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A New Approach of Stochastic Dominance for Ranking Transformations on the Discrete Random Variable

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Abstract

This paper presents some new stochastic dominance (SD) criteria for ranking transformations on a random variable, which is the first time that this is done for transformations under the discrete framework. By using the expected utility theory, the authors first propose a sufficient condition for general transformations by first degree SD (FSD), and further develop it into the necessary and sufficient condition for monotonic transformations. For the second degree SD (SSD) case, they divide the monotonic transformations into increasing and decreasing categories, and further derive their necessary and sufficient conditions, respectively. For two different discrete random variables with the same possible states, the authors obtain the sufficient and necessary conditions for FSD and SSD, respectively. The new SD criteria have the following features: each FSD condition is represented by the transformation functions and each SSD condition is characterized by the transformation functions and the probability distributions of the random variable. This is different from the classical SD approach where FSD and SSD conditions are described by cumulative distribution functions. Finally, a numerical example is provided to show the effectiveness of the new SD criteria.

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Keywords Stochastic dominance; transformation; utility theory; insurance

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1 Introduction

In real world, many human activities in insurance and financial fields induce risk transformations. For example, we assume that an investor owns a house where X denotes the value of the house (a random variable). The investor can insure the house with various levels of deductions. By choosing two different deduction policies, the investor creates different transformations $m(X)$ and $n(X)$. Then an interesting question occurs: which deduction policy (transformation) dominates the other? In other words, how to find an effective approach for ranking these transformations so as to choose the beneficial one?

Stochastic dominance (SD) is one of the most famous approaches to comparing pairs of prospects. Presented in the context of expected utility theory, the SD approach has the advantage that it requires no restrictions on probability distributions. Well-known specifications of SD are first degree SD (FSD) and second degree SD (SSD), which by far attract most of the attention in SD research. Due to the advantage mentioned above, the SD approach has been proved to be a powerful tool for ranking random variables and employed in various areas of finance, decision analysis, economics and statistics (See e.g., Meyer, 1989; Levy, 1992, 2006; Chiu, 2005; Li, 2009; Blavaskyy, 2010, 2011; Deutsch and Silber, 2011; Bibi, Duclos and Audrey, 2012; Yalonetzky, 2012; Tzeng et al., 2013; Loomes et al., 2014; Valentini, 2015; Tsetlin et al., 2015). Unfortunately, the SD approach is inefficient to rank transformations on random variables because it relies on the cumulative distribution functions (CDFs) of random variables, which are hard to calculate. In other words, the SD approach cannot be used directly to rank transformations.

To the best of the authors' knowledge, there are only two papers studying SD criteria for transformations on the continuous random variables under the conditions of increasing, continuous, and piecewise differentiable transformations (see Meyer, 1989 and Levy, 1992). However, there is little research which focuses on ranking transformations under the discrete framework. It should be pointed out that the outcomes of transformations for continuous random variables cannot be extended directly to the discrete system. In real life, we notice that the discrete random variables are ubiquitous and even the continuous random variables should be discretely handled in many cases, so it is necessary to find new SD criteria for ranking transformations on discrete random variables.

The paper aims to develop some new SD criteria for ranking transformations on the discrete random variable, which is the first time to investigate the ranking approach for the discrete system. In order to construct such theoretical paradigm, we start from the FSD rule by applying the expected utility theory, and derive a sufficient condition (See Theorem 1). We further extend the sufficient condition into a sufficient and necessary condition by restricting attention to monotonic transformations. In this case the FSD rule is determined exclusively by the difference between the compared transformation functions (See Theorem 2). For the case of SSD, we first divide the monotonic transformation functions into increasing and decreasing categories, respectively. For the increasing transformation functions, we respectively develop a sufficient condition and a sufficient and necessary condition for SSD by using the transformation functions and the probability distributions of the random variable (See Theorem 3 and Theorem 4). For decreasing transformations, we also respectively obtain a sufficient condition and a sufficient and necessary condition for SSD, which are different from those related to increasing transformations (See Theorem 5 and Theorem 6). In addition, for two different discrete random variables with the same possible states, we provide the sufficient and necessary conditions for FSD and SSD, respectively (See Theorem 7).

Compared with the classical SD rules, the new SD criteria have the following advantages: (1) the new SD criteria can directly rank transformations on a discrete random variable, while the classical SD rules do not work; (2) the new SD criteria make us avoid the tedious computation of CDFs, whereas this can not be done in the classical SD rules. In this sense, the new theoretical paradigm we derived can be regarded as a useful complement to the classical SD theory. Finally, a numerical example is provided to show the effectiveness of the new SD criteria.

The rest of this paper is organized as follows. Section 2 reviews the existing SD rules. Section 3 and Section 4 present the SD criteria of transformations by FSD and SSD, respectively. Section 5 makes a comparison between the new SD criteria and the classical SD rules. Section 6 gives a numerical example to show the efficiency of the new SD method and Section 7 draws the conclusions.

2 Preliminaries

This section introduces the definition of stochastic dominance, and the SD criteria for transformation on the continuous random variable.

Let X and Y be two random variables with support in the finite interval $[a,b]$, and their CDFs will be denoted by $F(x)$ and $G(x)$, respectively. Define $F^{(n)}(x)=\int_a^x F^{(n-1)}(x)dx$ ($n=2,3,\cdots$) with $F^{(1)}(x)=F(x)$, and define $G^{(n)}(x)$ similarly. Moreover, we denote U_n as the class containing all the functions u with $(-1)^{k+1}u^{(k)}\geq 0\ (k=1,2,\cdots,n)$, where u is a real-valued function and $u^{(k)}$ is the k-th derivative of u.

Definition 1. (Levy, 1992) (i) X dominates Y by FSD if $F(x)\leq G(x)$ for any real number x;

(ii) X dominates Y by SSD if $F^{(2)}(x)\leq G^{(2)}(x)$ for any real number x;

(iii) X dominates Y by n th degree SD ($n\geq 3$) if

$$F^{(k)}(b)\leq G^{(k)}(b) \text{ for } k=1,2,\cdots,n, \tag{1}$$

and

$$F^{(n)}(x)\leq G^{(n)}(x) \text{ for all } a\leq x\leq b. \tag{2}$$

The SD rules and the relevant class of preferences U_k are related in the following way:

X dominates Y by FSD if and only if $E[u(X)]\geq E[u(Y)]$ for any $u\in U_1$.

