ..., k\}$ index the different types of agents. Assume that each type of consumer, i, has mass $\frac{1}{nk}$.

In each period consumers value consumption in either sub-period one or sub-period two, but never in both sub-periods. When consumption is valued, every consumer's preferences are given by the period utility function,

$$U(c) = \sum_{j=1}^{n} u(c_j)$$

where c_j is the agent's consumption of good j. The underlying utility function, $u(c)$ is continuously differentiable, increasing and strictly concave. We assume that $u(0)$ is a finite number. Every consumer chooses consumption to maximize the expected sum of his discounted utility. As there is no discounting from one sub-period to the next, we can represent each consumer's objective function as:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor and c_t is the agent's consumption vector at date t.

With probability one half each agent will receive a first sub-period endowment and a utility shock such that consumption is only valued in the second sub-period and with probability one-half the agent will receive a second sub-period endowment and a utility shock such that consumption is only valued in the first sub-period. The probability distribution driving this process is non-atomic and i.i.d. Clearly the first group of agents sells their endowments in the first sub-period and purchases their consumption set in the second, while the second group of agents purchases and consumes in the first sub-period and sells their endowments in the second. We will call the first

group first sub-period sellers and the second group first sub-period buyers. Let Y be the aggregate endowment of each good in each sub-period, or $Y = \frac{1}{2nk}\sum_{y=1}^{k} y$.

Once the agents have been divided into first sub-period buyers and sellers at date t, each agent has a role in the goods market: $\chi_t \in \{B, S\}$ where B represents a first sub-period buyer and S a first sub-period seller. This uncertainty is realized at the start of each period and generates a history for each agent, $\{\chi_0, \chi_1, ...\}$. Let H_t be the set of possible histories at t and let an individual agent's history be represented by $h_t \in H_t$. $f_t(i, h_t)$ is the mass at date t of agents of type i with history, h_t. Observe that $f_t(i, h_t) = \frac{1}{nk}\frac{1}{2}^{t+1}$ is independent of both type and history. In this environment all of a consumer's choice variables can depend both on the consumer's type and on the consumer's history of being a first sub-period buyer or seller.

The goods market will have to clear in each sub-period. Then the market clearing conditions in the goods market are:

$$Y = \sum_{i \in I} \sum_{h_s \in H_{t-1}} c^i_{j't}(h_s, B) f_t(i, h_s, B) \text{ for all } j', t \tag{2a}$$

$$Y = \sum_{i \in I} \sum_{h_s \in H_{t-1}} c^i_{j't}(h_s, S) f_t(i, h_s, S) \text{ for all } j', t \tag{2b}$$

The Pareto optimal allocations of this environment are given by the solution to the social planner's problem. We will focus on the *ex ante* problem or on the set of optimal allocations that is chosen before the realization of histories. Let θ^i be the weight placed by a planner on each type i. Then the planner's problem is:

$$\max_{c^i_{jt}} \sum_{i=1}^{n} \theta^i \sum_{t=0}^{\infty} \beta^t \sum_{j=1}^{n} u\left(c^i_{jt}(h_t)\right) \tag{3}$$

subject to the market clearing conditions, $2a$ and $2b$. We define the solution to the planner's problem in the following proposition:

Proposition 1 *The set of Pareto optimal allocations has the following properties:*

(i) $c^i_{jt}(h_t) = c^i$ *for all* j, t, h_t *and*

(ii) $\theta^i u'(c^i) = \theta^{i'} u'(c^{i'})$ *for all* $i, i' \in I$.

1.1 The Enforceable Debt Solution

First we will find the competitive solution to this model by introducing privately issued bonds into the environment. An implicit assumption underlying standard competitive models is that private

debt is perfectly enforceable, and this is the assumption we make in this section of the paper.

We will use $b^i_{st}(h_{t-1}, \chi_t)$ to represent the bond holdings at date t of an agent of type i, history h_{t-1} and market role χ_t at the start of sub-period $s \in \{1, 2\}$. The price of good j in sub-period one of date t is given by p_{jt}, and the price of a similar good in sub-period two by q_{jt}. Then the first and second sub-period budget constraints for an agent of type $i = \{j', y\}$ with market role B are:

$$b^i_{2t}(h_{t-1}, B) = (1 + i_t)b^i_{1t}(h_{t-1}) - \sum_{j=1}^{n} p_{jt}c^i_{jt}(h_{t-1}, B) \tag{4a}$$

$$b^i_{1t+1}(h_{t-1}, B) = (1 + r_t)b^i_{2t}(h_{t-1}, B) + q_{j't}y \tag{4b}$$

where i_t is the interest rate paid on a bond held from sub-period 2 of date $t-1$ to sub-period 1 of date t and r_t is the interest paid on a bond held from sub-period 1 to sub-period 2 of date t. For an agent of type $i = \{j', y\}$ with market role S the budget constraints are:

$$b^i_{2t}(h_{t-1}, S) = (1 + i_t)b^i_{1t}(h_{t-1}) + p_{j't}y \tag{4c}$$

$$b^i_{1t+1}(h_{t-1}, S) = (1 + r_t)b^i_{2t}(h_{t-1}, S) - \sum_{j=1}^{n} q_{jt}c^i_{jt}(h_{t-1}, S) \tag{4d}$$

Because each agent chooses consumption and bond holdings *after* learning whether his market type is B or S in the current period, our statement of the consumer's objective function must take this fact into account. As there is no uncertainty in the current period, at date t each consumer of type i chooses c^i_{jt} and b^i_{st} to maximize the following objective function:

$$\beta^t \sum_{j=1}^{n} u\left(c^i_{jt}(h_t)\right) + E_t \sum_{s=t+1}^{\infty} \beta^s \left(\frac{1}{2}\sum_{j=1}^{n} u\left(c^i_{js}(h_{s-1}, B)\right) + \frac{1}{2}\sum_{j=1}^{n} u\left(c^i_{js}(h_{s-1}, S)\right) \right) \tag{5}$$

subject to the budget constraints, equations 4a, 4b, 4c and 4d and taking the initial endowment vector of bonds, $(1 + i_0)b^i_{10}(h_{-1}) \equiv b^i_0$, as given. We will call this the enforceable debt problem.

Definition 1 *An* ***enforceable debt equilibrium*** *is an allocation for each type and each possible history of goods, $\{c^i_{jt}(h_t)\}$, and of bonds, $\{b^i_{st}(h_t)\}$, and sequences of prices, $\{p_{jt}\}$ and $\{q_{jt}\}$, and of interest rates, $\{r_t\}$ and $\{i_t\}$, such that*

(i) given prices and interest rates, the enforceable debt problem is solved for each type i and history, h_t, and

(ii) markets clear

(a) in each sub-period of the goods market, equations 2a and 2b, and

(b) in the bond market: $0 = \sum_{i=1}^{n} b_{st}^{i}(h_t) \forall s, t.$

Definition 2 *A* ***stationary enforceable debt equilibrium*** *is an enforceable debt equilibrium in which* $c_{jt}^{i}(h_t) = c_j^i$ *for all* i, j, t, h_t.

