

## Almost first-degree stochastic dominance for transformations and its application in insurance strategy

*Feng Zhao, Jianwei Gao, and Yundong Gu*

### Abstract

Almost stochastic dominance is a relaxation of stochastic dominance, which allows small violations of stochastic dominance rules to avoid situations where most decision makers prefer one alternative to another but stochastic dominance cannot rank them. The authors first discuss the relations between almost first-degree stochastic dominance (AFSD) and the second-degree stochastic dominance (SSD), and demonstrate that the AFSD criterion is helpful to narrow down the SSD efficient set. Since the existing AFSD criterion is not convenient to rank transformations of random variables due to its relying heavily on cumulative distribution functions, the authors propose the AFSD criterion for transformations of random variables by means of transformation functions and the probability function of the original random variable. Moreover, they employ this method to analyze the transformations resulting from insurance and option strategy.

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**Keywords** Stochastic dominance; almost stochastic dominance; transformation; utility theory

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## **1 Introduction**

Modelling the portfolio selection criteria of rational decision makers represents a central problem in the area of finance and economic. The most common investment rules are certainly the mean–variance (MV) rule and stochastic dominance (SD) approach. However, they suffer from the following drawback that both the MV and the SD criteria may fail to determine dominance even in situations where all “reasonable” investors would clearly prefer one investment to another. For instance, consider the following example: suppose prospect portfolio  $Y$  yields a return of 2% with a probability of 0.5 and a return of 3% with a probability of 0.5. Portfolio  $X$  yields a return of 1% with a probability of 0.5 and a return of 100% with a probability of 0.5. There is no MV or SD dominance of one portfolio over the other in this case, i.e., both  $X$  and  $Y$  are in the efficient MV and SD sets. However, it is obvious that any “reasonable” investor would prefer portfolio  $X$ .

Leshno and Levy (2002) suggest that the stochastic dominance rules are unable to reveal such preferences because they account for extreme utility functions, which rarely represent preferences of investors in practice. They introduce the concept of almost stochastic dominance (ASD), which only considers preferences under non-extreme utility functions. Thus, the choice criteria of ASD focus on most decision makers rather than all of them, so it improve the discriminating power and robustness of the classical SD rules. Since Leshno and Levy’s study, many papers have been devoted to the theory and application research of ASD. Using data from the U.S. market, Bali et al. (2009) show that the ASD rules are consistent with the perception of investors that stocks are more attractive than bonds under a sufficiently long investment horizon. Levy et al. (2010) construct several experiments to show that ASD rule corresponds to sets of non-pathological preferences. Bali et al. (2011) further adopt the ASD rule to examine the practice of investing in stock market anomalies; they found that the ASD rule provides evidence for “the significance of size, short-term reversal, and momentum for short investment horizons and the significance of book-to-market and long-term reversal for longer term horizons” (p. 28). In addition, Bali et al. (2013) use the ASD rule to compare between the returns of various hedge fund strategies and the U.S. equity market. Regarding the theoretical research of ASD, Tzeng et al. (2013) revise the second-degree ASD rule to improve the correspondence with expected utility theory and develop higher-order ASD rules that impose additional preference assumptions. Guo et al. (2013) point out that Tzeng’s definition of ASD does not possess the hierarchy property. In order to resolve this problem, Tsetlin et al. (2015) further propose the generalized almost stochastic dominance. Moreover, Lizyayev and Ruszczyński (2012) propose a new almost stochastic dominance concept that is computationally tractable, and Niu and Guo (2014) develop this new ASD for higher order preferences.

Nearly all of the above literature is intend to generalize Leshno and Levy’s AFSD criterion to higher orders. In contrast to such studies, we will generalize this criterion from another angle. It is generally known that economic and financial activities usually induce transformations of an initial risk, which results in a new type of problem of how to rank transformations of random variables. In this case, the choice criteria of ASD, based on the cumulative distribution functions, are inconvenient to distinguish transformations of random variables. In consideration of the characteristic of this problem, we develop a new form of AFSD, which is expressed by the transformation functions and the probability function of the original random variable. Thus,

the new choice criterion of AFSD is more appropriate for ranking transformations of random variables. Moreover, we apply this method to analyze the transformations resulting from insurance and option strategies.

The remainder of this paper is organized as follows. Section 2 introduces the foundation knowledge of SD and ASD, and demonstrates that the AFSD criterion is helpful to narrow down the SSD efficient set. Section 3 presents the AFSD criterion for transformations of random variables. Section 4 proposes the applications of the new AFSD criterion in insurance and option strategies. Section 5 concludes the paper.