X dominates Y by SSD if and only if $E[u(X)]\geq E[u(Y)]$ for any $u\in U_2$.

X dominates Y by nth degree SD ($n \geq 3$) if and only if $E[u(X)] \geq E[u(Y)]$ for any $u \in U_n$ and $F^{(k)}(b) \leq G^{(k)}(b)$ for $k = 1, 2, \cdots, n$.

Integral conditions (1) and (2) mean that the SD approach relies on CDFs of the random variables, and it is inefficient to rank transformations on the random variable. To overcome this shortcoming, Meyer (1989) proposes the following results.

Lemma 1. (Meyer, 1989) Given a continuous random variable X with the density $f(x)$ and support in the interval $[a,b]$. If $m(x)$ and $n(x)$ are non-decreasing, continuous and piecewise differentiable functions, then

(i) the transformed random variable $m(X)$ dominates $n(X)$ by FSD if and only if

$$\{m(x) - n(x)\} f(x) \geq 0 \text{ for all } x \text{ in } [a,b], \tag{3}$$

(ii) the transformed random variable $m(X)$ dominates $n(X)$ by SSD if and only if

$$\int_a^x \{m(t) - n(t)\} f(t) dt \geq 0 \text{ for all } x \text{ in} [a,b]. \tag{4}$$

Lemma 1 provides the FSD and SSD criteria, which are only valid for non-decreasing, continuous and piecewise differentiable functions, and these SD criteria cannot be directly applied to ranking transformations on the discrete random variable. However, in the real world, discrete random variables are ubiquitous and even the continuous random variables should be discretely handled in many cases, so it is significant to find SD criteria for ranking transformations on discrete random variables. Considering that FSD and SSD have more practical implication than higher degree SD rules, this paper will focus on FSD and SSD rules in the remaining.

3 Dominance Conditions for FSD

Let X be a discrete random variable whose prospects are characterized by $\{p_1, x_1; \cdots, p_n, x_n\}$ with $x_1 < x_2 < \cdots < x_n$ and support in the finite interval $[a,b]$. Assume that two transformed random variables $m(X)$ and $n(X)$ are denoted as $\{p_1, m(x_1); \cdots, p_n, m(x_n)\}$ and $\{p_1, n(x_1); \cdots, p_n, n(x_n)\}$, or shortly as $\{p_1, m_1; \cdots, p_n, m_n\}$ and $\{p_1, n_1; \cdots, p_n, n_n\}$, respectively. Then, an interesting question arises: given a discrete random variable X and two transformed random variables $m(X)$ and $n(X)$, under what conditions will one transformation dominate the other for a given order?

Since Lemma 1 is only suitable for transformations on the continuous random variable, and it is invalid for the discrete random variable case, we need to develop new SD criteria for transformations on the discrete random variable.

We will first discuss the FSD conditions for ranking $m(X)$ and $n(X)$ using expected utility. That is, when is $E[u(m(X)]$ larger than or equal to $E[u(n(X)]$ for all increasing utility functions?

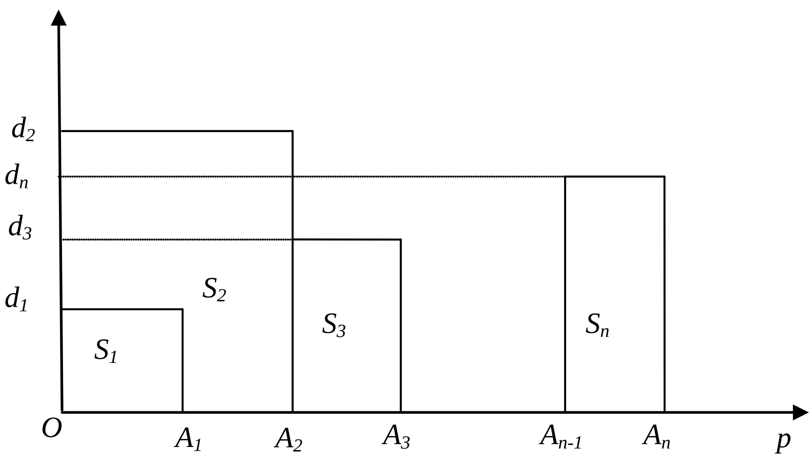
Theorem 1. The transformed random variable $m(X)$ dominates $n(X)$ by FSD if

$$m_i \geq n_i \text{ for all } i = 1, 2, \cdots, n. \tag{5}$$

Proof. See Appendix A. □

In order to better understand the meaning of Theorem 1, Figure 1 shows its graphical illustration. Differently from the classical SD rules, the areas $S_1, S_2, \cdots, S_n$ in Figure 1 are derived by the difference between $m(x)$ and $n(x)$ at x_i ($i = 1, 2, \cdots, n$) by multiplying the corresponding probability of x_i, rather than the CDFs. Here we let $d(x) = m(x) - n(x)$, then $d_i = m_i - n_i$ denotes the difference of $m(x)$ and $n(x)$ at $x_i (i = 1, 2, \cdots, n)$. Furthermore, we take n points $A_1, A_2, \cdots, A_n$ at the horizontal axis, such that $OA_1 = p_1$, $A_1A_2 = p_2$, $\cdots$, $A_{n-1}A_n = p_n$. We stipulate that a geometric area takes a positive value when it lies above the horizontal axis and takes a negative value when it lies below the horizontal axis, and then we can use the notation S_i to represent the algebra value of a rectangular area whose base and height are denoted as p_i and d_i ($i = 1, 2, \cdots, n$), respectively. From Figure 1 we notice that $m_i \geq n_i$ is equivalent to $S_i \geq 0$ ($i = 1, 2, \cdots, n$). Therefore, we can conclude that $m(X)$ dominates $n(X)$ by FSD if all the n areas $S_1, S_2, \cdots, S_n$ are all non-negative.

