Referee report in the paper by

Testing for unit roots with cointegrated data. Submitted to *e-Economics*

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Summary and General Comments

The paper shows by simulation that if data for $(x_t, y_t), t = 1, \ldots, T$ are generated by a cointegrated VAR in two dimensions, then various univariate tests for unit roots of x_t and y_t have rejections probabilities which deviate seriously from the nominal levels. This is of course a nice observation and should be followed up by an analysis of the two dimensional VAR to find out whether it helps to analyse the variables jointly.

Detailed Comments

Page 1: You show by simulations that all the performed tests have bad rejection probabilities when you do a univariate unit root test on Y. This is of course interesting, but in a wider perspective it should be pointed out that even in the joint analysis of the variables, by the CVAR say in two dimensions, the rank test can have bad rejection probabilities for some choice of parameters. In particular when the DGP is close to being $I(2)$, they can be large. As a check on the calculations it would also be interesting to see whether you get the correct rejection probability when you simulate a very long series. This would also be a safeguard against the possibility of programming errors.

Page 2, (6): Some pages are spent calculating back and forth between the CVAR in (6) and the ARDL formulation in (1). This is of course well known and some space would be saved by sticking to only one representation.

Page 4, line 32. It is unclear what you mean when saying that a VEC model is well behaved. You seem to mean that the diagonal elements the matrix

$$\begin{pmatrix} \delta_y & \theta\delta_y \\ \delta_x & \theta\delta_x \end{pmatrix}$$

are negative. For cointegrated $I(1)$ data we know that $y_t + \theta x_t$ is an $AR(1)$ process with coefficient $1 + \delta_y + \delta_x\theta$ which satisfies $-1 < 1 + \delta_y + \delta_x\theta < 1$. The right hand side is satisfied if $\delta_y + \delta_x\theta < 0$, so it is not sufficient that either δ_y or $\theta\delta_x$ is negative, but certainly sufficient that they both are, see comment to page 8 line 7, provided that $\delta_y + \delta_x\theta > -2$.

Page 4, line 6: You write "The first two columns of TABLE 2 describe the model parameters." This is obviously not so, as the constant terms β_{10} and β_{20} in equation (1) do not appear in TABLE 2. However, based on the text of TABLE 1, and "Case 1" on page 4, it appear that the constant terms are set to zero. Presumably they are also zero in the other simulations. This is of course important as under the null of nonstationarity, a constant term would generate a trend in the process, and change the asymptotic behaviour of the test statistics, see (1).

Page 5, line 2: You write "The Z column is useful for illustrating the range of deviations that can be expected from a sampling error." This is clearly not correct. The Z column illustrates the deviation you can get from testing in a true random walk model, but you are simulating much more complex structures which have many more lags. In these models the (small sample) deviations could be completely different, depending on the precise form of the DGP. Also, you estimate and calculate test statistics from models that are misspecified, because of the lags chosen.

Page 8, line 7: The cointegrating formulation is given in (6) as

$$\begin{aligned}\Delta y_t &= a_{10} + \delta_y(y_{t-1} + \theta x_{t-1}) + \varepsilon_{yt} \\ \Delta x_t &= a_{20} + \delta_x(y_{t-1} + \theta x_{t-1}) + \varepsilon_{xt}\end{aligned}$$

The differenced specifications (regression equations) are given by

$$\begin{aligned}\Delta y_t &= b_{10} + \rho_y y_{t-1} + error_{yt} \\ \Delta x_t &= b_{20} + \rho_x y_{t-1} + error_{xt}\end{aligned}$$

It is unclear how from these equations you can derive that $\rho_y = \delta_y$ and $\rho_x = \theta\delta_x$. Under this choice you would have that

$$error_{yt} = \delta_y \theta x_{t-1} + \varepsilon_{yt}$$

which seems to be against the specification that the error term is stationary (or even i.i.d.).

In fact it follows from the Granger representation theorem that

$$y_t = \frac{\theta}{\theta\delta_x + \delta_y} \sum_{i=1}^{t} \{\delta_x \varepsilon_{yi} - \delta_y \varepsilon_{xi}\} + \theta \frac{\delta_x a_{10} - \delta_y a_{20}}{\theta\delta_x + \delta_y} t + stationary\ terms \quad (1)$$

This shows that for the investigations to work you need the restriction $\delta_x a_{10} = \delta_y a_{20}$ to avoid a linear trend in the process, which otherwise would change the limit distributions of the statistics you work with.

Thus, the main term of y_t is a random walk $S_t = \sum_{i=1}^{t}\{\theta\delta_x\varepsilon_{yi} - \theta\delta_y\varepsilon_{xi}\}$. If you then consider a regression (or specification) equation of the form

$$\Delta y_t = a + by_{t-1} + c_1\Delta y_{t-1} + ... + c_k\Delta y_{t-k} + error_y, \tag{2}$$

it would be nice to express the values of the coefficients, $a, b, c_1, ..., c_k$, in terms of the parameters of the DGP. Note that y_t is nonstationary because of the contribution from S_t, and the only way the coefficients of (2) can be interpreted in terms of the coefficients of the VEC is by inserting the expression for y_t and equate coefficients. This will of course not give i.i.d. errors but stationary errors. Obviously the coefficient b must be zero, as you write, but that has nothing to do with the value of $\theta\delta_x$ and δ_y.

Page 13, line 3-: You write "Necessary conditions for the series to be well-behaved are $(i) - 1 < \delta_y < 0$ and $-1 < \theta\delta_x < 0$." You have to define the notion of "well-behaved" before you find necessary conditions.

You can find the necessary and sufficient condition for stationarity of $y_t + \theta x_t$ above.

The final remark is concerned with the statement on

Page 9, line 2-: "This suggests that the researcher ... opting for a more holistic approach."

It is a fact that although the CVAR methodology has been available for more than 30 years by now including the necessary software, many users still prefer to use unit root tests on individual variables. The standard (CVAR) approach suggests finding the rank first, and test for the stationarity of the individual variables by testing that a unit vector is a "cointegrating vector". I assume that this is what the author calls a "holistic" approach, and I find that the follow up analysis of the 2 dimensional CVAR is needed to show how the problems of the univariate unit roots can be solved.