Using $\lambda_{st}^{i}(h_{t-1})$ as the Lagrangian multiplier on the budget constraint for an agent of type i, history h_{t-1}, and market role S in sub-period s at date t and $\gamma_{st}^{i}(h_{t-1})$ for a similar agent with market role B, we find the following first-order conditions for the enforceable debt problem:

$$\frac{\beta^t u'\left(c_{jt}^{i}(h_{t-1}, B)\right)}{p_{jt}} \leq \gamma_{1t}^{i}(h_{t-1}) \forall j, t \qquad \frac{\beta^t u'\left(c_{jt}^{i}(h_{t-1}, S)\right)}{q_{jt}} \leq \lambda_{2t}^{i}(h_{t-1}) \forall j, t \tag{6a,b}$$

$$(1 + r_t)\gamma_{2t}^{i}(h_{t-1}) = \gamma_{1t}^{i}(h_{t-1}) \qquad (1 + r_t)\lambda_{2t}^{i}(h_{t-1}) = \lambda_{1t}^{i}(h_{t-1}) \tag{6c,d}$$

$$\frac{1 + i_{t+1}}{2}\left(\lambda_{1t+1}^{i}(h_{t-1}, S) + \gamma_{1t+1}^{i}(h_{t-1}, S)\right) = \lambda_{2t}^{i}(h_{t-1}) \tag{6e}$$

$$\frac{1 + i_{t+1}}{2}\left(\lambda_{1t+1}^{i}(h_{t-1}, B) + \gamma_{1t+1}^{i}(h_{t-1}, B)\right) = \gamma_{2t}^{i}(h_{t-1}) \tag{6f}$$

Note that the last four equations hold with equality because there is no non-negativity constraint on financial assets, so all choices of bond-holdings are interior solutions.

Before demonstrating that every stationary enforceable debt equilibrium is Pareto optimal, in lemmas 1 and 2 we will characterize prices and consumption in enforceable debt equilibria.

Lemma 1 *In any enforceable debt equilibrium at every date* t*, any pair of goods has the same price in sub-period one:* $p_{jt} = p_t$ *for all* j *and* t*, and in sub-period two:* $q_{jt} = q_t$ *for all* j *and* t

Lemma 2 *In any enforceable debt equilibrium at any date* t*, every agent of type* i *and history,* h_t*, consumes the same quantity of every good* j*:* $c_{jt}^{i}(h_t) = c_t^{i}(h_t)$ *for all* i, j, t, h_t.

Because our environment is symmetric in goods and agents, we find that in any sub-period s and at any date t, market clearing prices are the same for all goods and in equilibrium every agent chooses to consume the same quantity of every good. Thus symmetry simplifies our environment dramatically.

In lemma 3 we define equilibrium interest rates in a stationary enforceable debt equilibrium.

Lemma 3 *In a stationary enforceable debt equilibrium, the real interest rate at any date* t *from sub-period one to sub-period two is zero,* $(1 + r_t)\frac{p_t}{q_t} = 1$ *for all* t*, and the real interest rate from*

date t to date $t+1$ compensates the lender for the time value of money, $(1+i_{t+1})\frac{q_t}{p_{t+1}} = \frac{1}{\beta}$ for all t.

Because the consumption of buyers and sellers is the same in a stationary enforceable debt equilibrium, it must be the case that the within period real interest rate does not favor buyers or sellers. Thus in a stationary equilibrium $(1+r_t)\frac{p_t}{q_t} = 1$.

In lemma 4 we find that the only stationary level of consumption for an agent of type i that is consistent with the transversality condition is determined by the agent's initial asset position.

Lemma 4 *In a stationary enforceable debt equilibrium, the real savings of an agent of type i at each date t, is determined by b_0^i:*

$$c^i = \frac{1}{n}\left[y + (1-\beta)\frac{b_0^i}{p_0}\right] \quad \textit{for all } i$$

Proof. By lemmas 1 and 2, when $c_{jt}^i(h_t) = c_j^i$ for all j, t, h_t, then $c_j^i = c^i$ for all i, j. Rearranging the budget constraints for sub-periods one and two (and dropping the notation for histories as the equation is the same for buyers and for sellers), we find:

$$\frac{1}{q_t}\left[b_{1t+1}^i - (1+r_t)(1+i_t)b_{1t}^i\right] = y - nc^i \equiv \kappa^i$$

Iterating we find:

$$b_{1t+1}^i = \kappa^i q_t \sum_{s=0}^{t} \frac{1}{\beta^s} + \frac{1}{\beta^t}\frac{q_t}{p_0} b_0^i$$

The transversality condition for our problem is:

$$\lim_{t\to\infty} \frac{\beta^t u'\left(c^i\right)}{q_t} b_{1t+1}^i = 0 \quad \text{for all } i$$

Imposing the transversality condition on our expression for bonds we find:

$$u'\left(c^i\right)\left[\frac{\kappa^i}{1-\beta} + \frac{b_0^i}{p_0}\right] = 0 \quad \text{for all } i$$

Since $u'(c^i)$ is strictly greater than zero, the transversality condition holds if and only if:

$$\frac{b_0^i}{p_0} = \frac{-1}{1-\beta}\left(y - nc^i\right) \quad \text{for all } i$$

■

Now that we have characterized consumption, prices and interest rates in a stationary enforceable debt equilibrium, we can demonstrate that every one of these equilibria is a Pareto optimum. The initial endowment vector of assets, b_0, allows for wealth transfers from one type of agent to another.

Proposition 2 *Every stationary enforceable debt equilibrium is a Pareto optimum. The vector of intial assets, b_0, determines which of the Pareto optima can be reached competitively.*

Proof. To show that each stationary enforceable debt equilibrium is a Pareto optimum, we must first show that (i) a stationary debt equilibrium exists and then that in equilibrium (ii) $c_j^i = c^i$ for all i, j, and (iii) there exists a vector of weights, θ, such that planner's problem is maximized. Observe that (ii) follows from lemma 2.

To show (i) it is sufficient to show that when $c_{jt}^i(h_t) = c_j^i$ for all i, j, t, h_t (a) there exist sequences $\{b_{st}^i(h_t)\}$, $\{r_t\}$, $\{i_t\}$, and $\{p_{jt}\}$ such that the first order conditions 6a through 6f all hold, and (b) markets clear. The prices and interest rates consistent with a stationary enforceable debt equilibrium are found in lemmas 1 and 3.