Figure 1: The graphical illustration of Theorem 1



Theorem 1 presents a sufficient condition for determining FSD relations which only involves the transformation function. Apparently, it is much easier to compare the transformation functions than to compare the CDFs of transformed random variables.

However, we see that condition (5) is only a sufficient condition for $m(X)$ dominating $n(X)$ by FSD, rather than a necessary and sufficient condition. A natural question is whether condition (5) is also necessary. The following example shows that the answer is negative and the rest of this section is devoted to finding the necessary and sufficient condition.

Example 1. Suppose that a random variable X yields the outcomes -1, 0 and 1 with equal probabilities $\frac{1}{3}$. If $m(x)=1-x^2$, $n(x)=x^2-1$ and $p(x)=1-x$, then their probability distributions are shown as follows (See, Table 1).

(a) Table 1 reports that the transformed random variable $m(X)$ takes 0 with probability $\frac{2}{3}$ and 1 with probability $\frac{1}{3}$. Then, from Theorem 1, we see that $m(X)$ dominates X by FSD. However, it follows from Table 1 that $m(x_3)=0<1=x_3$. This fact shows that condition (5) is not necessary when the dominating transformation $m(x)$ is non-monotonic.

(b) From Table 1 we see that the transformed random variable $n(X)$ takes -1 with probability $\frac{1}{3}$ and 0 with probability $\frac{2}{3}$. Hence, by comparing CDFs of X and $n(X)$, from Theorem 1 we can deduce that X dominates $n(X)$ by FSD. On the other hand, Table 1 shows that $x_1=-1<0=n(x_1)$, which means that condition (5) is not necessary when the dominated transformation $n(x)$ is non- monotonic.

(c) We notice that from Table 1 the transformed random variable $p(X)$ takes values $0,1$ and 2 with equal probability $\frac{1}{3}$. Then, the relative position of CDFs of X and $p(X)$ indicates that $p(X)$ dominates X by FSD, whereas we have $p(x_3)=0<1=x_3$, which indicates that

Table 1: Probability distributions of $X, m(X), n(X)$ and $p(X)$

X	-1	0	1
$m(X)$	0	1	0
$n(X)$	0	-1	0
$p(X)$	2	1	0
$\Pr(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

condition (5) is still not necessary when one transformation is increasing and the other is decreasing.

Based on the above analysis, we conclude that condition (5) is not necessary if $m(x)$ and $n(x)$ are not comonotonic. The following theorem shows that condition (5) will be sufficient and necessary when the transformation functions are comonotonic.

Theorem 2. Supposing that the transformation functions $m(x)$ and $n(x)$ are comonotonic, $m(X)$ dominates $n(X)$ by FSD if and only if $m_i \geq n_i$ for all $i = 1, 2, \cdots, n$.

Proof. See Appendix A. □

Figure 1 can also illustrate the graphical presentation of Theorem 2, that is, $m(X)$ dominating $n(X)$ by FSD is equivalent to the situation that the n areas $S_1, S_2, \cdots, S_n$ are non-negative.

Remark 1. Theorem 2 provides a sufficient and necessary condition by restricting attention to monotonic transformations, and this condition only depends on the transformation function. In this sense it presents a simple way for determining FSD relations between two comonotonic transformations. This situation is different from the case of the classical SD rules for FSD in which we need to take a tedious calculation to obtain the CDFs of the transformed random variables. Therefore, Theorem 2 plays an active part in dealing with realistic problems via the new SD criterion we derived. The assumption of monotonous condition is appropriate because it is a common feature of transformations in the fields of insurance and decision analysis (see Meyer, 1989).

4 Dominance Conditions for SSD

In this section, we try to find some dominant criteria for SSD. It is well known that SSD condition in the classical SD approach is more complicated than the case of FSD. In order to find SSD criteria for ranking transformed discrete random variables, we will divide the monotonic transformation functions into increasing and decreasing ones, respectively.

Theorem 3. If $m(x)$ is increasing and

$$\sum_{i=1}^{k} (m_i - n_i) p_i \geq 0 \text{ for all } k = 1, 2, \cdots, n, \tag{6}$$

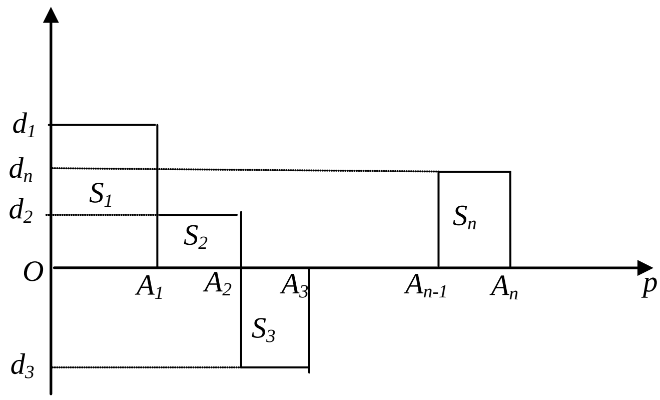
then the transformed random variable $m(X)$ dominates $n(X)$ by SSD.

Proof. See Appendix B. □

Figure 2 makes a graphical explanation about Theorem 3. We investigate these n rectangular areas from left to right in Figure 2 and find that if the sum of the first k ($k = 1, 2, \cdots, n$) rectangular areas are all non-negative, then we can conclude that $m(X)$

dominates $n(X)$ by SSD. In other words, the condition of $\sum_{i=1}^{k}(m_i - n_i)p_i \geq 0$ is equivalent to the case of $\sum_{i=1}^{k} S_i \geq 0$ ($i = 1, 2, \cdots, n$, $k = 1, 2, \cdots, n$).

Figure 2: The graphical illustration of Theorem 3



Remark 2. Theorem 3 presents a sufficient condition for one transformation dominating the other by SSD via the transformation functions and the probability distributions of the random variable, which again makes us avoid the tedious computation of CDFs, whereas this can not be done in the classical SD theory.

