Comments on E-Journal Article on the Gollier and Weitzman 2010 Economic Letters article.

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1 General Comments

The paper seems to make an interesting point about the paper by Gollier and Weitzman (2010). The basic claim is that when the agent is risk-neutral in teh G&W framework, the appropriate certainty equivalent discount rate is the same as that discussed (but not really proposed) by Gollier (2004) coming from the Expected Net Future Value (ENFV) framework. This latter term structure is the exact opposite of Weitzman (1998) in the sense that it starts at the expected discount rate and for ever longer time horizons increases to the maximum possible realisation of the discount rate. Weitzman (1998) proposes one that, conversely, declines to the lowest possible realisation. In the process of showng this, there are several statements are made concerning whether G&W actually resolve the so-called Gollier-Weitzman Puzzle.

However, I am pretty sure that the claim made in the paper is far from general, despite what the later sections of the paper claim. In truth, and as shown below, the risk-neutral agent would only deploy the ENFV social discount rate if, in the two-state-two-period model presented, the rates of return are greater than the pure time preference rate and all consumption is postponed to the future date. So the result is a special case, as I show below.

The proof of this critique is shown below. The paper rather attempts to prove its point using numerical simulations. These do not withstand theoretical scrutiny and describe a special case.

I dare say, though, that some of the other critiques of the Weitzman (1998) model have some merit. However, these critiques must now take into account the fact that the theoretical model used as motivation does not support their claims, even though some other model might. However, the claims agains the ENPV approach must also take into account the fact that other models of risk-neutral agents have better explained the Gollier-Weitzman puzzle (Freeman 2010, e-journal), and show that the ENFV approach is inferior. It must also reconcile the fact that the Local Expectations Hypothesis, which comes from a general equilibrium model in finance, leads to a Weitzman type model of pricing and discount rates (See Cox Ingersoll and Ross 1981). Ang and Liu (2004) also have a proof of this pricing formula, albeit not for risk-free assets.

So, the paper makes claims that are too strong considering the broader literature in this area and the absence of theoretical results. There are many other critiques of the Weitzman approach to discounting risk free projects, some of which explicitly take a risk neutral approach, and are far more general in their theoretical treatment of uncertainty. As well as missing out some important references in the literature, the paper also suffers from selective interpretation of previous work. So, there are many possible improvements that could be made to the paper to make it more readible and make clear where the results stand in relation to the other work in the areas.

The paper also suffers from being poorly organised. It was difficult to navigate.

In conclusion, there is a chance that the proof below contains a mistake. If so I look forward to the corrections. However, if the proof withstands scrutiny, I think it shows that the paper is not of sufficient quality to be published.

2 Specific points

2.1 On G&W (2010)

G&W (2010) start with the following intertemporal welfare function $V(\mathbf{C})$ for $\mathbf{C} = (C_0, C_1, ...,)$. Supposing that the welfare function is discounted utilitarian then:

$$V(\mathbf{C}) = \Sigma_t \exp(-\rho t) U(C_t)$$

which is a very special, but standard case. In this context, G&W have the following expression for the certainty equivalent discount rate under the ENPV approach of Weitzman:

$$R_*^W(t) = -\frac{1}{t} \ln\left(\Sigma_i q_i^W \exp(-r_{it})\right)$$

where:

$$q_i^W = \frac{p_i U'(C_{0i})}{\Sigma_i p_i U'(C_{0i})}$$

In the iso-elastic case explored by G&W, this becomes:

$$q_i^W = \frac{p_i C_{0i}^{-\eta}}{\Sigma_i p_i C_{0i}^{-\eta}}$$

So $R_*^W(t)$ becomes:

$$R_*^W(t) = -\frac{1}{t} \ln\left(\Sigma_i \frac{p_i C_{0i}^{-\eta}}{\Sigma_i p_i C_{0i}^{-\eta}} \exp(-r_{it})\right)$$

and the future valued version (the Gollier approach) has a similar structure.

In G&W the term structure of the discount rate is solved by deriving C_{0i} using the budget constraint over an infinite horizon. That is, noting that the optimality condition requires that:

$$C_{0i}^{-\eta} = C_{ti}^{-\eta} \exp(-\rho t) \exp(r_i t)$$

and that:

$$K_0 = \Sigma_t C_{ti} \exp(-r_i t)$$

This leads to:

$$C_{0i} = K_0 \left(1 - \exp\left(\frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta}\right)\right)$$