By lemma 4 we know that:

$$\frac{b_0^i}{p_0} = \frac{-1}{1-\beta}\left(y - nc^i\right) \qquad \text{for all } i$$

As consumption is strictly non-negative, this implies that there is a lower bound on the initial level of debt:

$$b_0^i \geq \frac{-p_0 y}{1-\beta} \qquad \text{for all } i$$

The market clearing condition will impose additional constraints on the vectors, c^i and b_0^i in equilibrium. First, observe that when calculating market demand, we can sum over the histories to find:

$$\sum_{i\in I}\sum_{h_s\in H_{t-1}} c_{j't}^i(h_s,\chi_t) f_t(i,h_s,\chi_t) = \sum_{i\in I} c^i \frac{1}{nk}\frac{1}{2}$$

Then market clearing gives us:

$$Y = \sum_{y=1}^{k} y \frac{1}{nk}\frac{1}{2} = \sum_{i\in I} c^i \frac{1}{nk}\frac{1}{2}$$

or

$$\sum_{i\in I} c^i = \sum_{y=1}^{k} y$$

which given the transversality condition is equivalent to:

$$\sum_{i \in I} b_0^i = 0$$

We can conclude that a stationary enforceable debt equilibrium exists whenever the following conditions hold: (i) $p_{jt} = p_t$ for all j and t, (ii) $q_{jt} = q_t$ for all j and t, (iii) $1 + i_{t+1} = \frac{1}{\beta}\frac{p_{t+1}}{q_t}$ for all t, (iv) $1 + r_t = \frac{q_t}{p_t}$ for all t, (v) $b_0^i \geq \frac{-p_0 y}{1-\beta}$ for all i and (vi) $\sum_{i \in I} b_0^i = 0$. In this equilibrium, $c_{jt}^i(h_t) = c^i = \frac{1}{n}\left[y + (1-\beta)\frac{b_0^i}{p_0}\right]$ for all i, j, t, h_t.

To show (iii), that there exists a vector of weights, θ, such that planner's problem is maximized, let $\theta^{i'} = 1$. Let $\theta^{\hat{\imath}} = u'(c^{i'})/u'(c^{\hat{\imath}})$ for all $i', \hat{\imath} \in I$. Then θ is a vector with the property that $\theta^i u'(c^i) = \theta^{i'} u'(c^{i'})$ for all $i, i' \in I$. ■

When consumption is Pareto optimal, bonds are used to transfer purchasing power from one sub-period to the next. For example if $b_0^i = 0$ for all i, then there are no transfers of wealth from one type of agent to the next and $b_{1t}^i = 0$ for all i, t. However, in this case, bonds are still used between sub-periods to give agents the wherewithal to make purchases: $b_{2t}^i(h_{t-1}, B) = -p_t y$ and $b_{2t}^i(h_{t-1}, S) = p_t y$ for all i, t, h_{t-1}. In order for an *ex ante* Pareto optimal allocation to be achieved, it must be the case that the transfer of purchasing power within periods takes place at no cost (that is, $(1 + r_t)\frac{p_t}{q_t} = 1$). Any other intratemporal interest rate would make sellers better off than buyers or vice versa.

In this section of the paper we have found the set of *ex ante* Pareto optimal allocations in our environment and demonstrated that when debt is enforceable, every stationary equilibrium is a Pareto optimum. Effectively we have demonstrated that when we take an *ex ante* approach to welfare and restrict our interest to stationary equilibria, the first welfare theorem holds in our environment.

2 Fiat Money

While perfectly enforceable debt is a solution to the problem of buying and selling at different points in time, the assumption of perfect enforceability is very strong. In the absence of an explicit institutional structure that could make debt enforceable, the more realistic assumption is arguably that private debt is not enforceable at all. For this reason, it is standard procedure in many areas of the monetary literature to assume that agents are anonymous: Agents who have defaulted in the past can not be distinguished from those who have not, and thus default is optimal and borrowing

is impossible.[1] In this section of the paper we assume that agents are anonymous, and therefore the economy has no bonds.

The means of exchange in this section of the paper is not debt, but fiat money. In the first section of the paper we found that, whether a planner chooses to redistribute wealth or not – that is, whether b_0 is a vector of zeros or not – the stationary equilibrium of the economy is a Pareto optimum. We will find that in the fiat money environment, a Pareto optimum can only be reached by a government that treats the different types of agents differently.

To introduce fiat money into this environment, assume that each consumer of type i has m_0^i units of fiat money at date 0, and thus the aggregate date 0 money supply is $M_0 = \frac{1}{nk}\sum_i m_0^i$. The government controls the money supply by imposing a tax, τ_t^i, on each type i paid at the end of date $t \geq 0$. So, the aggregate money supply changes as follows: $M_{t+1} = M_t - \frac{1}{nk}\sum_i \tau_t^i$. The government burns the proceeds of the tax – or, if the tax is negative, costlessly prints fiat money to transfer to every consumer.

Definition 3 *A **government policy** is a series of initial money supplies, m_0^i, and a sequence of taxes, $\{\tau_t^i\}$.*

In the first sub-period the consumers of type B use their money holdings to buy goods and those of type S sell their endowment for cash. In the second sub-period type B agents sell their endowment while type S agents use their cash to purchase of goods.

Let $\eta_t^i(h_t) \geq 0$ be the money holdings that an agent of type $i = \{j', y\}$ and history h_t carries at date t from sub-period one to sub-period two, and $m_{t+1}^i(h_t) \geq 0$ the money holdings carried by this agent from date t to date $t+1$. Then the budget constraints for the fiat money problem faced by a first sub-period sellers are:

$$\eta_t^i(h_{t-1}, S) = m_t^i(h_{t-1}) + p_{j't}y \tag{7a}$$

$$m_{t+1}^i(h_{t-1}, S) = \eta_t^i(h_{t-1}, S) - \tau_t^i - \sum_{j=1}^{n} q_{jt}c_{jt}^i(h_{t-1}, S) \tag{7b}$$

And for a first sub-period buyer:

$$\eta_t^i(h_{t-1}, B) = m_t^i(h_{t-1}) - \sum_{j=1}^{n} p_{jt}c_{jt}^i(h_{t-1}, B) \tag{7c}$$

$$m_{t+1}^i(h_{t-1}, B) = \eta_t^i(h_{t-1}, B) - \tau_t^i + q_{j't}y \tag{7d}$$

[1] Kocherlakota (2002) emphasizes this point.

Observe that, because money holdings must be non-negative at date t, equation $7c$ is the equivalent of a cash-in-advance constraint.

The fiat money problem is for an agent to choose c^i_{jt}, η^i_t and m^i_{t+1} to maximize equation 5 subject to the fiat money budget constraints, equations 7a, 7b, 7c and 7d.

Definition 4 *A **fiat money equilibrium** is an allocation for each type and each possible history of goods $\{c^i_{jt}(h_t)\}$ and of money, $\{\eta^i_t(h_t)\}$ and $\{m^i_t(h_{t-1})\}$, and sequences of prices, $\{p_{jt}\}$ and $\{q_{jt}\}$, initial money endowments, $\{m^i_0\}$ and taxes, $\{\tau^i_t\}$, such that*

(i) given the government policy and prices, the fiat money problem is solved for agents of all types and histories, and

(ii) markets clear

(a) in the goods market, equations 2a and 2b,

(b) $M_t = \sum_{i\in I}\sum_{h_t\in H_t} m^i_t(h_{t-1})f_t(i,h_t)$ for all t.