## 2 Stochastic dominance and almost stochastic dominance

This section will first introduce some preliminary knowledge of stochastic dominance and almost stochastic dominance. Let  $X$  and  $Y$  be two random variables with the cumulative distribution functions  $F(x)$  and  $G(x)$ , respectively, and support in the interval  $[a, b]^*$ . Moreover, we denote  $U_1$  as the set of all increasing utility functions and  $U_2$  as the set of all increasing and concave utility functions.

**Definition 1.** (Levy, 1992) (i)  $X$  dominates  $Y$  by first-degree stochastic dominance (FSD) if  $F(x) \leq G(x)$  for any real number  $x$ , with at least one strong inequality.

(ii)  $X$  dominates  $Y$  by second-degree stochastic dominance (SSD) if

$$\int_a^x F(t)dt \leq \int_a^x G(t)dt \text{ for any real number } x, \text{ with at least one strong inequality.}$$

The SD rules and the relevant class of preferences  $U_k$  are related in the following way:

$X$  dominates  $Y$  by FSD if and only if  $E[u(X)] \geq E[u(Y)]$  for any  $u \in U_1$  with a strong inequality for at least one  $u_0 \in U_1$ .

$X$  dominates  $Y$  by SSD if and only if  $E[u(X)] \geq E[u(Y)]$  for any  $u \in U_2$  with a strong inequality for at least one  $u_0 \in U_2$ .

Stochastic dominance has been originally developed in the traditional expected utility framework, and is widely used in finance, economics and many other areas of science (see Levy, 2006 for a review). The SD rules provide decision-making criteria for entire preference classes. For example, FSD ranks the distributions for all decision makers with increasing utility functions and SSD works for all decision makers with increasing and concave utility functions, and they attract most of the attention in the SD research.

However, the SD approach also has inherent limitations. While “most” decision makers may prefer one distribution over another, the SD rules may not reveal this preference due to some extreme utility functions in the case of even a very small violation of these rules. Then, the almost stochastic dominance has been developed as an extension of the classical stochastic dominance framework to solve such drawback.

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\*Footnote: For simplicity, we assume that the range of the random variable is finite. Actually, the stochastic dominance criteria can easily be extended to the infinite range by mathematical skills (see Hanoch and Levy, 1969).

**Definition 2.** (Leshno and Levy, 2002) For  $0 < \varepsilon < 0.5$ ,  $X$  dominates  $Y$  by  $\varepsilon$ -AFSD if and only if

$$\int_S [F(x) - G(x)] dx \leq \varepsilon \int_a^b |F(x) - G(x)| dx, \quad (1)$$

where  $S = \{x \in [a, b] \mid F(x) > G(x)\}$ .

The AFSD relation of  $X$  over  $Y$  means that  $F(x) \leq G(x)$  for most outcomes in the support interval  $[a, b]$ , except for a relatively small segment that violates the FSD condition. The parameter  $\varepsilon$  highlights the degree of FSD violation allowed. The smaller  $\varepsilon$ , the stronger this dominance. In particular, Equation (1) is satisfied if  $X$  dominates  $Y$  by FSD (i.e., no FSD violation).

Next, we introduce the set of utility functions that are accounted for under  $\varepsilon$ -AFSD:

$$U_1^*(\varepsilon) = \{u(x) \in U_1 \mid u'(x) \leq \inf_x \{u'(x)\} [\frac{1}{\varepsilon} - 1], \forall x \in [a, b]\}. \quad (2)$$

Leshno and Levy (2002) prove that  $X$  dominates  $Y$  by  $\varepsilon$ -AFSD if and only if  $E[u(X)] \geq E[u(Y)]$  for all  $u$  in  $U_1^*(\varepsilon)$ .

It is easy to see that utility functions with extreme preferences, e.g., zero and/or infinite marginal utility, are considered pathological and are eliminated from  $U_1^*(\varepsilon)$ , and only the “well-behaved” or “reasonable” non-decreasing utility functions are kept in  $U_1^*(\varepsilon)$ . Then, AFSD criterion effectively overcomes the above-mentioned defect of the SD approach that the SD rules cannot distinguish two distributions even if “most” decision makers may prefer one distribution over another.

In what follows, we will discuss the relations between FSD, SSD and AFSD.

