OBSERVATIONS ON THE REFEREE'S DERIVATIONS

In this note I make observations about the Referee's comments that relate to the paper's derivation of risk averse CERs, and discuss the scope of the applicability of the results derived by the paper in the risk neutral case. I refer to the Section numbers of the Referee's comments. As his expressions are not all numbered, they will be reproduced when needed for clarity.

Observations on section 2.1

Observation 1

The Referee states:

> In G&W the term structure of the discount rate is solved by deriving C_{0i} using the budget constraint over an infinite horizon. That is, noting that the optimality condition requires that:
>
> $$C_{0i}^{-\eta} = C_{ti}^{-\eta} \exp(-\rho t) \exp(r_i t)$$
>
> and that:
>
> $$K_0 = \Sigma_t C_{ti} \exp(-r_i t)$$
>
> This leads to:
>
> $$C_{0i} = K_0 \left(1 - \exp\left(\frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta}\right)\right)$$

The term structure of the discount rate is not solved in G&W. They limit themselves to deriving an expression for the risk averse CER that is morphologically similar to Weitzman's expression for risk neutral CER, and as the latter is known to be declining, claim that this proves the validity of the recommendation that discount rates should be declining. At no point in G&W is C_{0i} derived, as the utility function referred to in their expression (9) is never defined nor parametrized.

It is not clear how the Referee arrived at the above expression for C_{0i}. The paper, in contrast, derived it as follows:

$$C_0 = K_0 - x \tag{15}$$

where

$$x = \frac{e^{\frac{rt}{\sigma}} K_0}{e^{\frac{\rho t}{\sigma} + rt} + e^{\frac{rt}{\sigma}}} \tag{43}$$

The two expressions are not the same. The paper's derivation follows an explicit utility maximization, which is not referred to in the Referee's derivation.

I evaluated the Referee's expression for C_{0i} with the data of the example used in the paper (as well as for several other combinations of risk aversions and time preferences), and in no case did the Referee's expression compute the right C_{0i}.

Observation 2

G&W derive their results assuming a risk averse investor. In that light, it is hard to interpret the following statements of the Referee:

The comparative statics which determine the initial consumption levels in each state are immediate. In particular the corner solutions are: 1) $C_{0i} = K_0$ if $\exp\left(\frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta}\right) = 0$; 2) $C_{0i} = 0$ (or some subsistence level of consumption) if $\exp\left(\frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta}\right) \geq 1$. In other words:

$$C_{0i} \begin{cases} = 0 \ \text{ if } \frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta} \geq 0 \\ K_0 > C_{0i} > 0 \ \text{ if } \frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta} < 0 \\ = K_0 \ \text{ if } \frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta} \to -\infty \end{cases}$$

There is no finite corner solution for risk averse investors. Where do indifference curves touch the axes? His expression for $C_{0i} = 0$ is right for risk neutral investors, for then it boils down to the condition that the interest rate exceed the time preference, but not for risk averse investors. For them C_{0i} is never zero.

Observations on section 2.2

The Referee states "The discount rate is derived in a different way in the current paper, compared to G&W. Rather than deriving the weights above, and deriving the CER as a probability and risk adjusted weighted average of the discount factors exp ($-r_i t$) (or equivalently, following Gollier with alternative weights for the compound factor exp ($r_i t$))."

This statement is incorrect for two reasons:

1. The paper derives the discount rates *exactly* as in G&W. Expression (17) of the paper is the equivalent of expression (12) of G&W. Maybe the Referee failed to understand that both the Gollier and Weitzman approach discount rates can be derived directly from these expressions, without having to calculate what G&W call "risk adjusted probabilities."
2. None the less, the paper works through those transformations as well, and the probability weighting that the Referee claims was not used is there in Section A.3 of the paper. See expression (50) for scenario specific discount factor exp ($-r_i t$), and expression (54) for scenario specific compound factor exp ($r_i t$).