Once again we use $\lambda^i_{st}(h_{t-1})$ as the Lagrangian multiplier on the budget constraint for a first sub-period seller of type i and history h_{t-1} in sub-period s of date t, and $\gamma^i_{st}(h_{t-1})$ as the multiplier for a similar first sub-period buyer. We find the following first-order conditions for an agent of type $i = (j', y)$ who is solving the fiat money problem:

$$\frac{\beta^t u'\left(c^i_{jt}(h_{t-1},B)\right)}{p_{jt}} \leq \gamma^i_{1t}(h_{t-1})\forall t \qquad \frac{\beta^t u'\left(c^i_{jt}(h_{t-1},S)\right)}{q_{jt}} \leq \lambda^i_{2t}(h_{t-1})\forall j,t \tag{8a, 8b}$$

$$\gamma^i_{2t}(h_{t-1}) \leq \gamma^i_{1t}(h_{t-1}) \qquad \lambda^i_{2t}(h_{t-1}) \leq \lambda^i_{1t}(h_{t-1}) \tag{8c, 8d}$$

$$\frac{1}{2}\left(\lambda^i_{1t+1}(h_{t-1},S) + \gamma^i_{1t+1}(h_{t-1},S)\right) \leq \lambda^i_{2t}(h_{t-1}) \tag{8e}$$

$$\frac{1}{2}\left(\lambda^i_{1t+1}(h_{t-1},B) + \gamma^i_{1t+1}(h_{t-1},B)\right) \leq \gamma^i_{2t}(h_{t-1}) \tag{8f}$$

Notice that our first order conditions are similar to those for the enforceable debt problem, except that we now have non-negativity constraints on all of our choice variables.

As is typical in a cash-in-advance environment, we must also impose the transversality condition to ensure that over the infinite horizon our agents are neither saving assets that they never intend to spend, nor borrowing money that they never intend to repay.[2]

$$\lim_{t\to\infty}\inf \lambda^i_{2t}(h_{t-1})m^i_t(h_{t-1}) = 0 \text{ for all } i, h_t$$

[2] Cole and Kocherlakota (1998) demonstrate the sufficiency of this condition in a standard cash-in-advance environment.

We will study the conditions under which a Pareto optimal equilibrium can be obtained in this environment. First, we will observe that in a Pareto optimal equilibrium it must be the case that every agent with consumption, $c^i > 0$ carries sufficient money balances at every date t.

Lemma 5 *In a Pareto optimal fiat money equilibrium, at every date t and for every type i money holdings must equal or exceed the expenditure a first sub-period buyer requires in order to purchase the Pareto optimal allocation.*

Proof. To reach a Pareto optimal allocation, it must be the case that $\eta_t^i(h_{t-1}, B) \geq 0$ and therefore (as is clear from equation 7c) it must be the case that $m_t^i(h_{t-1}) \geq p_t n c^i$ for all i, t, h_t. This must be true of every consumer, because there is no way of knowing *ex ante* whether the consumer will be a first sub-period buyer or seller in period t. ■

It is no surprise that we find that the only prices consistent with the Pareto optima of section 1 are deflationary.

Proposition 3 *In a Pareto optimal fiat money equilibrium:*

(i) $p_{jt} = p_t$ *for all* j, t

(ii) $q_{jt} = q_t$ *for all* j, t

(iii) $q_t = p_t$ *for all* t *and*

(iv) $p_{t+1} = \beta p_t$ *for all* t.

Let $p_t^* = q_t^* = p_{jt} = q_{jt}$ for all j, t and $p_{t+1}^* = \beta p_t^*$ for all t. Then $\{p_t^*, q_t^*\}_{t=0}^{\infty}$ represents a sequence of prices that is in the set of prices consistent with a Pareto optimal allocation. To study the government policies that implement a Pareto optimal allocation we will take Pareto optimal prices as given.

In Proposition 4 we demonstrate that any government policy that implements a Pareto optimal allocation is type specific. The proof is by contradiction. Any uniform government policy will give every agent the same monetary endowment and the same path of taxation. The transversality condition will then imply that the real value of every agent's savings is the same in every period – and market clearing will mean that this value must be zero. Thus the only candidate for a Pareto optimum implemented by government policy is the allocation that does not involve wealth transfers from one type of agent to another. In this allocation every agent spends the whole value of his endowment in every period. This in turn means that the each agent's end of period money

holdings change only due to taxation. In other words, given a uniform government policy every type of agent holds the same quantity of money in every period.

The Pareto optimal allocation without wealth transfers, however, requires that agents with high endowments consume more than agents with low endowments. Lemma 5 makes it clear that government policy must give enough money to the wealthy agents to purchase their optimal consumption bundle. But now we find that the lower endowment agent will have extra cash in every period. And it stands to reason, that an optimizing agent with extra cash in every period will choose to spend some of that extra cash today and thus will not choose the Pareto optimal allocation. This is the logic behind proposition 4.[3]

Proposition 4 *A Pareto optimal allocation cannot be implemented in a fiat money equilibrium equilibrium if the government policy treats all consumers uniformly:*
Given a Pareto optimal price sequence, $\{p_t^*, q_t^*\}_{t=0}^{\infty}$*, if* $m_0^i = m_0$ *and* $\tau_t^i = \tau_t$ *for all* i *and* t*, then the equilibrium allocation is not a Pareto optimum.*

In an environment with an endogenous cash-in-advance constraint we have found that a government policy that implements a Pareto optimum must treat agents with low endowments differently from those with high endowments. Since the requisite differential treatment will give agents an incentive to misrepresent their endowments, there is reason to doubt that such a government policy would be successful. As in Sissoko (2007) where we study a standard cash-in-advance environment, we use this fact to motivate the exploration of credit-based money. We go beyond the argument in the previous paper by proposing a specific institutional framework that can serve as a starting point for analyzing the monetary role of financial intermediaries.

3 Default Free Intermediaries

In this section of the paper, we consider a different form of money. Here claims drawn on private financial intermediaries take the place of fiat money. The exploration of the coexistence of fiat money and intermediated money will be left to future work. Because our financial intermediaries provide liquidity to the economy, we will often call them banks.

[3] Bhattacharya, Haslag and Martin (2005) also find that type-specific lump-sum taxes implement the first-best in several heterogeneous agent environments.

In this paper we will make explicit an assumption underlying much of the literature that studies the circulation of private liabilities issued by intermediaries: Our intermediaries are default free. While this assumption may seem strong, papers on the circulation of private liabilities typically focus on the incentive problem faced by individual agents in the economy and not on the incentive problem of bankers who are in a position to defraud the public. By not modelling the incentive problem faced by banks, these papers effectively assume that banks are default free. Examples include Williamson (1999, 2004), Bullard and Smith (2003) and Berentsen, Camera and Waller (2005).

To our knowledge, only Cavalcanti and Wallace (1999a,b) endogenize the problem of the banker whose liabilities serve as means of exchange. The Cavalcanti and Wallace papers study a random-matching model of money where bankers are distinguished from non-bankers by the fact that their histories are public knowledge. They find that when bankers are sufficiently patient, they will honor their liabilities and make possible a higher level of welfare in the economy.[4] We too demonstrate that intermediaries can improve welfare, but in an environment that is simpler and more easily compared to a standard general equilibrium model. Furthermore, whereas bankers in Cavalcanti and Wallace can only issue bank notes, our bankers actively lend to the general public, so the nature of their activities is different. On the other hand, we have not endogenized the banker's problem – although we hope to do so in the future.