**Proposition 1.** If  $X$  dominates  $Y$  by FSD, then for any positive number  $\varepsilon < 0.5$ , we have  $X$  dominates  $Y$  by  $\varepsilon$ -AFSD.

This conclusion directly follows from Definition 2 and it shows that the AFSD efficient set is included in the FSD efficient set. That is, compared with FSD, the AFSD criterion will narrow down the menu of investments that need to be considered.

**Proposition 2.** (i) Let  $X$  and  $Y$  be any pair of random variables with different mathematical expectations. If  $X$  dominates  $Y$  by SSD, then there exists a positive number  $\varepsilon < 0.5$  such that  $X$  dominates  $Y$  by  $\varepsilon$ -AFSD, but not vice versa.

(ii) Let  $X$  and  $Y$  be any pair of random variables with the same mathematical expectation. If  $X$  dominates  $Y$  by SSD, then for any positive number  $\varepsilon < 0.5$ , we have  $X$  non-dominates  $Y$  by  $\varepsilon$ -AFSD.

Proof. See Appendix A.

Proposition 2 describes the relations between AFSD and SSD. Whether the SSD relation will result in AFSD relation is determined by the mathematical expectations of the compared random variables. The answer will be yes if they have different mathematical expectations, otherwise will be no. Comparing the AFSD condition with the SSD condition, we can further get the AFSD relation between random variables while we determine their SSD relation, which indicates that we can further narrow down the SSD efficient sets by AFSD criterion with few additional computations.

As mentioned above, AFSD has great potential for practical applications as it eliminates unrealistic “pathological” preferences, hence it allows for a considerable reduction of the stochastic dominance efficient set, narrowing down the menu of investments that need to be considered. Attracted by its great charm, many authors have been devoted to develop the AFSD criterion. Leshno and Levy (2002) first propose the almost second-degree stochastic dominance (ASSD) and Tzeng et al. (2013) modify it to give another form. However, Guo et al. (2013) point out that the former has the hierarchy property but not the expected-utility maximization, whereas the latter has the expected-utility maximization but not the hierarchy property. Levy (2006) gives a new definition of ASSD and Tsetlin et al. (2015) indicate it is also not the expected-utility maximization. Tsetlin et al. (2015) propose the generalized almost stochastic dominance, which satisfy both the expected-utility maximization and the hierarchy property.

Unlike all the existing literature, we will generalize Leshno and Levy’s definition of AFSD from a new perspective. Considering that the AFSD criterion is not convenient to rank transformations for its relying heavily on cumulative distribution functions, we will adapt the AFSD approach to include the ranking of transformations of random variables. Due to the enormous controversy over the definitions of the second-degree and higher-degree almost stochastic dominance, and more significantly, they cannot be directly extended to the transformations case, this paper will only focus on the AFSD criterion for transformations of random variables.

### **3 AFSD criterion for transformations**

Almost all human activities, especially economic and financial activities, will induce risk transformations which are viewed as beneficial or detrimental by the decision maker. For example, the insurance policy may alter the value of the insured asset when one buys some insurance to hedge risks of real world; in financial option, the sale of the call option while holding the common stock will convert the value of the total investment. When a random variable is transformed from one form into another, the risk of the asset will also be changed correspondingly. In many practical problems, we need to rank such risk transformations and accordingly choose the beneficial one, so it is significant to find an effective approach to study risk transformations in practice.

It is well-known that stochastic dominance is one of the most famous approaches to compare pairs of prospects. Unfortunately, the SD approach is not convenient to rank transformations. Then, some authors try to adapt the SD approach to include the ranking of transformations of random variables (see Meyer, 1989; Levy, 1992; Gao and Zhao, 2017; Gao et al., 2018). Since the almost stochastic dominance can better distinguish random variables than the SD approach, we will develop such SD criteria for transformations into almost stochastic dominance criteria.

**Theorem 1.** Suppose that  $X$  is a continuous random variable with density function  $f(x)$  and support in the finite interval  $[a, b]$ . If  $m(x)$  and  $n(x)$  are both increasing (or decreasing) in  $[a, b]$ , then we have  $m(X)$  dominates  $n(X)$  by  $\varepsilon$ -AFSD if and only if

$$\int_T [n(x) - m(x)]f(x)dx \leq \varepsilon \int_a^b |m(x) - n(x)| f(x)dx, \quad (3)$$

where  $T = \{x \in [a, b] \mid m(x) < n(x)\}$ .

Proof. See Appendix B.