Similar to Theorem 1, condition (6) in Theorem 3 is only sufficient rather than necessary and sufficient. Again is natural top ask whether condition (6) is also necessary. The following example shows that the answer is no.

Example 2. We assume that a random variable X yields the outcomes -1, 0 and 1 with equal probability $\frac{1}{3}$, and that $n(x) = x^2 - 1$ and $q(x) = -2x$. Their probability distributions are listed in Table 2.

Table 2: Probability distributions of $X, n(X)$ and $q(X)$

X	-1	0	1
$n(X)$	0	-1	0
$q(X)$	2	0	-2
$\Pr(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

By analyzing the data in Table 2, we can make the following statements.

(a) X dominates $n(X)$ by SSD via the hierarchical property of the SD rules. However, $[x_1 - n(x_1)]p_1 = -\frac{1}{3} < 0$, which implies that condition (6) is not necessary if the dominating transformation is increasing and the dominated transformation is non-monotonic.

(b) X dominates $q(X)$ by SSD, while $[x_1 - q(x_1)]p_1 = -1 < 0$. This means that the condition (6) is not necessary if the dominating transformation is increasing and the dominated transformation is decreasing.

(c) $\sum_{i=1}^{k}[q(x_i) - x_i]p_i \geq 0 (k = 1, 2, 3)$. However, step (b) tells us that X dominates $q(X)$ by SSD, rather than $q(X)$ dominates X. This implies that the condition (6) could not be sufficient if the dominating transformation is not increasing.

Based on the above statements (a)-(c), we conclude that if either of the two transformations is not increasing, the condition (6) will not be sufficient and necessary. This situation indicates that we should concentrate on the case of two increasing transformations.

Theorem 4. Suppose that $m(x)$ and $n(x)$ are both increasing. Then, $m(X)$ dominates $n(X)$ by SSD if and only if $\sum_{i=1}^{k}(m_i - n_i)p_i \geq 0$ for all $k = 1, 2, \cdots, n$.

Proof. See Appendix B. □

Theorem 4 can also be illustrated by Figure 2. Recall that in Figure 2 the SSD conditions depend on the n rectangular areas, we investigate these rectangular areas from left to right and find that if the cumulative sum of the first $k (k = 1, 2, \cdots, n)$ rectangular areas is non-negative, then $m(X)$ dominates $n(X)$ by SSD; If the cumulative sum of the first $k (k = 1, 2, \cdots, n)$ rectangular areas is non-positive, then $n(X)$ dominates $m(X)$ by SSD; Otherwise, there is no SSD relation between $m(X)$ and $n(X)$.

Compared with Theorem 2, Theorem 4 reduces the requirement of the n rectangular areas. That is, any of the n rectangular areas except for the first one may take negative value, but the cumulative sum of the first $k (k = 1, 2, \cdots, n)$ rectangular areas (from left to right) must be non-negative. It also means that if $m(X)$ dominates $n(X)$ by FSD, then $m(X)$ dominates $n(X)$ by SSD, which is in accordance with the hierarchical property of SD rules.

Remark 3. Similarly to Theorem 3, Theorem 4 shows that we can determine the SSD relations between two transformed random variables by the transformation functions and the probability function of the original random variable, rather than by CDFs of the transformed random variables. More precisely, the signs of the tail conditional expectations $E[m(X) - n(X) \mid X \leq x_k] (k = 1, 2, \cdots, n)$ determine the SSD relations between the two transformed random variables. Compared with the classical SSD rule, CDFs of random variables are absent in Theorem 4, which allows us avoid the tedious computation of CDFs. As a result, Theorem 4 provides us a simple way to determine the SSD relations only by means of the transformation functions and the probability function of the random variable.

Recall that when $m(x)$ and $n(x)$ are comonotonic, there exists a unified necessary and sufficient condition for FSD case (see Theorem 2). However, this statement is not valid for SSD. That is, we need to seek a necessary and sufficient condition for SSD if $m(x)$ and $n(x)$ are both decreasing. We first provide a sufficient condition as follows.

Theorem 5. If $m(x)$ is decreasing and

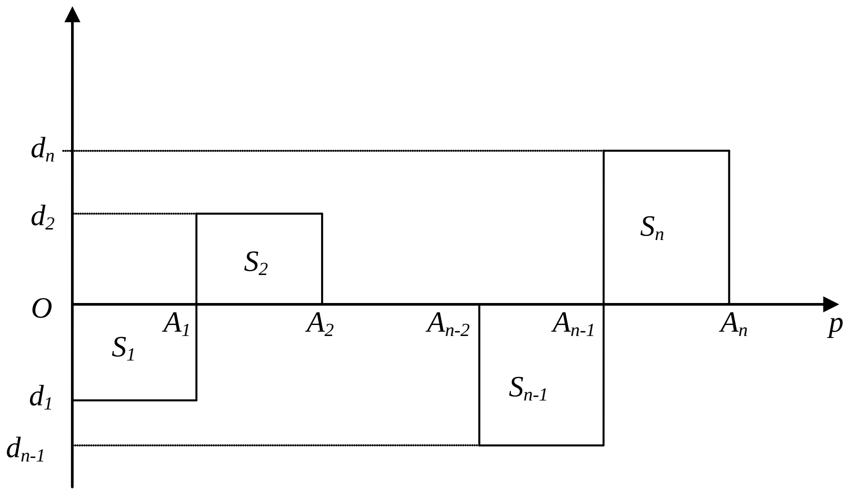
$$\sum_{i=k}^{n}(m_i - n_i)p_i \geq 0 \text{ for all } k = 1, 2, \cdots, n, \tag{7}$$

then the transformed random variable $m(X)$ dominates $n(X)$ by SSD.

Proof. See Appendix B. □

In order to better interpret the meaning of Theorem 5, Figure 3 gives the graphical illustration of Theorem 5. We investigate these n rectangular areas from right to left (just the opposite directions as what we do with Figure 2), and obtain that if the cumulative sum of the first $k(k = 1, 2, \cdots, n)$ rectangular areas is non-negative, then $m(X)$ dominates $n(X)$ by SSD. In other words, the condition of $\sum_{i=k}^{n}(m_i - n_i)p_i \geq 0$ can be expressed by $\sum_{i=k}^{n} S_i \geq 0$ ($k = 1, 2, \cdots, n$).