The comparative statics which determine the initial consumption levels in each state are immediate. In particular the corner solutions are: 1) $C_{0i} = K_0$ if $\exp\left(\frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta}\right) = 0$; 2) $C_{0i} = 0$ (or some subsistence level of consumption) if $\exp\left(\frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta}\right) \geq 1$. In other words:

$$C_{0i} \begin{cases} = 0 \ \text{ if } \frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta} \geq 0 \\ K_0 > C_{0i} > 0 \ \text{ if } \frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta} < 0 \\ = K_0 \ \text{ if } \frac{1-\eta}{\eta} r_i - \frac{\rho}{\eta} \to -\infty \end{cases}$$

So, in any given state, if the rate of return r_i is high, ρ is low (patient social planner) and there is a low preference for consumption smoothing (η close to zero), then consumption will be low at time zero. For $\eta > 1$, and positive rate of return r_i, C_{0i} will never be zero. For $\eta \leq 1$ the value of C_{0i} depends on the relative values of r_i and ρ. Also, if $r_i = \rho$ then $C_{0i} = K_0 (1 - \exp(-\rho)) = K_0 (1 - \exp(-r_i))$, and so is not affected by risk aversion η. This is also true if $\eta = 1$.

2.2 Current paper

All of the foregoing is important for understanding the 2 period model of the current paper. In the two period two state model the equivalent formula for C_{0i} :

$$C_{0i} = K_0 \left(1 - \frac{\exp\left(\frac{r_i}{\eta} t\right)}{\exp\left(\frac{\rho}{\eta} t + r_i t\right) + \exp\left(\frac{r_i}{\eta} t\right)}\right)$$

which has similar comparative statics for each C_{0i} in each state of the world i. In particular meaning that as $\eta \to \infty$, $C_{0i} \to K_0$ and as $\eta \to \bar{\eta}$, $C_{0i} = 0$ where $\bar{\eta}$ is defined by $\exp\left(\frac{r_i}{\bar{\eta}} t\right) = \exp\left(\left(\frac{\rho}{\bar{\eta}} + r_i\right) t\right) + \exp\left(\frac{r_i}{\bar{\eta}} t\right)$.

The discount rate is derived in a different way in the current paper, compared to G&W. Rather than deriving the weights above, and deriving the CER as a probability and risk adjusted weighted average of the discount factors $\exp(-r_i t)$ (or equivalently, following Gollier with alternative weights for the compund factor $\exp(r_i t)$), the current paper rather defines the CER discount rate, $R_d(t)$, via a certainty equivalent discount factor defined as follows:

$$\begin{aligned}\exp(-R_d(t)\,t) &= \frac{p_1\exp(-\rho t)\,U'(C_{t1})+p_2\exp(-\rho t)\,U'(C_{t2})}{p_1U'(C_{01})+p_2U'(C_{02})}\\ &= \frac{p_1\exp(-\rho t)\left((\exp(r_1t)(K_0-C_{01})\right)^{-\eta}+p_2\exp(-\rho t)\left((\exp(r_2t)(K_0-C_{02})\right)^{-\eta}}{p_1C_{01}^{-\eta}+p_2C_{02}^{-\eta}}\end{aligned}$$

where the second line assumes iso-elastic utility. This leads to a certainty equivalent rate $R_d(t)$ of the following form:

$$R_d(t)=-\frac{1}{t}\ln\left(\frac{p_1\exp(-\rho t)\left((\exp(r_1t)(K_0-C_{01})\right)^{-\eta}+p_2\exp(-\rho t)\left((\exp(r_2t)(K_0-C_{02})\right)^{-\eta}}{p_1C_{01}^{-\eta}+p_2C_{02}^{-\eta}}\right)$$

This definition, in the two period case, is the object of analysis in this paper, and it differs in structure from the object analysed by Gollier and Weitzman (2010), which in the iso-elastic case two state case would be as follows:

$$R_*^W(t)=-\frac{1}{t}\ln\left(\frac{p_1C_{01}^{-\eta}}{p_1C_{01}^{-\eta}+p_2C_{02}^{-\eta}}\exp(-r_1t)+\frac{p_2C_{02}^{-\eta}}{p_1C_{01}^{-\eta}+p_2C_{02}^{-\eta}}\exp(-r_2t)\right) \tag{1}$$

In this paper a numerical example illustrates that $R_*^W(t)=R_d(t)$ (and this this is also equal to the equivalent 'Gollier type' future value definition $R_c(t)$).