The paper explicitly explains this: "[The CERs were] calculated directly from (17), without going through the transformations that Gollier and Weitzman (2010) used to obtain expressions morphologically similar to the original Weitzman (1998) and Gollier (2004) formulations, or using 'risk adjusted probabilities.' The same result is obtained, however, if one works through those transformations (A.3)." This must have escaped the Referee's attention, for he states:

This definition, in the two period case, is the object of analysis in this paper, and it differs in structure from the object analysed by Gollier and Weitzman (2010), which in the iso-elastic case two state case would be as follows:

$$R_*^W(t) = -\frac{1}{t} \ln \left(\frac{p_1 C_{01}^{-\eta}}{p_1 C_{01}^{-\eta} + p_2 C_{02}^{-\eta}} \exp(-r_1 t) + \frac{p_2 C_{02}^{-\eta}}{p_1 C_{01}^{-\eta} + p_2 C_{02}^{-\eta}} \exp(-r_2 t) \right) \quad (1)$$

...

His expression (1) above, which he claims to be correct, corresponds exactly to one of the ways in which the paper calculates R_d, see its expression (53). Thus, he is suggesting that the paper's method of CER calculation is wrong while explicitly showing it to be right.

Observations on section 2.2.1

The Referee claims that the main result of the paper "although not stated in these terms" is that as the degree of risk aversion declines, CERs converge towards the correct risk neutral discount factor. As stated in my previous replies, this is not the main result of the paper, it is just an illustration of one of its points. I challenge anyone, however, to find a set of numerical parameters for the G&W type model defined in the paper for which this result is not true.

Observations on section 2.3

Observation 1

The issue raised in this Section has already been addressed in my previous replies, but because it is so central to the Referee's opinion of the paper, it will be addressed again in greater detail.

We are now back to analysis under risk neutrality, and therefore no longer dealing with the G&W paper, which only treats the case of risk aversion. Here the Referee examines what happens to risk neutral investors CERs as a function of their pure rate of time preference. He distinguishes three cases:

1. The pure rate of time preference is lower than the lowest possible market interest rate. The decision-maker will invest all of his endowment, postponing all consumption to time *t*.
2. The pure rate of time preference is higher than the highest possible interest rate. The decision-maker will not invest at all. He will consume all of his endowment in the present.
3. The pure rate of time preference falls between the lowest possible and the highest possible interest rates.

What can be concluded from this?

- I case 2, the decision maker is not an investor. He therefore is not a subject of the Weitzman (1998) model. This is not a limitation of the model, since the model has been set up to examine the behavior of investors and to propose decision making tools for them. The fact that there are many non-investors alive does not say anything about the correctness of the decision tools proposed to investors.
- For case 1, the Referee agrees with the paper's assertion that under the Weitzman model's assumptions risk neutral CERs are growing functions of time.
- For case 3 the Referee writes: "The third case is the obvious intermediate case, the characteristics of which need to be proven, but which often sees the discount rate decline to the lowest possible realisation of r_i, as in Weitzman (1998). So, the results in the two state two period model are not as general as claimed in the paper."

This last point was already addressed earlier, but the argument is worth restating more formally. Depending on where a case 3 decision-maker's pure rate of time preference falls within the range defined by the lowest and the highest interest rates, he will either decide not to invest at all, as in case 2 above, or he will be an investor, as in case 1, who will postpone all consumption to time *t*. The cases therefore effectively reduce to two: Case 1 corresponds to those who do not invest and are therefore not subjects of the Weitzman (1998) model or the paper; and Case 2, which corresponds to investors, whose decision rule is the subject of both Weitzman (1998), Gollier (2004) and the paper.

The Referee states that the characteristics of case 3 investors need to be proven. This is done below for those who are investors. Therefore, the following applies to case 2 investors as well.