Figure 3: The graphical illustration of Theorem 5



Remark 4. Compared with Theorem 3, a new condition (7) in Theorem 5 is introduced to substitute condition (6) in Theorem 3. This new formulation is suitable for discussing the SSD relations of the decreasing transformations.

Theorem 6. Suppose that $m(x)$ and $n(x)$ are both decreasing. Then, $m(X)$ dominates $n(X)$ by SSD if and only if $\sum_{i=k}^{n}(m_i - n_i)p_i \geq 0$ for all $k = 1, 2, \cdots, n$.

Proof. See Appendix B. □

The graphical illustration of Theorem 6 can also be expressed by Figure 3. We investigate these n rectangular areas in Figure 3 from right to left. If the cumulative sum of the first $k(k = 1, 2, \cdots, n)$ rectangular areas is non-negative (resp. non-positive), then $m(X)$ dominates $n(X)$ (resp. $n(X)$ dominates $m(X)$) by SSD. Otherwise, there is no SSD relation between $m(X)$ and $n(X)$.

In addition, from Figure 1, Figure 2 and Figure 3, we find that Theorem 6 and Theorem 4 both reduce the condition requirement of Theorem 2 in which all the n rectangular areas are non-negative, that is, Theorem 6 and Theorem 4 argue that these n rectangular areas can take negative values, but the precondition is that the cumulative sum of the first $k(k = 1, 2, \cdots, n)$ rectangular areas should be non-negative. The difference between Theorem 6 and Theorem 4 is that Theorem 6 sums up the rectangular areas from right to left, while Theorem 4 does it from left to right. Undoubtedly, Theorem 6 and Theorem 4 are homogeneous in essence except that the former deals with increasing transformations while the latter copes with decreasing ones.

5 Comparison of the New SD Criteria and the Classical SD Rules

Up to now, we have constructed some new SD criteria to rank transformations on a discrete random variable. We must point out that the stochastic dominance for transformations is not a new concept, but a necessary and useful supplement to the classical stochastic dominance. More specifically, the stochastic dominance for transformations is used to rank the transformed random variables, while the classical stochastic dominance can be used to compare the risk of any two random variables, but the classical stochastic dominance is ineffective when dealing with transformations on the same random variable. Since Meyer's SD criteria can only rank transformations of the continuously distributed random variables, the new SD criteria developed in Section 3 and Section 4 overcome the weakness of the existing SD approach.

To better understand these two types of SD rules, Table 3 shows their main differences from three aspects: application scope, expression approach and the specific criteria.

Remark 5. It should be pointed out that as a partial order relation, the SD approach is not useful to rank all the random variables. However, as a screening device, the SD approach can divide the whole decision making set into an efficient subset and an inefficient one, and then the decision maker choose within the efficient former (See, e.g., Li 2009; Blavaskyy, 2010, 2011; Tzeng et al.,2013; Loomes et al., 2014; Tsetlin et al., 2015). Such statements are also suitable to our new SD criteria.

Table 3: Comparison of the New SD Criteria and the Classical SD Rules

	The new SD criteria	The classical SD rules
Application scope	two transformed random variables $m(X)$ and $n(X)$	two ordinary random variables X and Y
Expression approach	transformation functions $m(x), n(x)$ and probability function $p(x)$ of X	difference of CDFs of X and Y
FSD rule	$m(x_i) \geq n(x_i)$ for all $i = 1, 2, \cdots, n$ ($m(x)$ and $n(x)$ are comonotonic)	$F(x) \leq G(x)$ for all real numbers x
SSD rule	(1) $\sum_{i=1}^{k}(m_i - n_i)p_i \geq 0$ for all $k = 1, 2, \cdots, n$ ($m(x)$ and $n(x)$ are both increasing) (2) $\sum_{i=k}^{n}(m_i - n_i)p_i \geq 0$ for all $k = 1, 2, \cdots, n$ ($m(x)$ and $n(x)$ are both decreasing)	$F^{(2)}(x) \leq G^{(2)}(x)$ for all real numbers x

We further study the intrinsic links between the two types of SD rules in the following. Note that the two transformations $m(X)$ and $n(X)$ on a random variable can also be regarded as two special random variables, then there should exist the corresponding random variables X and Y generating similar SD rules as $m(X)$ and $n(X)$ do. Given two comonotonic transformation functions $m(X)$ and $n(X)$, if we apply these functions to a discrete random variable, we get the same pair of random variable $X = \{p_1, x_1; \cdots, p_n, x_n\}$ with $x_1 < x_2 < \cdots < x_n$ and $Y = \{p_1, y_1; \cdots, p_n, y_n\}$ with $y_1 < y_2 < \cdots < y_n$. In contrast, if the above two discrete random variables have the same support, they can be regarded as two increasing transformation functions applied to a discrete random variable. Following Theorem 2 and Theorem 4, we draw the following conclusion:

Theorem 7. If the two discrete random variables $X = \{p_1, x_1; \cdots, p_n, x_n\}$ with $x_1 < x_2 < \cdots < x_n$ and $Y = \{p_1, y_1; \cdots, p_n, y_n\}$ with $y_1 < y_2 < \cdots < y_n$ have the same possible future states, then we have:

(1) X dominates Y by FSD if and only if $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$;

(2) X dominates Y by SSD if and only if $\sum_{i=1}^{k}(x_i - y_i)p_i \geq 0$ for all $k = 1, 2, \cdots, n$.

Theorem 7 shows that the new SD criteria can rank not only the transformations on a discrete random variable, but also different random variables with the same possible states.

Compared with the FSD and SSD algorithms provided by Levy (2006) for equal-probability distributed and discrete random variables, Theorem 7 further extends these two algorithms to more general discrete random variables on condition that they have the same possible states.

Numerical Example

This section provides a numerical example to illustrate how to determine the SD relations for transformations by using the new SD criteria and further determine the efficient set of decision making.