2.2.1 The main claimed result:

Although not stated in these terms, the main "result" or claim of the paper is:

$$\lim_{\eta\to 0}R_*^W(t)=R^G(t) \tag{2}$$

where the definition of $R^G(t)$ is:

$$\begin{aligned}R^G(t) &= \frac{1}{t}\ln\left(p_1\exp(r_1t)+p_2\exp(r_2t)\right)\\ &= -\frac{1}{t}\ln\left(\frac{1}{p_1\exp(r_1t)+p_2\exp(r_2t)}\right)\end{aligned}$$

which, as is well known from Gollier (2004), is increasing with the time horizon, counter to the literature on declining discount rates such as Weitzman (1998). In words, the paper tries to show that the risk-neutral certainty equivalent discount rate emerging from the paper by Gollier and Weitzman (2010), $R_*^W(t)$, converges to that associated with the expected net future value criterion (ENFV) of Gollier (2004), making it the appropriate social discount rate for evaluating risk free projects from the perspective of a risk free social planner: the term structure of discount rates should be increasing rather than decreasing. Indeed, Weitzman (1998) proposed the ENPV formula:

$$R^W(t)=-\frac{1}{t}\ln\left(p_1\exp(-r_1t)+p_2\exp(-r_2t)\right)$$

which is declining with the time horizon.

2.3 Problem

The numerical example in Table 3 illustrates that the claim in (2) is at least a possibility: the example is reproducible at least. However, the problem is that this **does not constitute a proof**. The parameter values that are used for the numerical exercise are quite specific, so, we cannot be sure that the strong statements made in this paper are indeed true in general, or whether they are just a special case. What is required is a complete formal statement of the properties of $R_*^W(t)$ with respect to η. This would go as follows.

A proof of $\lim_{\eta\to 0} R_*^W(t)$ First, one needs to find the limit of $R_*^W(t)$ in (1). Recalling the structure of (1), first find $\lim_{\eta\to 0} C_{0i}^{-\eta}$ using:

$$C_{0i} = K_0\left(1 - \frac{\exp\left(\frac{r_i}{\eta}t\right)}{\exp\left(\frac{\rho}{\eta}t + r_i t\right) + \exp\left(\frac{r_i}{\eta}t\right)}\right)$$

Then with some rearrangment:

$$\begin{aligned} C_{0i}^{-\eta} &= K_0^{-\eta}\left(1 - \frac{\exp\left(\frac{r_i}{\eta}t\right)}{\exp\left(\frac{\rho}{\eta}t + r_i t\right) + \exp\left(\frac{r_i}{\eta}t\right)}\right)^{-\eta} \\ &= K_0^{-\eta}\left(\frac{\exp(-\rho t)\exp(-r_i t\eta)}{\left(\exp\left(\frac{\rho}{\eta}t + r_i t\right) + \exp\left(\frac{r_i}{\eta}t\right)\right)^{-\eta}}\right) \end{aligned}$$

To find $\lim_{\eta\to 0} C_{0i}^{-\eta}$, take the numerator and denominator separately. First the numerator:

$$\lim_{\eta\to 0}\exp(-\rho t)\exp(-r_i t\eta) = \exp(-\rho t)$$

and then the denominator:

$$\begin{aligned} &\lim_{\eta\to 0}\left(\exp\left(\frac{\rho}{\eta}t + r_i t\right) + \exp\left(\frac{r_i}{\eta}t\right)\right)^{-\eta} \\ = \ &\exp\left(-\lim_{\eta\to 0}\eta\ln\left(e^{\frac{r_i}{\eta}t} + e^{\frac{\rho}{\eta}t}e^{r_i t}\right)\right) \end{aligned}$$

But we know from Weitzman (1998) and Gollier (2004) and others that since taking the limit (from the right) as $\eta \to 0$ is similar in this context to taking the limit as $t \to \infty$ that:

$$-\lim_{\eta\to 0}\eta\ln\left(e^{\frac{r}{\eta}t} + e^{\frac{\rho}{\eta}t}e^{rt}\right) = -t\max[\rho, r_i]$$

That is, the limiting value of this limit is the maximum discount rate of ρ or r_i. So finally, since $\lim_{\eta\to 0} K_0^{-\eta} = 1$:

$$\lim_{\eta\to 0} C_{0i}^{-\eta} = \frac{\exp(-\rho t)}{\exp(-t\max[\rho, r_i])}$$

This leaves two cases[1]:

$$\lim_{\eta\to 0} C_{0i}^{-\eta} = \begin{cases} = \frac{\exp(-\rho t)}{\exp(-r_i t)} = \exp((r_i - \rho)t) & \text{if } r_i > \rho \\ = \frac{\exp(-\rho t)}{\exp(-\rho t)} = 1 & \text{if } r_i \leq \rho \end{cases}$$