We will show here that private institutions that are default-free can serve as the infrastructure of a financial/monetary system and argue that the assumption that intermediaries are default-free is a good first approximation to the environment in a modern developed economy. The most obvious real world examples of this phenomenon are American Express, Visa and MasterCard – almost all merchants in the United States accept credits in accounts with these financial intermediaries as payment and almost none buy insurance to protect these accounts in case of default. According to *The 2004 Federal Reserve Payments Study* the value of credit card transactions in the U.S. is currently more than three times the value of ATM withdrawals.

In our model we find that when these default-free intermediaries offer deposit accounts with credit lines to the consumers in our economy and play a trigger strategy – withdrawing credit in case of consumer default – the intermediaries make it possible for the economy to reach a Pareto

[4] Mills (2007) distills the Cavalcanti Wallace model to its essentials and then generalizes it to find allocations in which both inside money and fiat money are essential.

optimum. If the consumers in our economy are sufficiently patient, a contract that does not distinguish between types of consumers can be used to reach a Pareto optimum. Because first sub-period sellers must accept a banker's liability in exchange for their endowment, this means of exchange is only possible when sellers have confidence in the banking system. This is the sense in which our equilibrium depends on the assumption that bankers are default free.

In many ways our approach to credit is similar to that of Berentsen, Camera and Waller (2005), although the underlying model of money is very different. Banks in both papers do not issue notes, but instead act as financial record-keepers for the economy and share information. Consumer debt is supported by the threat that access to financial markets will be withdrawn in case of default. Thus it is no surprise that the results in both papers depend on the level of the discount factor. The key difference between the two papers is the relationship between financial transactions and the goods market. In the Berentsen et al. model fiat money and credit coexist because agents have identities in financial markets, but are anonymous in the goods market – thus, debt cannot be used to purchase goods. In this paper debt is issued in order to purchase goods, and it completely displaces fiat money.

We introduce into our model, a competitive banking system that takes deposits and offers loans. Because the industry is competitive, banks make zero profits and the rate of interest on loans and on deposits is the same. Thus banks offer the consumers in our economy accounts, which when positive are interest bearing deposits and when negative are credit lines on which interest must be paid. Interest payments are credited or charged at the end of every sub-period. Banks share costlessly verifiable information on defaulters, and any agent who fails to pay a debt at date t' is shut out of credit markets for all dates $t > t'$. All accounts start with an initial value of zero.

Let $d^i_{st}(h_t)$ be the value of the account of an agent of type $i = \{j', y\}$ and history h_t at the start of sub-period s of date t, let i_{dt} be the interest rate paid or charged on accounts from sub-period 2 of date $t-1$ to sub-period 1 of date t and r_{dt} be the interest paid on an account held from sub-period 1 to sub-period 2 of date t. Then at date t a first sub-period buyer and a first sub-period seller face the following budget constraints:

$$d^i_{2t}(h_{t-1}, B) = (1 + i_{dt})d^i_{1t}(h_{t-1}) - \sum_{j=1}^{n} p_{jt}c^i_{jt}(h_{t-1}, B) \tag{9a}$$

$$d^i_{1t+1}(h_{t-1}, B) = (1 + r_{dt})d^i_{2t}(h_{t-1}, B) + q_{j't}y \tag{9b}$$

$$d^i_{2t}(h_{t-1}, S) = (1 + i_{dt})d^i_{1t}(h_{t-1}) + p_{j't}y \tag{9c}$$

$$d^i_{1t+1}(h_{t-1}, S) = (1 + r_{dt})d^i_{2t}(h_{t-1}, S) - \sum_{j=1}^{n} q_{jt}c^i_{jt}(h_{t-1}, S) \tag{9d}$$

It should be no surprise that these budget constraints are relabelled versions of the budget constraints in the enforceable debt problem. In order to make the enforcement of debt endogenous, banks will have to impose credit limits on the amount of debt that the various types of agents can borrow to ensure that over the infinite horizon default is always more costly than repayment. Thus there is an additional set of constraints:

$$d^i_{st}(h_t) \geq \bar{d}^i_{st}(h_t) \tag{10}$$

where $\bar{d}^i_{st}(h_t)$ is a non-positive number that constrains the debt of an agent of type i and history h_t in sub-period s of date t. The consumer's intermediated credit problem is to choose c^i_{jt} and d^i_{st} to maximize the objective function, equation 5, subject to the budget constraints, equations 9a, 9b, 9c and 9d, and to the debt constraints defined in equation 10.

Definition 5 *An* ***intermediated credit equilibrium*** *is an allocation of goods,* $\{c^i_{jt}(h_t)\}$*, and of account balances,* $\{d^i_{st}(h_t)\}$*, and sequences of prices,* $\{p_{jt}\}$ *and* $\{q_{jt}\}$*, of interest rates,* $\{i_{dt}\}$ *and* $\{r_{dt}\}$*, and of credit constraints,* $\{\bar{d}^i_{st}(h_t)\}$*, such that*
(i) given prices, interest rates and credit constraints, the consumer's intermediated credit problem is maximized for consumers of every type i *and history,* h_t
(ii) the goods markets clear, equations 2*a and* 2*b,*
(iii) for all $d^i_{st}(h_t) \geq \bar{d}^i_{st}(h_t)$*, a consumer of type* i *and history* h_t *will choose to repay debt at date* t*.*

Observe that this problem has the same first order conditions as the enforceable debt problem – with the caveat that all of these conditions may hold as inequalities in the present environment. We wish to establish the circumstances in which a Pareto optimal allocation can be implemented in an intermediated credit equilibrium. Since all accounts start with an initial value of zero, $d^i_{10} = 0$ for all i. Then Lemma 4 (which depends only on the budget constraints and the transversality condition) tells us that the only Pareto optimal allocation consistent with the equilibrium is the allocation in which no borrowing or saving takes place from one period to the next. This allows us to focus on the special case in which banks permit no borrowing at the end of sub-period two or by first sub-period sellers in sub-period one. In other words we will assume that banks offer debt

contracts with the following constraints:

$$\bar{d}^i_{1t+1}(h_t) = 0$$

$$\bar{d}^i_{2t}(h_{t-1}, S) = 0$$

This assumption greatly simplifies our exposition without imposing a binding constraint on our agents given the equilibrium allocation we seek to obtain.

The Pareto optimal allocation that does not involve transfers of wealth has $c^i_{jt}(h_t) = \frac{y}{n}$ for all j, t, h_t and $i = \{j', y\}$. Observe that the prices and interest rates consistent with a stationary equilibrium that were found in lemmas 1 and 3 are equally as valid in this environment as in the enforceable debt environment.