With the accelerating trend of population aging, the pension fund gap of China is becoming increasingly wide. For example, the World Bank stated that the size of China's 2001 to 2075 pension fund gap is 9.15 trillion Yuan (Wang et al. 2014). To effectively control the pension fund gap, one of the important approaches is to increase the investment return of the pension fund. However, in real world, the government pension sector of China can not directly invest the pension fund, so it authorizes several institutional investors to invest. Then, there is a principal-agent relationship between the government pension sector of China and each institutional investor. For a given return rate on investment, different institutional investors may provide different revenue-sharing proposals except for the common commission for the agency. As mentioned, the new SD criteria can divide the whole choice set into an efficient subset and an inefficient one.

Owing to the uncertainty of the stock market, there is an enormous risk for investing money in the stock market. Let X denote the rate of return on investing money in stock market and its probability distribution is shown as in Table 4. Assume that there are four institutional investors who can provide revenue-sharing proposals denoted by X, $m(X)$, $n(X)$ and $r(X)$, respectively. Table 5 shows the probability distributions of the four different revenue-sharing proposals.

Table 4: The probability distribution of the rate of return on investment X

X	−50%	−10%	5%	20%	50%
$\Pr(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

We assume that revenue-sharing proposals $m(X)$, $n(X)$ and $r(X)$ are as follows:

$$m(x)=\begin{cases}\frac{2}{5}x, x\le 0\\ \frac{4}{5}x, x>0\end{cases},\ n(x)=\begin{cases}\frac{4}{5}x, x\le 0\\ x, \quad x>0\end{cases},\ \text{and}\ r(x)=\frac{3}{5}x+5\%.$$

According to $m(X)$, the institutional investor A will allocate four fifths of investment return rate to the government pension sector of China if the return rate is positive, and he will allocate two fifths of the the return rate to the government pension sector if the return rate is negative. Similarly, according to $n(X)$, the institutional investor B will distribute the total return rate to the government pension sector if the return rate is positive (meaning he only takes the commission in this situation), otherwise, he will distribute four fifths to the government pension sector if the return rate is negative. According to $r(X)$, the institutional investor C will allocate three fifths of the return rate plus five percent (regarded as risk-free interest rate) to the government pension sector. In addition, the implication of the revenue-sharing proposal X is that the institutional investor D will assign the total return rate to the government pension sector and he only takes the commission.

Table 5: Probability distributions of revenue-sharing proposals $X, m(X), n(X), r(X)$

X	−50%	−10%	5%	20%	50%
$m(X)$	−20%	−4%	4%	16%	40%
$n(X)$	−40%	−8%	5%	20%	50%
$r(X)$	−25%	−1%	8%	17%	35%
$\Pr(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

To find the efficient set and the inefficient set, we need to make pairwise comparisons. Note that the four revenue-sharing proposal functions are all increasing, so we can determine the FSD and SSD relations between them by Theorem 2 and Theorem 4, respectively. We now perform some comparisons ((a1) to (a3) below) to determine the FSD relations and some other comparisons ((b1) to (b2) below) to determine the SSD relations.

(a1) The revenue-sharing proposal X does not dominate any other revenue-sharing proposals by FSD because $x_1 = -50\%$ is far less than the values of m_1, n_1 and r_1; The proposal $m(X)$ does not dominate X by FSD because $m_4 < x_4$; The proposal $n(X)$ dominates X by FSD because $n_i \geq x_i$ holds for $i = 1, 2, \cdots, 5$. Therefore, X should be ruled out from the efficient set of the revenue-sharing proposals, and only $m(X), n(X)$ and $r(X)$ are kept in the efficient set.

(a2) There is no FSD relation between $m(X)$ and $n(X)$ because $m_1 > n_1$ while $m_5 < n_5$. Similarly, there is no FSD relation between $m(X)$ and $r(X)$ for $m_1 > r_1$ while $m_3 < r_3$. So, no revenue-sharing proposals in this step will be deleted from the efficient set.

(a3) There is no FSD relation between $n(X)$ and $r(X)$ because $n_1 < r_1$ but $n_5 > r_5$. Therefore, in this step, there will be no revenue-sharing proposals removed from the efficient set.

(b1) In this step, we will determine the SSD relations between $m(X)$ and $n(X)$.

Since $(m_1 - n_1)p_1 = 0.025$, $(m_2 - n_2)p_2 = 0.01$, $(m_3 - n_3)p_3 = -0.0025$, $(m_4 - n_4)p_4 = -0.01$ and $(m_5 - n_5)p_5 = -0.0125$, it is easy to verify that $\sum_{i=1}^{k}(m_i - n_i)p_i \geq 0$ for all $k = 1, 2, \cdots, 5$. Therefore, according to Theorem 4, we can conclude that $m(X)$ dominates $n(X)$ by SSD. That is, $n(X)$ should be ruled out from the efficient set, and only $m(X)$ and $r(X)$ are kept in the efficient set.

(b2) In this step, we will determine the SSD relations between $m(X)$ and $r(X)$.

On the one hand, from Table 5 we obtain that $(m_1 - r_1)p_1 = 0.00625$, $(m_2 - r_2)p_2 = -0.0075$ and $(m_1 - r_1)p_1 + (m_2 - r_2)p_2 = -0.00125 < 0$, so $m(X)$ does not dominate $r(X)$ by SSD. On the other hand, $r_1 < m_1$ implies that $r(X)$ does not dominate $m(X)$ by SSD.

Therefore, according to Theorem 2 and Theorem 4, we can determine the FSD relations and SSD relations among the four revenue-sharing proposals. The above computational procedure shows that $n(X)$ dominates X by FSD, and $m(X)$ dominates $n(X)$ by SSD, so X and $n(X)$ are ruled out from the efficient set. That is, $m(X)$ and $r(X)$ are kept in the efficient set, and X and $n(X)$ are included in the inefficient set.

6 Conclusion

Very often insurance activities induce transformations of an initial risk, which results in a new problem of how to rank transformations on the same random variable. Although the SD approach has been proved to be a powerful tool for ranking random variables, it is inefficient in ranking transformations for it relies heavily on the cumulative distribution functions.