Theorem 1 proposes the AFSD criterion for transformations of a continuous random variable with the transformation functions and the probability density function of the original random variable, rather than the accustomed cumulative distribution functions of the transformed random variables. It indicates that if the “mathematical expectation” of  $n(X) - m(X)$  at  $T = \{x \in [a, b] \mid m(x) < n(x)\}$  makes up only a small proportion (not larger than  $\varepsilon$ ) of the whole mathematical expectation of  $|m(X) - n(X)|$ , then  $m(X)$  dominates  $n(X)$  by  $\varepsilon$ -AFSD. Compared with Definition 1, this new choice criterion of AFSD can rank transformations directly by transformation functions and then we don’t need to compute the cumulative distribution functions of the transformed random variables. So it is more convenient to rank transformations by Theorem 1.

Moreover, from the proof of Theorem 1, it is easy to find that the monotonicity of  $m(x)$  and  $n(x)$  is not needed in the “if” part. Then we will further give a sufficient condition for one transformation dominating another by AFSD.

**Corollary 1.** Suppose that  $X$  is a continuous random variable with density function  $f(x)$  and support in the finite interval  $[a, b]$ . For any two functions  $m(x)$  and  $n(x)$  defined in  $[a, b]$ , if  $\int_T [n(x) - m(x)]f(x)dx \leq \varepsilon \int_a^b |m(x) - n(x)| f(x)dx$ , then  $m(X)$  dominates  $n(X)$  by  $\varepsilon$ -AFSD.

Obviously, there is no restriction to the compared transformations, so Corollary 1 gives a general condition for determining the AFSD relation between transformations.

**Remark 1.** Meyer (1989), Levy (1992) and Gao et al. (2018) have discussed the SD criteria for transformations of continuous random variables, Theorem 1 and Corollary 1 extend such criteria to the ASD case.

It should be pointed out that the above results would only apply to transformations of continuous random variables, and cannot be used directly in the discrete case. While in real life, the discrete random variables are ubiquitous and even the continuous random variables should be discretely handled in many cases, so it is advantageous to investigate the AFSD criterion for transformations of discrete random variables.

**Theorem 2.** Suppose that  $X$  is a discrete random variable with the probability mass function  $P(X = x_i) = p_i, i = 1, 2, \dots, r$  and support in the finite interval  $[a, b]$ . If  $m(x)$  and  $n(x)$  are both increasing (or decreasing) in  $[a, b]$ , then we have  $m(X)$  dominates  $n(X)$  by  $\varepsilon$ -AFSD if and only if

$$\sum_{i \in I} [n(x_i) - m(x_i)]p_i \leq \varepsilon \sum_{i=1}^r |n(x_i) - m(x_i)| p_i, \quad (4)$$

where  $I = \{i \mid m(x_i) < n(x_i), 1 \leq i \leq r\}$ .

The proof of Theorem 2 is very similar to that of Theorem 1 and therefore is omitted.

The interpretation of Theorem 2 is just the same as that of Theorem 1. By replacing the probability density function with the probability mass function and the integral operation with summation, Theorem 2 presents the necessary and sufficient condition for  $\varepsilon$ -AFSD relation between transformation of discrete random variables.

Once again, we can get a sufficient condition for  $\varepsilon$ -AFSD under the discrete case.

**Corollary 2.** Suppose that  $X$  is a discrete random variable with the probability mass function  $P(X = x_i) = p_i, i = 1, 2, \dots, r$  and support in the finite interval  $[a, b]$ . For any two functions  $m(x)$  and  $n(x)$  defined in  $[a, b]$ , if  $\sum_{i \in I} [n(x_i) - m(x_i)] p_i \leq \varepsilon \sum_{i=1}^r |n(x_i) - m(x_i)| p_i$ , then  $m(X)$  dominates  $n(X)$  by  $\varepsilon$ -AFSD.

**Remark 2.** Gao and Zhao (2017) first propose the SD criteria for transformations of random variables, Theorem 2 and Corollary 2 further develop such SD criteria into ASD criteria. Meanwhile, Theorem 2 and Corollary 2 are necessary complements to Theorem 1 and Corollary 1 and they together constitute the AFSD criterion for transformations of random variables.