Definition 6 *A **pareto optimal price system** is composed of sequences of prices, $\{p_{jt}\}$ and $\{q_{jt}\}$, and sequences of interest rates, $\{i_{dt}\}$ and $\{r_{dt}\}$, such that:*
(i) $p_{jt} = p_t$ *for all* j, t
(ii) $q_{jt} = q_t$ *for all* j, t
(iii) the real intra-temporal interest rate is zero, $(1 + r_{dt})\frac{p_t}{q_t} = 1$ *for all* t*, and*
(iv) the real inter-temporal interest rate compensates for holding an asset over time,
$(1 + i_{dt+1})\frac{q_t}{p_{t+1}} = \frac{1}{\beta}$ *for all* t*.*

The parameter which remains to be determined in an intermediated credit equilibrium consistent with our transfer-free Pareto optimal allocation is the vector of credit constraints, $\bar{d}^i_{2t}(h_{t-1})$. The following discussion will assume that our agent faces a pareto optimal price system.

Let V^{id} be the continuation value of default for an agent of type i. This is just the utility an agent gets from consuming nothing forever:

$$V^{id} = \frac{1}{1-\beta} nu(0)$$

Let V^{ic} be the continuation value of consuming the Pareto optimal allocation forever for an agent of type $i = \{j', y\}$:

$$V^{ic} = \frac{1}{1-\beta} nu\left(\frac{y}{n}\right)$$

An agent, who chooses to default at time T, will borrow as much as possible at T and due to the concavity of the utility function will consume equal amounts of all goods at date T. Thus the

value of default to an agent of type $i = (j', y)$ is: $nu(\frac{-\bar{d}}{np_t}) + \beta V^{id}$ where $\bar{d}$ is the credit constraint. His utility when he does not default is: $nu\left(\frac{y}{n}\right) + \beta V^{ic}$. Let $\hat{d}_t^i(\beta, y)$ be the d at which

$$nu\left(\frac{y}{n}\right) + \beta V^{ic} - nu(\frac{-d}{np_t}) - \beta V^{id} \geq 0 \tag{11}$$

holds with equality. Since the left hand side of this equation is strictly increasing in d, for all $d > \hat{d}_t^i(\beta, y)$, an agent of type i will choose not to default in a Pareto optimal equilibrium for any asset level that is greater than $\hat{d}_t^i(\beta, y)$. In other words, $\hat{d}_t^i(\beta, y)$ is the equilibrium credit constraint when the equilibrium allocation is the transfer-free Pareto optimum. The properties of this credit constraint are established in the following lemma:

Lemma 6 *Given a transfer-free Pareto optimal allocation, the credit constraint, $\hat{d}_t^i(\beta, y)$, for an agent of type $i = \{j', y\}$*

(i) does not depend on an agent's history,

(ii) is decreasing in an agent's endowment level, y, where $\hat{d}_t^i(\beta, y) < -p_t y$ for all i,

(iii) is decreasing in β and

(iv) in the limit as $\beta \to 1$, $\hat{d}_t^i(\beta, y) = -\infty$.

This result allows us to demonstrate, first, that a Pareto optimum can be implemented by enforceable debt contracts in an intermediated credit equilibrium, and, second, that if β is sufficiently high this Pareto optimal allocation can be implemented using a schedule of credit constraints that does not distinguish between the different types of agents, i.

Proposition 5 *The transfer-free Pareto optimal allocation can be attained in an intermediated credit equilibrium.*

Proposition 6 *For $\beta > \beta^*$, the transfer-free Pareto optimal allocation can be implemented in an intermediated credit equilibrium by a uniform debt contract.*

Proof. By lemma 6 we know that $\max_y \hat{d}_t^i(\beta, y) = \hat{d}_t^i(\beta, 1)$. Thus, a uniform debt contract with $\bar{d}_{2t}^i(h_{t-1}) = \hat{d}_t^i(\beta, 1)$ will guarantee that no agent defaults.

The transfer-free Pareto optimal allocation can only be obtained, however, if the highest income type can afford to buy the allocation or if $\hat{d}_t^i(\beta, 1) \leq -p_t k$. From equation 11 we know that $\hat{d}_t^i(\beta, 1) = -p_t k$ when

$$\beta^* = \frac{u\left(\frac{k}{n}\right) - u\left(\frac{1}{n}\right)}{u\left(\frac{k}{n}\right) - u(0)}$$

As $\hat{d}_t^i$ is decreasing in β, we can conclude that whenever $\beta > \beta^*$ our allocation can be implemented in an intermediated credit equilibrium given the following uniform debt constraints for all i, t and h_t:

$$\begin{aligned} \bar{d}_{2t}^i(h_{t-1}) &= \hat{d}_t^i(\beta, 1) \\ \bar{d}_{1t+1}^i(h_t) &= 0 \\ \bar{d}_{2t}^i(h_{t-1}, S) &= 0 \end{aligned}$$

■

Because the banks communicate with each other and can force a defaulter into autarky forever, they can offer every consumer contract terms such that he will choose not to default. If the consumers are sufficiently patient, they can offer the same terms to every type and still have be in an equilibrium without default.

While the autarkic penalty to default that we impose may seem excessive, it is important to note that the results above do not depend on the specific form of the penalty, but only on the existence of a one-period penalty that is repeated forever. Thus there is no reason to believe that the addition of fiat money to the economy will change the character of the results.

The intermediated environment that we propose in this paper has several important properties: First, unlike the fiat money economy, a Pareto optimum can be reached in the intermediated money environment without type-specific policies – the elastic nature of credit makes it a better solution to our monetary problem.[5] Second, when our agents are sufficiently patient the default penalty is off the equilibrium path of play and the problem becomes identical to the competitive problem with enforceable debt. In short, we model intermediaries as agents who use the fact that they are perceived to be default-free to resolve the liquidity problem in the economy and thereby make it possible for the economy to reach a first-best allocation.

Finally, the Pareto optimal intermediated credit equilibrium can be viewed as a self-confirming equilibrium. Because agents believe that the banks set the credit constraints correctly, they believe that the bank's borrowers will repay their loans and therefore that the banks will not default. This equilibrium is delicate in the sense that, if the consumers in the economy stop believing in the intermediated credit equilibrium, the equilibrium will no longer exist. This property of the

[5] Williamson (1999, 2004) also finds that the benefits of private money derive from its elasticity. Bullard and Smith (2003) arrive at a similar result, but use the terminology of the real-bills doctrine. Sissoko (2007) also makes this point.

equilibrium is, however, realistic: when consumers lose faith in the viability of a banking system, they withdraw deposits and a financial crisis follows. In future work we hope to investigate this property of our model further.

4 Conclusion

Because credit in our environment solves a liquidity problem, we find that debt can be sustained by nothing more than the threat of losing the right to borrow in the future. This result stands in stark contrast to results of the existing literature on self-enforcing debt contracts – see for example Bulow and Rogoff (1989) and Kehoe and Levine (1993). The different results derive from different assumptions. The existing literature assumes that spot markets work perfectly in the absence of financial intermediation, whereas we assume that liquidity constraints can affect market outcomes and that financial intermediation is needed to make markets work. Because withdrawing credit in our model is equivalent to imposing a liquidity constraint on an agent, it is a severe penalty that is sufficient to support an equilibrium with debt.[6]

We argue in the introduction that this model is like the competitive model except that the role of financial intermediaries has been made explicit. Our reasoning is as follows: the trading friction that we introduce can be inserted into many competitive models with a continuum of agents by (i) dividing each period into two sub-periods and allowing goods to perish in each sub-period and (ii) giving one half of the agents their endowments (or production opportunities) in the first sub-period and their preference for consumption in the second and half of the agents the reverse. Simple application of the folk theorem implies that as the discount factor converges to one, agents will choose to repay intra-period debt, and thus there will always be an endogenous debt equilibrium that is equivalent to the competitive equilibrium of the initial model.