In this paper we have developed some new FSD and SSD criteria for ranking transformations on a discrete random variable, and this is the first time that this is done for the discrete case. With these new SD criteria, we can determine the SD relations between transformations directly by the transformation functions and the probability distribution of the original random variable. Since there is no need to tediously compute the transformations of CDFs and their integral, the new SD criteria are more efficient to rank transformations than the classical SD rules. In this sense, the newly established theoretical paradigm, so far as ranking transformations on the discrete random variable, can be viewed as a good substitute for the cirteria based on the cumulative distribution functions in the classical SD approach. Moreover, by using the new SD criteria, we provide new FSD and SSD algorithms for any two discrete random variables with the same support, which extend Levy's FSD and SSD algorithms for equal-probability distributed random variables to a more general case.

The new SD criteria we developed only compare transformations of the same random variable. It would be useful to extend the analysis to the ranking of transformations on more than one random variables and to consider higher-degree SD rules for transformations.

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Appendix A

Proof of Theorem 1. For any utility function $u(x) \in U_1$, we have

$$E[u(m(X)] - E[u(n(X)] = \sum_{i=1}^{n}[u(m_i) - u(n_i)]p_i = \sum_{i=1}^{n} u'(\xi_i)(m_i - n_i)p_i \geq 0, \quad \text{(A1)}$$

where ξ_i is among m_i and n_i for all $1 \leq i \leq n$.□

Proof of Theorem 2. The sufficiency is obvious from Theorem 1. We then only need to prove the necessity.

Suppose that $m(x)$ and $n(x)$ are both increasing, we will use the reduction to absurdity to prove this conclusion. If there exists a number *j* such that $1 \leq j \leq n$ and $m_j < n_j$, then we can define the utility function with the following form:

$$u(x) = \begin{cases} m_j, x < m_j \\ x, \quad m_j \leq x \leq n_j \\ n_j, x > n_j \end{cases} . \quad \text{(A2)}$$

From the function (A2), it is easy to see that $u(x) \in U_1$, $u(m_j) = m_j$, $u(n_j) = n_j$ and

(a) if $i < j$, then $u(m_i) = m_j$ and $u(m_i) \leq u(n_i)$ since m_j is the minimum of $u(x)$;

(b) if $i > j$, then from the monotonous property of $u(x)$ and $n(x)$, we get $u(n_i) = n_j$ and $u(m_i) \leq u(n_i)$ since n_j is the maximum of $u(x)$.

Therefore, from the analyses of (a) and (b), we can conclude that $u(m_i) \leq u(n_i)$ for all $1 \leq i \leq n$ and $i \neq j$. Combining it with the assumption $m_j < n_j$, we derive that

$$E[u(m(X)] - E[u(n(X)] = \sum_{i=1}^{n}[u(m_i) - u(n_i)]p_i$$

$$= \sum_{i=1}^{j-1}[u(m_i) - u(n_i)]p_i + [u(m_j) - u(n_j)]p_j + \sum_{i=j+1}^{n}[u(m_i) - u(n_i)]p_i$$

$$\leq [u(m_j) - u(n_j)]p_j = (m_j - n_j)p_j < 0, \quad \text{(A3)}$$

which is a contradiction with the assumption of $m(X)$ dominating $n(X)$ by FSD.

If $m(x)$ and $n(x)$ are both decreasing, it is obvious that $-m(x)$ and $-n(x)$ are both increasing. Since X dominates Y by FSD if and only if $-Y$ dominates $-X$ by FSD (see Shaked and Shanthikumar, 2007), we have

$m(X)$ dominates $n(X)$ by FSD

$\Leftrightarrow -n(X)$ dominates $-m(X)$ by FSD

$\Leftrightarrow -n(x_i) \geq -m(x_i)$ for all $i = 1, 2, \cdots, n$

$\Leftrightarrow m(x_i) \geq n(x_i)$ for all $i = 1, 2, \cdots, n$.□

Appendix B

Proof of Theorem 3. If $m_i \geq n_i$ holds for all $i = 1, 2, \cdots, n$, then by Theorem 1, we can derive that $m(X)$ dominates $n(X)$ by FSD. Hence, $m(X)$ dominates $n(X)$ by SSD via the hierarchical property of SD rules. Otherwise, let

$\Phi = \{i \mid 1 \leq i \leq n, \textit{and } m_i < n_i\} = \{j_1, j_2, \cdots, j_r\}$ denote the set of all the subscripts which violate the condition (1), where $1 \leq j_1 < j_2 < \cdots < j_r \leq n$.

According to the condition (6), we have $\sum_{i=1}^{j_1}(m_i - n_i)p_i \geq 0$ and $\sum_{i=1}^{j_1-1}(m_i - n_i)p_i \geq (n_{j_1} - m_{j_1})p_{j_1}$. Then, there exist j_r numbers $p_{11}, p_{21}, \cdots, p_{j_r 1}$, such that

$$0 \leq p_{i1} \leq p_i (1 \leq i < j_1),\ p_{i1} = 0 (j_1 \leq i \leq j_r),\ \text{(B1)}$$

and

$$\sum_{i=1}^{j_1-1}(m_i - n_i)p_{i1} = (n_{j_1} - m_{j_1})p_{j_1}\ . \tag{B2}$$

Similarly, from $\sum_{i=1}^{j_2}(m_i - n_i)p_i \geq 0$, we get

$$\sum_{i=1}^{j_1-1}(m_i - n_i)p_i + (m_{j_1} - n_{j_1})p_{j_1} + \sum_{i=j_1+1}^{j_2-1}(m_i - n_i)p_i \geq (n_{j_2} - m_{j_2})p_{j_2}\ . \tag{B3}$$

Substituting Equations (B1) and (B2) into (B3), we obtain

$$\sum_{i=1}^{j_1-1}(m_i - n_i)(p_i - p_{i1}) + \sum_{i=j_1+1}^{j_2-1}(m_i - n_i)(p_i - p_{i1}) \geq (n_{j_2} - m_{j_2})p_{j_2}\ .$$