## 4 Applications in insurance strategy

We have presented the new choice criterion of AFSD for transformations of random variables in Section 3. Since the AFSD criterion can eliminate unrealistic “pathological” preferences and allow for a considerable reduction of the SD efficient set, the AFSD criterion for transformations has practical application in ranking transformations which arise in either the insurance or option market applications. This will be demonstrated by discussing two examples. The first one is concerned with different forms of insurance policies, and the second one is a call option. Transformations are used to model these risk altering instruments.

Consider the situation of owing an asset such as a house or an automobile whose value is random. This may be due to events such as natural catastrophes like wind damage of flood, or losses caused by other agents through theft or collision. Let  $X$  denote this random value and assume that its support is the interval  $[a, b]$ , where  $b$  is the value of the asset if no loss occurs. One type of insurance policy that can be purchased to alter this random variable  $X$ , is the deductible policy in which the insurance company promises to reimburse all losses in excess of some stated amount  $d$  (the deductible) in return for a payment of size  $\delta$  (the premium). Such an insurance policy is represented by transformation

$$m(x) = \begin{cases} b - d_m - \delta_m, & x < b - d_m \\ x - \delta_m, & x \geq b - d_m \end{cases}.$$

A similar insurance policy with deductible  $d_n$  and premium  $\delta_n$  defines transformation  $n(x)$ . It is certainly that  $m(x)$  and  $n(x)$  are both increasing in  $[a, b]$ . Then, how to choose the better insurance policy?

To answer this question, we first form the difference of  $m(x)$  and  $n(x)$ . If we assume that  $d_m < d_n$ , then we have

$$m(x) - n(x) = \begin{cases} d_n - d_m + \delta_n - \delta_m, & a \leq x \leq b - d_n \\ b - d_m - \delta_m - x + \delta_n, & b - d_n < x \leq b - d_m \\ \delta_n - \delta_m, & b - d_m \leq x \leq b \end{cases} \quad (5)$$

Of course, experience in choosing among policies with varying sizes for the deductible indicates that it is unlikely for the premium charged to be smaller for policies with lower deductibles. Furthermore, it is typical for the reduction in the premium to be a fraction of the increase in the deductible. Thus it is further assumed that  $\delta_m > \delta_n$ , and that  $d_n - d_m > \delta_m - \delta_n$ . Under this restriction, the difference  $m(x) - n(x)$  is first positive and constant, then declines with slope minus 1, and finally is constant and negative. It is easy to see that  $m(x) - n(x)$  is positive in  $[a, b - d_m - \delta_m + \delta_n)$  and is negative in  $(b - d_m - \delta_m + \delta_n, b]$ . From Theorem 1 in Meyer (1989) we know that there is no FSD relation between  $m(X)$  and  $n(X)$ . However, by Theorem 1 of this paper, we deduce that

$m(X)$  dominates  $n(X)$  by  $\varepsilon$ -AFSD

$$\begin{aligned} \Leftrightarrow & \int_{b-d_m-\delta_m+\delta_n}^{b-d_m} [x - (b - d_m - \delta_m + \delta_n)] f(x) dx + \int_{b-d_m}^b (\delta_m - \delta_n) f(x) dx \\ & \leq \varepsilon \int_a^b |m(x) - n(x)| f(x) dx \end{aligned} \quad (6)$$

and

$n(X)$  dominates  $m(X)$  by  $\varepsilon$ -AFSD

$$\begin{aligned} \Leftrightarrow & \int_a^{b-d_n} (d_n - d_m + \delta_n - \delta_m) f(x) dx + \int_{b-d_n}^{b-d_m-\delta_m+\delta_n} (b - d_m - \delta_m - x + \delta_n) f(x) dx \\ & \leq \varepsilon \int_a^b |m(x) - n(x)| f(x) dx, \end{aligned} \quad (7)$$

where  $f(x)$  denote the probability distribution function of  $X$ . The above expressions indicate that the better insurance policy is not only concerned with the premium and deductible, but also concerned with the probability distribution function of the random variable. For example, if these three factors satisfy (6), then we have  $m(X)$  dominates  $n(X)$  by  $\varepsilon$ -AFSD, in spite of the fact that there is no FSD relations between  $m(X)$  and  $n(X)$ . In other words, the “well-behaved” (not zero and/or infinite marginal utility) agent will choose the insurance policy  $m(X)$ .