We also argue that the endogenous debt equilibrium is intermediated by default free banks. We recognize that other institutional structures can underlie endogenous debt; however, the traditional emphasis on the importance of confidence in the banking system to monetary and financial stability motivates our approach. Relaxing the assumption that bankers are default free will lead us to define a reaction function for our consumers that states how their beliefs about intermediaries are updated in the case of observed default. It is possible that a single default will cause the whole system

[6] As noted above Berentsen, Camera and Waller (2005) also use the threat of future borrowing to support debt in an environment with a liquidity problem – and find a result comparable to ours.

of intermediated credit to collapse – or that it will have no effect whatsoever. From a historical perspective, this range of possibilities may be appropriate when discussing financial markets. We hope that future research will indicate that the elastic nature of credit gives it an advantage over fiat money that contributes to the resilience of financial markets even in the event of unanticipated default.

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Appendices

A Proofs of Lemmas

Lemma 1 *In any enforceable debt equilibrium at every date t, any pair of goods has the same price in sub-period one: $p_{jt} = p_t$ for all j and t, and in sub-period two: $q_{jt} = q_t$ for all j and t*

Proof. First observe that when $c^i_{jt} > 0$, $u'(0) > u'(c^i_{jt})$. Thus equation $6a$ tells us that if any good is not consumed, its price is greater than the price of any good that is consumed. Assume that $p_{j't} = \min_j\{p_{jt}\}$ and there exists $p_{\hat{j}t} > p_{j't}$. Then equation $6a$ indicates that $c^i_{j't}(h_{t-1}, B) > c^i_{\hat{j}t}(h_{t-1}, B)$ for all i, t, h_t. This, however, produces a contradiction because it is impossible that the markets for goods $\hat{j}$ and j' both clear. Therefore we can conclude that $p_{jt} = p_t$ for all j and t. The demonstration that $q_{jt} = q_t$ for all j and t follows the same logic. ■

Lemma 2 *In any enforceable debt equilibrium at any date t, every agent of type i and history, h_t, consumes the same quantity of every good j: $c^i_{jt}(h_t) = c^i_t(h_t)$ for all i, j, t, h_t.*

Proof. Combining lemma 1 with equations $6a$ and $6b$ we find that $c^i_{jt}(h_t) = c^i_t(h_t)$ for all i, j, t, h_t. ■

Lemma 3 *In a stationary enforceable debt equilibrium, the real interest rate at any date t from sub-period one to sub-period two is zero, $(1 + r_t)\frac{p_t}{q_t} = 1$ for all t, and the real interest rate from date t to date $t + 1$ compensates the lender for the time value of money, $(1 + i_{t+1})\frac{q_t}{p_{t+1}} = \frac{1}{\beta}$ for all t.*

Proof. When $c^i_{jt}(h_t) = c^i_j$ for all j, t, h_t, then $\lambda^i_{2t}(h_{t-1}) = \frac{p_t}{q_t}\gamma^i_{1t}(h_{t-1})$ for all i, t, h_t. Substituting out the multipliers in equations $6e$ and $6f$, we find:

$$\begin{aligned} \frac{2}{q_t} &= \beta(1 + i_{t+1})\left(\frac{1 + r_{t+1}}{q_{t+1}} + \frac{1}{p_{t+1}}\right) \\ \frac{2}{(1 + r_t)p_t} &= \beta(1 + i_{t+1})\left(\frac{1 + r_{t+1}}{q_{t+1}} + \frac{1}{p_{t+1}}\right) \end{aligned}$$

Then in a stationary enforceable debt equilibrium $1 + r_t = \frac{q_t}{p_t}$ for all t and $1 + i_{t+1} = \frac{1}{\beta}\frac{p_{t+1}}{q_t}$. ■

Lemma 6 *Given a transfer-free Pareto optimal allocation, the credit constraint, $\hat{d}^i_t(\beta, y)$, for an agent of type $i = \{j', y\}$*

(i) does not depend on an agent's history,

(ii) is decreasing in an agent's endowment level, y, where $\hat{d}^i_t(\beta, y) < -y$ for all i,

(iii) is decreasing in β and

(iv) in the limit as $\beta \to 1$, $\hat{d}^i_t(\beta, y) = -\infty$.

Proof. Inspection of equation 11 demonstrates (i).

To demonstrate (ii) first assume that $\hat{d}^i_t(\beta, y) \geq -p_t y$. Then $nu\left(\frac{y}{n}\right) + \beta V^{ic} - nu(\frac{-\hat{d}}{np_t}) - \beta V^{id} < 0$ and we have a contradiction.

Now rewrite equation 11 as follows:

$$u\left(\frac{y}{n}\right) - \beta u(0) \geq (1 - \beta)u\left(\frac{-d}{np_t}\right)$$

The left hand side of this equation is increasing in y and thus $\hat{d}^i_t(\beta, y)$ is decreasing in y.

To show (iii) rewrite equation 11 as follows:

$$\frac{\beta}{1 - \beta}\left(u\left(\frac{y}{n}\right) - u(0)\right) \geq u\left(\frac{-d}{np_t}\right) - u\left(\frac{y}{n}\right)$$

Since the left hand side of this equation is increasing in β, $\hat{d}^i_t(\beta, y)$ is decreasing in β. In the limit as $\beta \to 1$, $\frac{\beta}{1-\beta} \to \infty$. Therefore, there is no finite value of $\hat{d}^i_t(\beta, y)$ as $\beta \to 1$. ■

B Proof of Proposition 3

Proposition 3 *In a Pareto optimal fiat money equilibrium*

(i) $p_{jt} = p_t$ *for all* j, t

(ii) $q_{jt} = q_t$ *for all* j, t

(iii) $q_t = p_t$ *for all* t *and*

(iv) $p_{t+1} = \beta p_t$ *for all* t.

Proof. In a Pareto optimal equilibrium, positive quantities are consumed of every good, so equation $8a$ holds with equality. Assume that $p_{jt} > p_{j't}$. Then by equation $8a$ we know that first sub-period buyers consume more of good j' than of good j, and the consumption allocation is not Pareto optimal. Conclusion: in a Pareto optimal equilibrium, $p_{jt} = p_t$ for all j, t. The proof that $q_{jt} = q_t$ for all j, t follows the same logic.

Dividing equations $8a$ and $8b$ by lagged versions of themselves, we find:

$$\frac{q_{t+1}}{\beta q_t} = \frac{\lambda^i_{2t}(h_{t-1})}{\lambda^i_{2t+1}(h_t)} \text{ for all } i, t, h_t$$

$$\frac{p_{t+1}}{\beta p_t} = \frac{\gamma^i_{1t}(h_{t-1})}{\gamma^i_{1t+1}(h_t)} \text{ for all } i, t, h_t$$

Observe that according to equation $7c$ the non-negativity constraint on $\eta^i_t(h_{t-1}, B)$ implies that Pareto optimal consumption is only possible if $m^i_t(h_{t-1}) > 0$ for all i, t, h_t. Then equations $8e$ and $8f$ hold with equality. Furthermore according to equation $7a$, $\eta_t(h_{t-1}, S) > 0$.