As stated in an earlier reply, the pure rate of time preference makes no difference in the case of risk averse investors, because time preference affects both market and project results alike. This can be proven following the method used by G&W. We have already established that the optimal consumption path of risk averse investors is to consume only in the future. Their welfare function for any scenario *i* is as follows:

$$V_i() = K_0 \exp(t(r_i - \rho)) \tag{1}$$

where $V_i()$ is the investor's welfare in scenario *i*, r_i is the scenario specific interest rate, ρ is the pure rate of time preference, and *t* is time.

In the two-scenario case, expected welfare is given by:

$$E[V_i()] = K_0 [\, p_1 \exp(t(r_1 - \rho)) + p_2 \exp(t(r_2 - \rho))] \tag{2}$$

where the p_i are the probabilities of the scenarios.

The CER can be established by finding the yield of a small investment δ such that $E[V_i()]$ remains unchanged. Let's define r^* as the rate of return of the yield that meets this condition. Then:

$$(K_0 - \delta) [\, p_1 \exp(t(r_1 - \rho)) + p_2 \exp(t(r_2 - \rho))] + \delta \exp(t(r^* - \rho))$$

$$= K_0 [\, p_1 \exp(t(r_1 - \rho)) + p_2 \exp(t(r_2 - \rho))] \tag{3}$$

We can carry out the multiplication in the left-hand side, which will then be equal to

$$K_0 [\, p_1 \exp(t(r_1 - \rho)) + p_2 \exp(t(r_2 - \rho))]$$

$$- \delta [\, p_1 \exp(t(r_1 - \rho)) + p_2 \exp(t(r_2 - \rho))] + \delta \exp(t(r^* - \rho)) \tag{4}$$

Notice that the first term of (4) is the same as the right-hand side of (3). Expression (3) therefore becomes:

$$\delta \exp(t(r^* - \rho)) = \delta [\, p_1 \exp(t(r_1 - \rho)) + p_2 \exp(t(r_2 - \rho))] \tag{5}$$

Dividing through by δ yields:

$$\exp(t(r^* - \rho)) = p_1 \exp(t(r_1 - \rho)) + p_2 \exp(t(r_2 - \rho)) \tag{6}$$

Dividing through by $\exp(t\rho)$ yields:

$$\exp(t\, r^*) = p_1 \exp(t\, r_1) + p_2 \exp(t\, r_2) \tag{7}$$

Solving for r* we get:

$$r^* = (1/t) \ln [\, p_1 \exp(t\, r_1) + p_2 \exp(t\, r_2) \tag{8}$$

This expression is exactly the same as R^G (t) in the Referee's note, which is a growing function of time. This is what in the paper is called *r**, the only correct discount rate that can be derived from the assumptions of the Weitzman (1998) model.

This formally proves what was already stated in an earlier reply, and which should be self-evident: if the market conditions are such that a risk neutral decision maker becomes and investor, then his pure rate of time preference, if any, makes no difference.

To summarize the conclusions about the three cases that the Referee has distinguished, we can see that in fact there are only two relevant cases: that of investors, who have chosen to invest, and that of non-investors. Non-investors don't need investment advice, and are not the subjects of the

Weitzman (1998) model. As for the investors, the conclusions of this paper apply, regardless of their rates of pure time preference. The statement by the Referee "So, the results in the two state two period model are not as general as claimed in the paper" is therefore false, because the results apply to *any* risk averse investor.

Observation 2

The Referee makes some statements that are either inconsistent or raise doubts about his understanding of the paper's thesis. On the one hand, he states:

- "in this case it is true to say that $\lim_{\eta\to 0} R_*^W(t) = R^G(t)$"

On the other hand, he states:

- "The third case is the obvious intermediate case, the characteristics of which need to be proven, but which often sees the discount rate decline to the lowest possible realisation of r_i, as in Weitzman (1998)."

So, which is it? Has he accepted the fact that Weitzman discounting is wrong, as proven in Section 5.1 of the paper or has he not? The first of the two preceding quotes suggests that he has, while the second suggests that he hasn't. Is "as in Weitzman (1998)" correct in his view?

If he feels that Weitzman discounting is correct, disproving the paper's assertion to the contrary should have been his main goal.