A short note on multiplicative processes subordinated to renewal counting processes

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(Dated: December 14, 2018)

Abstract

This note is an erratum on equation (14) for the paper *A Parsimonious Model for Intraday European Option Pricing* by Mauro Politi and myself, available as a discussion paper at http://www.economics-ejournal.org/economics/discussionpapers/2012-14 and later published as Scalas, Enrico and Politi, Mauro (2013) A note on intraday option pricing. International Journal of Applied Nonlinear Science, 1 (1). pp. 76-86. ISSN 1752-2862.

PACS numbers: 02.50.-r, 02.50.Ey, 89.65.Gh

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I. DEFINITIONS AND BASIC PROPERTIES

Definition 1: Let $\{U_i\}_{i=1}^{\infty}$ and $\{J_i\}_{i=1}^{\infty}$ be two sequences of independent and identically distributed positive random variables. Let $F_U(w)$ and $F_J(w)$ denote their respective cumulative distribution functions. Let $\mathbb{E}(U_1) < \infty$. Further assume that the two sequences are mutually independent.

Definition 2: The *epochs* are defined as the sequence $\{T_n\}_{n=1}^{\infty}$ of cumulative sums of the random variables J_is.

$$T_n := \sum_{i=1}^{n} J_i. \tag{1}$$

Definition 3: The *counting process* $N(t)$ is

$$N(t) := \max\{k : T_k \leq t\}. \tag{2}$$

Definition 4: Let Y_n denote the following product

$$Y_n := U_0 \prod_{i=1}^{n} U_i, \tag{3}$$

with $U_0 = 1$.

Definition 5: The *multiplicative process* $Y(t)$ is defined as the (càdlàg) random product of $N(t)$ of the U_is

$$Y(t) := Y_{N(t)} = U_0 \prod_{i=1}^{N(t)} U_i, \tag{4}$$

with $Y(0) = U_0 = 1$.

Proposition 1. The multiplicative process $Y(t)$ belongs to the class of semi-Markov processes.

Proof. This is true by construction. However, one can see that, for any Borel set $A \in \mathbb{R}^+$

$$\mathbb{P}(Y_n \in A, J_n \leq t | Y_1, \ldots, Y_{n-1}, J_1, \ldots, J_{n-1}) = \\ \mathbb{P}(Y_n \in A, J_n \leq t | Y_{n-1}) = P(Y_n \in A | Y_{n-1}) F_J(t). \tag{5}$$

□

Lemma 1. The expected value of Y_n is given by

$$\mathbb{E}(Y_n) = [\mathbb{E}(U_1)]^n. \tag{6}$$

Proof. This is an immediate consequence of the fact that the U_is are i.i.d.. $\square$

Definition 6: The *natural filtration* $\mathcal{F}_t$ is the σ-field generated by the process $Y(t)$. We can write

$$\mathcal{F}_t := \sigma(Y(s), s \leq t). \tag{7}$$

Definition 7: Let $\mathcal{G}_n := \sigma(J_1, \ldots, J_n, U_1, \ldots, U_n)$.

Lemma 2. We have

$$\mathcal{F}_t := \sigma(Y(s), s \leq t) = \sigma(J_1, \ldots, J_k, U_1, \ldots, U_k : k \leq N(t)) = \mathcal{G}_{N(t)}. \tag{8}$$

Proof. By definition

$$\sigma(Y(s), s \leq t) = \sigma(Y_{N(s)}, s \leq t) = \sigma\left(U_0 \prod_{i=1}^{N(s)} U_i, s \leq t\right) \subseteq \sigma(J_1, \ldots, J_k, U_1, \ldots, U_k : k \leq N(t)). \tag{9}$$

In the other direction, if we know all the values of $Y(s)$ for s up to time t, we can find the corresponding sequences $J_1, \ldots, J_k$ and $U_1, \ldots, U_k$ with $k \leq N(t)$. In other words, the sequences are measurable with respect to $\sigma(Y(s), s \leq t)$ and $\sigma(Y(s), s \leq t) \supseteq \sigma(J_1, \ldots, J_k, U_1, \ldots, U_k : k \leq N(t))$. $\square$

Lemma 3. Consider the ratio $Y(t)/Y(s)$ with $s < t$, this is given by

$$\frac{Y(t)}{Y(s)} = 1 \tag{10}$$

if $N(t) < N(s) + 1$ (no events between s and t) and

$$\frac{Y(t)}{Y(s)} = \prod_{i=N(s)+1}^{N(t)} U_i \tag{11}$$

if $N(t) \geq N(s) + 1$ (at least one event between s and t). The conditional expectation $\mathbb{E}(Y(t)/Y(s)|\mathcal{F}_s)$ is given by

$$\mathbb{E}(Y(t)/Y(s)|\mathcal{F}_s) = \sum_{n=0}^{\infty} \mathbb{P}(N(t) - N(s) = n|\mathcal{F}_s)[\mathbb{E}(U_1)]^n = \sum_{n=0}^{\infty} \mathbb{P}(N(t) - N(s) = n|\mathcal{F}_s)\mathbb{E}(Y_n). \tag{12}$$

Proof. As a consequence of Lemma 2, the U_is are independent of $\mathcal{F}_s$ for $i > N(s)$. Incidentally, you can convince yourself that this is the case, by noting that, if you are sitting at any time s, the size of the next U_is does not depend on the previous history of the process and hence on $\mathcal{F}_s$. Then remember that the the U_is and J_is are mutually independent (see Definition 1). The expected value of the product of n U_is is given by Lemma 1. Between s and t, there can be n events with any non-negative value of n. Then, the expectations of the product of the U_i must be multiplied by the probability of the event $\mathbb{P}(N(t) - N(s) = n)$. This probability does depend on $\mathcal{F}_s$ because it depends on the *residual lifetime* (a.k.a. *forward recurrence time*) $T_{N(s)+1} - s$ which, in turn, depends on the previous history as discussed in [1]. Then, the sum over all the possible values of n gives the desired expectation. □

Corollary 1. If $\mathbb{E}(U_1) = 1$ then $\mathbb{E}(Y(t)/Y(s)|\mathcal{F}_s) = 1$.

Proof. This is an immediate consequence of Lemma 3. □

II. MARTINGALE PROPERTY

Theorem Let $Y(t)$ be a multiplicative process as defined in Definition 5 and further assume that $\mathbb{E}(U_1) = 1$, then $Y(t)$ is a martingale with respect to its natural filtration $\mathcal{F}_t$.

Proof. Considering Definition 4, one has that $|Y(t)| = Y(t)$ and one finds that $|Y(t)|$ is integrable, namely

$$\mathbb{E}(|Y(t)|) = \mathbb{E}(Y(t)) = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n)\mathbb{E}(Y_n) = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n) = 1 < \infty. \tag{13}$$

Using the fact that it is possible to take out what is known in conditional expectations, it is possible to write the following chain of equalities

$$\mathbb{E}(Y(t)|\mathcal{F}_s) = \mathbb{E}(Y(s)Y(t)/Y(s)|\mathcal{F}_s) = Y(s)\mathbb{E}(Y(t)/Y(s)|\mathcal{F}_s). \tag{14}$$

Now, since $E(U_1) = 1$, using Corollary 1, we can conclude that

$$\mathbb{E}(Y(t)|\mathcal{F}_s) = Y(s). \tag{15}$$

□

[1] M. Politi, T. Kaizoji and E. Scalas (2011). Full characterization of the fractional Poisson process. Europhysics Letters **96**(2), 20004.