Similarly, put and call option contracts can modify the value of common stock. These contracts provides the buyer of the option with the right to either buy (call) or sell (put) shares of common stock at a fixed price referred to as the strike price. On the other hand, the seller of such an option contract incurs the obligation to either sell or buy the common stock at the agreed upon strike price if the contract purchaser decides to exercise the option. To model one such option transaction using the transformation notation, let  $X$  represent the random value of



100 shares of a given common stock. An investor who owns the common stock can sell a call contract (100 shares) with strike price  $x_m$  for a price of  $p_m$ . This investment of selling a call option while owning the common stock can be represented by the following transformation

$$m(x) = \begin{cases} x + p_m, & x < x_m \\ x_m + p_m, & x \geq x_m \end{cases} .$$

The original value  $x$  becomes  $m(x)$  when the stock is held and the call option is sold. A similar option strategy with strike price  $x_n$  and option price  $p_n$  defines transformation  $n(x)$ . Assume that  $x_m < x_n$ , then we have

$$m(x) - n(x) = \begin{cases} p_m - p_n, & a \leq x \leq x_m \\ (x_m + p_m) - (x + p_n), & x_m \leq x \leq x_n \\ (x_m + p_m) - (x_n + p_n), & x_n < x \leq b \end{cases} . \quad (8)$$

This is exactly the same as in the deductible insurance examples above. Perform similar analysis, we further assumed that  $p_m > p_n$  and  $x_n - x_m > p_m - p_n$ . Obviously,  $m(x) - n(x)$  is positive in  $[a, x_m + p_m - p_n)$  and becomes negative in  $(x_m + p_m - p_n, b]$ . Although there is no FSD relation between  $m(X)$  and  $n(X)$ , if we apply Theorem 1 in the paper to this example, then we can conclude that

$$\begin{aligned} & m(X) \text{ dominates } n(X) \text{ by } \varepsilon\text{-AFSD} \\ \Leftrightarrow & \int_{x_m + p_m - p_n}^{x_n} (x + p_n - x_m - p_m) f(x) dx + \int_{x_n}^b (x_n + p_n - x_m - p_m) f(x) dx \\ & \leq \varepsilon \int_a^b |m(x) - n(x)| f(x) dx \end{aligned} \quad (9)$$

and

$$\begin{aligned} & n(X) \text{ dominates } m(X) \text{ by } \varepsilon\text{-AFSD} \\ \Leftrightarrow & \int_a^{x_m} (p_m - p_n) f(x) dx + \int_{x_m}^{x_m + p_m - p_n} (x_m + p_m - x - p_n) f(x) dx \\ & \leq \varepsilon \int_a^b |m(x) - n(x)| f(x) dx . \end{aligned} \quad (10)$$

These two expressions show that the “well-behaved” investor will choose option strategy  $m(X)$  if (9) is satisfied or choose  $n(X)$  if (10) is satisfied.

Recall that Gao et al. (2018) have ever used the SD approach to analyze the transformations resulting from holding a stock with the corresponding call option. We reanalyze this problem with the AFSD criteria for transformations and this new choice criterion of AFSD can narrow down the menu of option strategies that need to be considered. Besides this, we also apply the new AFSD criterion to analyze the transformations resulting from insurance.

## **5 Conclusion**

Very often economic and financial activities induce transformations of an initial risk, which results in a new type of problem of how to rank transformations of the same random variable. The SD approach is one of the most important tools for ranking distributions, and Leshno and Levy (2002) further develop the SD approach into ASD, which further improves the discrimination of the SD approach. However, the existing ASD approach is not suitable for ranking transformations of random variables. Then, based on expected utility theory, we propose the new AFSD criterion for transformations of random variables. Finally, we introduce the application of the new AFSD criterion in insurance and option strategy.

Although Tsetlin et al. (2015) extend Leshno and Levy's definition of AFSD to second-degree and higher-degree cases, it is much more difficult to adapt them to transformations of random variables. Similarly, the convex stochastic dominance, which is of great value in the field of securities investment, also cannot be directly expanded to transformations of random variables. How to extend second-degree and higher-degree ASD criterion and the convex stochastic dominance to transformations of random variables is the most important topic in our future research.

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## Appendix A

**Proof of Proposition 2.** Let  $F(x)$  and  $G(x)$  be the cumulative distribution functions of  $X$  and  $Y$ , and  $S = \{x \in [a, b] \mid F(x) > G(x)\}$  be the set of all the outcomes which violate the FSD condition that  $F(x) \leq G(x)$ . If  $X$  dominates  $Y$  by SSD, then  $\int_a^x (G(t) - F(t))dt \geq 0$  for all real numbers  $x$ .