To obtain expressions for $\lim_{\eta\to 0} R_*^W(t)$ we can insert this result into (1). There are 3 possible cases in the two-state world:

$$\lim_{\eta\to 0} R_*^W(t) = \begin{cases} \frac{1}{t}\ln(p_1\exp(r_1 t) + p_2\exp(r_2 t)) = R^G(t) & \text{if } r_1 \text{ and } r_2 > \rho \\ -\frac{1}{t}\ln(p_1\exp(-r_1 t) + p_2\exp(-r_2 t)) = R^W(t) & \text{if } r_1 \text{ and } r_2 < \rho \\ -\frac{1}{t}\ln\left(\frac{p_1\exp(r_1-\rho)}{p_1\exp(r_1-\rho)+p_2}\exp(-r_1 t) + \frac{p_2}{p_1\exp(r_1-\rho)+p_2}\exp(-r_2 t)\right) & \text{if } r_1 > \rho \text{ and } r_2 < \rho \end{cases}$$

Now we can see that the numerical example falls into the first category: r_1 and $r_2 > \rho$, since $\rho = 0$ and $r_1 = 5\%$ and $r_2 = 1\%$. Even the example in the appendix only has $\rho = 0.5\%$, so it too falls into the first category. In this case, in the two period example, it is true to say that $\lim_{\eta\to 0} R_*^W(t) = R^G(t)$. However, as the theoretical analysis shows, this is just one case of several. In fact, when the social planner is more impatient, the risk-neutral social discount rate is that of Weitzman (1998): $\lim_{\eta\to 0} R_*^W(t) = R^W(t)$. The third case is the obvious intermediate case, the characteristics of which need to be proven, but which often sees the discount rate decline to the lowest possible realisation of r_i, as in Weitzman (1998). So, the results in the two state two period model are not as general as claimed in the paper.

The basic problem with the model is that a risk neutral planner does not need to smooth. Consumption will either all happen in the future, or in the present depending on the whether returns are higher than impatience or not. This is what determines the social discount rate in this model.

3 Minor Points

- The paper has a selective reading of the literature. The paper emphasises the tendency of some papers to talk about risk aversion and the location of risk when the context is risk neutrality. In fact there are two papers that have addressed the "puzzle" (it's not really a puzzle if a model is underspecified, but anyway) in its original terms. First, Freeman (2010)

[1] It should be noted that the limit is strictly indeterminate since the limit taken from the left differs from that taken from the right. The limits here are taken from the right, assuming a risk averse planner becomes more risk neutral. This seems like a reasonable approach, and is illustrative at least.

"Yes, we should discount at the lowest possible rate..." which is published in the economics e-journal. This paper solves the puzzle by separating out risk aversion from preferences for consumption smoothing and shows that the Gollier-Weitzman puzzle is simply a manifestation of an old problem in Finance. My view is that this is the most convincing solution of the problem in the risk neutral terms that the original problem was presented. T also shows that the ENPV approach is superior to the ENFV approach. Second, Hepburn and Groom (2007, JEEM), while a less convincing solution also make the point that the location of risk in time is not the source of the puzzle at all. So, despite the length of the paper, the paper is not complete in the way it sets up the problem/solution.

- The functon $V(.)$ needs to be introduced. The casual reader will have no idea what this is supposed to represent. Nor why it should vary depending upon the state of the world. This is a completely different framework to that discussed by G&W who rather define $V(\mathbf{C}_i)$ to show that the measure of welfare does not change depending on the resolution of uncertainty, but the consumption paths do. This looks rather sloppy, unless there is some other meaning to this approach which has not been explained properly. Are we to believe that in one state of the world we are a discounted utilitarian, and in another we are quasi-hyperbolic, for instance?

- Page 5: the rationale for using $V_i''(C_0)$, that is, the second derivative of the fucntion $V(.)$, needs to be explained. This is not how G&W (2010) present their work, and it would be clearer for all if the notation coincided with previous work.

- In section 4, the author states that G&W choose the discounted utilitarian form of preferences to make their point. It is possibly the phrasing of this point, but one of the clever things about G&W is that they solve the "puzzle" without specifying the preferences, particularly the nature of pure time preference and the precise extent of risk aversion. The DU version of their model is a very special case that is worked through at the end, with the even more special logarithmic case specifying the original ENPV case. As written, this paper seems to suggest that only DU is considered by G&W.

- On Page 7 it is stated that Weitzman (1998) only considered 2 time periods. This is incorrect. That paper has a limiting result, and also characterises a declining forward rate rather than the average rate of discount. So the 2-period statement (irrespective of the zero minus epsilon moment) is inaccurate. This inaccuracy reduces the credibility of the research by making it seem out of touch with the general literature. Perhaps the meaning is that in effect only 2 time periods are treated in Weitzman (1998), but if so this needs to be explained.