Since $\eta_t(h_{t-1}, S) > 0$, equation $8d$ holds with equality. Then equations $8e$ and $8f$ tell us that $\gamma^i_{2t}(h_{t-1}) = \lambda^i_{2t}(h_{t-1})$, and therefore that $q_t \geq p_t$ with equality when $\eta_t(h_{t-1}, B) > 0$.

When $q_t = p_t$, we find that $q_t = \beta q_{t-1}$.

Assume that at date t, $q_t > p_t$ and consumption is Pareto optimal. Then the budget constraints tell us that $m^i_{t+1}(h_{t-1}, S) > m^i_{t+1}(h_{t-1}, B)$ for all i and h_{t-1}. If the equilibrium is Pareto optimal, by lemma 5 we know that $m^i_{t+1}(h_{t-1}, B) \geq np_{t+1}c^i$. Then when the agent of type i whose history at date t is $h_t = \{h_{t-1}, S\}$ makes purchases at date $t+1$, he finds $m^i_{t+1}(h_{t-1}, S) > np_{t+1}c^i$ with the result that $\eta_{t+1}(h_{t-1}, S, B) > 0$ and $q_{t+1} = p_{t+1}$. Furthermore, the budget constraints tell us that $m^i_{t+2}(h_{t-1}, S, B) > m^i_{t+2}(h_{t-1}, B, B)$ for all i and h_{t-1}. Then by iterating this argument we find that this agent continues to carry extra cash into every period in the future and therefore we can conclude that $q_{t'} = p_{t'}$ for all $t' > t$.

Conclusion $q_t = p_t$ for all t and therefore $q_{t+1} = \beta q_t$ for all t. ■

C Proof of Proposition 4

Proposition 4 *A Pareto optimal allocation cannot be implemented in a fiat money equilibrium if the government policy treats all consumers uniformly:*
Given a Pareto optimal price sequence, $\{p_t^*, q_t^*\}_{t=0}^{\infty}$, if $m_0^i = m_0$ and $\tau_t^i = \tau_t$ for all i and t, then the equilibrium allocation is not a Pareto optimum.

Proof. Assume a Pareto optimal allocation, a Pareto optimal price sequence, and $m_0^i = m_0$ and $\tau_t^i = \tau_t$ for all i and t. First we will show that the only Pareto optimal allocation consistent with the transversality condition is the allocation where $c^i = \frac{y}{n}$ for all $i = \{j', y\}$. Next we will show that this allocation is not a fiat money equilibrium.

Combine the first and second sub-period budget constraints for either first sub-period sellers or buyers (repressing the history notation) to find:

$$\begin{aligned} m_t^i &= p_{t-1}^*(y - nc^i) + m_{t-1}^i - \tau_{t-1} \\ &= (y - nc^i)p_0^* \sum_{s=0}^{t-1} \beta^s + m_0 - \sum_{s=0}^{t-1} \tau_s \end{aligned}$$

where the last equality is found by iteration. Impose the transversality condition on money (taking into account the fact that $p_t^* = \beta^t p_0^*$) to find:

$$\lim_{t\to\infty} \frac{u'(c^i)}{p_0^*} \left[(y - nc^i)p_0^* \sum_{s=0}^{t-1} \beta^s + m_0 - \sum_{s=0}^{t-1} \tau_s \right] = 0 \quad \text{for all } i$$

which in turn implies:

$$y - nc^i = \frac{1-\beta}{p_0^*} \left[\sum_{s=0}^{\infty} \tau_s - m_0 \right] \quad \text{for all } i$$

Thus the uniformity of the government policy, implies that all agents have the same real savings at every date t. But, market clearing implies that the only possible level of savings is therefore zero. Since agents save nothing, we have only one candidate allocation for a Pareto optimal equilibrium, $c^i = \frac{y}{n}$ for all $i = \{j', y\}$.

Given $c^i = \frac{y}{n}$, every agent spends the whole of value of his endowment in every period and our budget constraints tell us that

$$m_{t+1}^i(h_t) = m_t^i(h_{t-1}) - \tau_t^i \text{ for all } i, t, h_t$$

In other words, each agent's money holdings change only due to taxation. Since we have assumed that there is a single monetary policy for all types of agents, this immediately implies that every

agent holds the same quantity of money at every date t. Combining this fact with lemma 1, we find that

$$m_t^i(h_{t-1}) = M_t = m_0 - \sum_{s=0}^{t-1} \tau_s \geq p_t^* y \quad \text{for all } i = \{j, y\} \text{ and } t$$

Recall that the maximum value of y is k and consider the behavior of an agent of type $i' = \{j, y'\}$ where $y' < k$. This agent knows that at every date t, $m_0 - \sum_{s=0}^{t-1} \tau_s \geq p_t^* k > p_t^* y'$. Thus this agent knows that if he chooses to carry $m_{t+1} < m_0 - \sum_{s=0}^{t-1} \tau_s$ into the next period, he will still be able to consume $c^i = \frac{y'}{n}$ for all dates in the future. Then there is some ε such that at date t he can spend some of his excess money and consume $c_{jt}^i(h_t) = \frac{y+\varepsilon}{n}$ for all j and $c^i = \frac{y'}{n}$ for all dates greater than t. Since this allocation is both affordable and preferred by an agent of type i', the allocation $c^i = \frac{y}{n}$ for all $i = \{j, y\}$ can not be an equilibrium allocation. ■

D Proof of Proposition 5

Proposition 5 *The transfer-free Pareto optimal allocation can be attained in an intermediated credit equilibrium.*

Proof. The intermediated credit equilibrium is composed of the following for each agent of type $i = \{j', y\}$: $c_{jt}^i(h_t) = \frac{y}{n}$ for all i, j, t, h_t, a pareto optimal price system and debt constraints such that $\bar{d}_{2t}^i(h_{t-1}) = \hat{d}_t^i(\beta, y)$, and $\bar{d}_{1t+1}^i(h_t) = \bar{d}_{2t}^i(h_{t-1}, S) = 0$ for all i, t, h_t.

(i) Since $\hat{d}_t^i(\beta, y) < -p_t y$, every agent can purchase the allocation. Lemmas 1 and 3 demonstrate that this allocation is optimal given the pareto optimal price system. The budget constraints make it clear that the only time an agent will go into debt is as a first sub-period buyer in which case $d_{2t}^i(h_{t-1}, B) = -p_t y$ for all i, t. First sub-period sellers will choose to lend $d_{2t}^i(h_{t-1}, S) = p_t y$ for all i, t. All other asset positions are zero.

(ii) Markets clear for all Pareto optimal allocations.

(iii) We demonstrated above that $\hat{d}_t^i(\beta, y)$ precludes default and the other credit constraints preclude both debt and default. ■