(i) If  $EX \neq EY$ , we have  $\int_a^b (G(t) - F(t))dt = \int_S (G(t) - F(t))dt + \int_{S^c} (G(t) - F(t))dt > 0$ ,

where  $S^c$  denotes the complementary set of  $S$ . Then, it follows that

$$\int_S (F(t) - G(t))dt < \frac{1}{2} \int_a^b |G(t) - F(t)|dt.$$

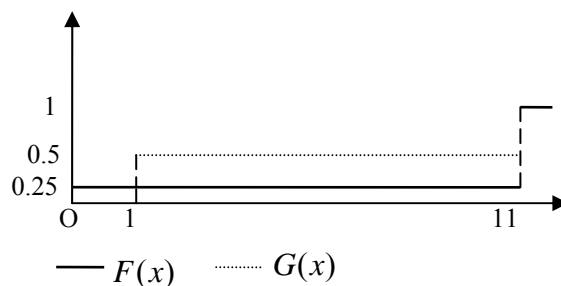
That is, there exists a positive number  $\varepsilon < 0.5$  such that

$$\int_S (F(t) - G(t))dt < \varepsilon \int_a^b |G(t) - F(t)|dt, \text{ i.e., } X \text{ dominates } Y \text{ by } \varepsilon\text{-AFSD.}$$

On the contrary, if  $X$  dominates  $Y$  by  $\varepsilon$ -AFSD, the AFSD condition (1) cannot guarantee the SSD condition that  $\int_a^x (G(t) - F(t))dt > 0$  holds for all real numbers  $x$ . For example, if

$X = \{(\frac{1}{4}, 0), (\frac{3}{4}, 1)\}$ ,  $Y = \{(\frac{1}{2}, 1), (\frac{1}{2}, 1)\}$ , it is obvious that  $X$  dominates  $Y$  by 0.1-AFSD (see Figure 1). However, since  $F(x) > G(x)$  when  $x \in (0, 1)$ , we have  $X$  non-dominates  $Y$  by SSD.

Figure 1: Area between  $F(x)$  and  $G(x)$



(ii) If  $EX = EY$ , then  $\int_a^b (G(t) - F(t))dt = 0$

and  $\int_S (F(t) - G(t))dt = \frac{1}{2} \int_a^b |G(t) - F(t)|dt$ .

That is, for any positive number  $\varepsilon < 0.5$ ,  $X$  non-dominates  $Y$  by  $\varepsilon$ -AFSD.

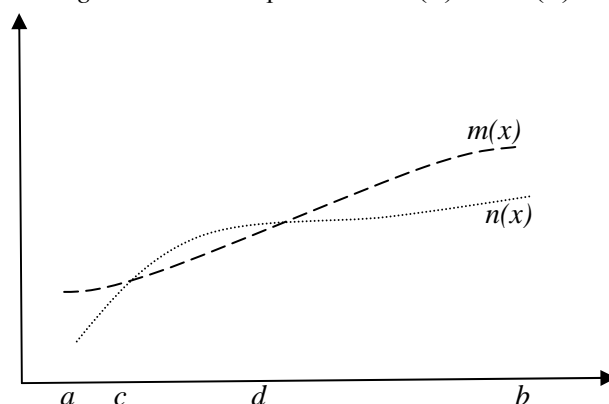
## Appendix B

**Proof of Theorem 1.** (1) “Only if” part: Assume  $m(x)$  and  $n(x)$  are both increasing in  $[a, b]$ .

We only need to prove that if  $\int_T [n(x) - m(x)]f(x)dx > \varepsilon \int_a^b |m(x) - n(x)| f(x)dx$ , then there exists a utility function  $u(x) \in U_1^*(\varepsilon)$ , such that  $E[u(m(X))] - E[u(n(X))] < 0$ .

For convenience, we suppose  $T = (c, d) \subseteq [a, b]$ . From the known conditions, we deduce that (i)  $m(x) \geq n(x)$  if  $x \in [a, c]$ ; (ii)  $m(x) < n(x)$  if  $x \in (c, d)$ ; (iii)  $m(x) \geq n(x)$  if  $x \in [d, b]$  (see Figure 2).

Figure 2: Relative position of  $m(x)$  and  $n(x)$



Define

$$u(x) = \begin{cases} l_1 x, & x \leq m(c) \\ l_2 x - (l_2 - l_1)m(c), & m(c) < x < n(d) \\ l_1 x + (l_2 - l_1)[n(d) - m(c)], & x \geq n(d) \end{cases},$$

where  $l_2 > l_1 > 0$  and  $\frac{l_1}{l_1 + l_2} = \varepsilon$ . It is obvious that  $u(x) \in U_1^*(\varepsilon)$  and

- (i) for any  $x \leq c$ , we have  $m(x) \leq m(c)$ ;
- (ii) for any  $x \in (c, d)$ , we have  $m(x) \geq m(c)$  and  $m(x) < n(x) \leq n(d)$ ;
- (iii) for any  $x \geq d$ , we have  $m(x) \geq m(d) \geq n(d)$ . Hence, we get

$$u(m(x)) = \begin{cases} l_1 m(x), & x \leq c \\ l_2 m(x) - (l_2 - l_1)m(c), & c < x < d \\ l_1 m(x) + (l_2 - l_1)[n(d) - m(c)], & x \geq d \end{cases}$$

Similarly, we have

$$u(n(x)) = \begin{cases} l_1 n(x), & x \leq c \\ l_2 n(x) - (l_2 - l_1)m(c), & c < x < d \\ l_1 n(x) + (l_2 - l_1)[n(d) - m(c)], & x \geq d \end{cases}$$

Therefore, we obtain

$$u(m(x)) - u(n(x)) = \begin{cases} l_1[m(x) - n(x)], & x \leq c \\ l_2[m(x) - n(x)], & c < x < d \\ l_1[m(x) - n(x)], & x \geq d \end{cases}$$

and

$$\begin{aligned} E(u(m(X)) - E(u(n(X))) &= \int_a^b [u(m(x)) - u(n(x))]f(x)dx \\ &= l_1 \int_a^c [m(x) - n(x)]f(x)dx + l_2 \int_c^d [m(x) - n(x)]f(x)dx + l_1 \int_d^b [m(x) - n(x)]f(x)dx \\ &= l_1 \int_a^b |m(x) - n(x)| f(x)dx - (l_1 + l_2) \int_c^d [n(x) - m(x)]f(x)dx \\ &= (l_1 + l_2) \left\{ \varepsilon \int_a^b |m(x) - n(x)| f(x)dx - \int_c^d [n(x) - m(x)] f(x)dx \right\} \\ &< 0. \end{aligned}$$

Similarly, if  $m(x)$  and  $n(x)$  are both decreasing in  $[a, b]$ , we only need to redefine

$$u(x) = \begin{cases} l_1x, & x \leq m(d) \\ l_2x - (l_2 - l_1)m(d), & m(d) < x < n(c), \\ l_1x + (l_2 - l_1)[n(c) - m(d)], & x \geq n(c) \end{cases}$$

where  $l_2 > l_1 > 0$  and  $\frac{l_1}{l_1 + l_2} = \varepsilon$ . By repeating the above process, we can also deduce that  $E(u(m(X)) - E(u(n(X))) < 0$ .

(2) “If” part: Suppose  $\int_T [n(x) - m(x)]f(x)dx \leq \varepsilon \int_a^b |m(x) - n(x)| f(x)dx$ , then for any  $u(x) \in U_1^*(\varepsilon)$ , if  $\inf_{x \in [a, b]} u'(x) = l_1$  and  $\sup_{x \in [a, b]} u'(x) = l_2$ , we get  $\varepsilon \leq \frac{l_1}{l_1 + l_2}$ . Thus, we deduce that

$$\begin{aligned} E(u(m(X)) - E(u(n(X))) &= \int_a^b [u(m(x)) - u(n(x))]f(x)dx \\ &= \int_a^b u'(\xi_x)[m(x) - n(x)]f(x)dx \quad (\text{where } \xi_x \text{ is among } m(x) \text{ and } n(x)) \\ &= \int_T u'(\xi_x)[m(x) - n(x)]f(x)dx + \int_{T^c} u'(\xi_x)[m(x) - n(x)]f(x)dx \\ &\geq l_2 \int_T [m(x) - n(x)]f(x)dx + l_1 \int_{T^c} [m(x) - n(x)]f(x)dx \\ &= l_1 \int_a^b |m(x) - n(x)| f(x)dx - (l_1 + l_2) \int_T [n(x) - m(x)]f(x)dx \\ &\geq (l_1 + l_2) \left\{ \varepsilon \int_a^b |m(x) - n(x)| f(x)dx - \int_T [n(x) - m(x)]f(x)dx \right\} \\ &\geq 0. \square \end{aligned}$$

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