Therefore, there exist j_r numbers $p_{12}, p_{22}, \cdots, p_{j_r 2}$ such that

$$0 \leq p_{i2} \leq p_i - p_{i1} (1 \leq i < j_2),\ p_{j_1 2} = 0,\ p_{i2} = 0 (j_2 \leq i \leq j_r), \tag{B4}$$

and

$$\sum_{i=1}^{j_2-1}(m_i - n_i)p_{i2} = (n_{j_2} - m_{j_2})p_{j_2}\ . \tag{B5}$$

Repeating this process for r times, we can conclude that there exist j_r numbers $p_{1r}, p_{2r}, \cdots, p_{j_r r}$ such that

$$0 \le p_{ir} \le p_i - p_{i(r-1)} (1 \le i < j_r) \,,\; p_{j_{r-1}r} = p_{j_r r} = 0 \,, \tag{B6}$$

and

$$\sum_{i=1}^{j_r - 1} (m_i - n_i) p_{ir} = (n_{j_r} - m_{j_r}) p_{j_r} \,. \tag{B7}$$

For any utility function $u(x) \in U_2$, by using the differential mean value theorem, we have

$$\begin{aligned} & E[u(m(X)] - E[u(n(X)] \\ & = \sum_{i=1}^{n} [u(m_i) - u(n_i)] p_i = \sum_{i=1}^{n} u'(\xi_i)(m_i - n_i) p_i \\ & = \sum_{i=1}^{j_r} u'(\xi_i)(m_i - n_i) p_i + \sum_{i=j_r+1}^{n} u'(\xi_i)(m_i - n_i) p_i \,, \end{aligned} \tag{B8}$$

Where ξ_i is among m_i and n_i for all $1 \le i \le n$.

Obviously, $u'(\xi_i) \ge 0$ and $(m_i - n_i) p_i \ge 0$ for all $j_r + 1 \le i \le n$. Then, we have

$$\begin{aligned} & E[u(m(X)] - E[u(n(X)] \ge \sum_{i=1}^{j_r} u'(\xi_i)(m_i - n_i) p_i \\ & = \sum_{i=1}^{j_r - 1} u'(\xi_i)(m_i - n_i) p_i + u'(\xi_{j_r})(m_{j_r} - n_{j_r}) p_{j_r} \\ & = \sum_{i=1}^{j_r - 1} u'(\xi_i)(m_i - n_i) p_i - u'(\xi_{j_r}) \sum_{i=1}^{j_r - 1} (m_i - n_i) p_{ir} \,. \end{aligned} \tag{B9}$$

For any $1 \le i \le j_r - 1$, it follows from the increasing property of $m(x)$ and $n(x)$ that $m_i \le m_{j_r - 1}$ and $n_i \le n_{j_r - 1}$. Hence, we can derive that $\xi_i \le \max\{m_{j_r - 1}, n_{j_r - 1}\} = m_{j_r - 1} \le m_{j_r}$. Due to the decreasing property of $u'(x)$ and $m_{j_r} \le \xi_{j_r}$, we can conclude that

$$E[u(m(X)] - E[u(n(X)] \ge [u'(m_{j_r}) - u'(\xi_{j_r})] \sum_{i=1}^{j_r - 1} (m_i - n_i)(p_i - p_{ir}) \ge 0 \,.\square \tag{B10}$$

Proof of Theorem 4. The sufficiency of this theorem can be immediately obtained from Theorem 3. We then only need to prove the necessity.

Suppose that the condition of $\sum_{i=1}^{k}(m_i - n_i)p_i \geq 0$ (for all $k = 1, 2, \cdots, n$) is invalid, we then let S denote the set of all subscripts violating this condition, i.e., $S = \{k \mid \sum_{i=1}^{k}(m_i - n_i)p_i < 0, 1 \leq k \leq n\}$. Let r be the minimum of S, we then have $m_r < n_r$. Define $u(x) = \begin{cases} x, & x \leq n_r \\ n_r, & x > n_r \end{cases}$, we find that $u(x) \in U_2$. According to the definition of $u(x)$ and the monotonicity of $m(x)$, $n(x)$ and $u(x)$, we conclude that

(a) if $1 \leq i \leq r$, we get $u(m_i) = m_i$, $u(n_i) = n_i$;

(b) if $r < i \leq n$, we have $u(m_i) \leq u(n_i)$ for the reason that $u(n_i) = n_r$ and n_r is the maximum of $u(x)$.

So,

$$E[u(m(X)] - E[u(n(X)] = \sum_{i=1}^{n}[u(m_i) - u(n_i)]p_i$$

$$= \sum_{i=1}^{r}[u(m_i) - u(n_i)]p_i + \sum_{i=r+1}^{n}[u(m_i) - u(n_i)]p_i$$

$$\leq \sum_{i=1}^{r}(m_i - n_i)p_i < 0,$$

which is a contradiction with $m(X)$ dominating $n(X)$ by SSD.□

Proof of Theorem 5. If $m_i \geq n_i$ holds for all $i = 1, 2, \cdots, n$, then by Theorem 1, we have that $m(X)$ dominates $n(X)$ by FSD. We further conclude that $m(X)$ dominates $n(X)$ by SSD via the hierarchical property of SD rules. Otherwise, let $\Psi = \{i \mid 1 \leq i \leq n, \textit{and}\ m_i < n_i\} = \{l_1, l_2, \cdots, l_r\}$ denote the set of all the subscripts which violate condition (7), where $1 \leq l_r < \cdots < l_2 < l_1 \leq n$.

According to condition (7), we have $\sum_{i=l_1}^{n}(m_i - n_i)p_i \geq 0$ and $\sum_{i=l_1+1}^{n}(m_i - n_i)p_i \geq (n_{l_1} - m_{l_1})p_{l_1}$. Then there exist $n - l_r$ numbers $p_{11}, \cdots, p_{(n-l_r)1}$, such that $0 \leq p_{i1} \leq p_i\,(l_1 < i \leq n)$, $p_{i1} = 0\,(l_r < i \leq l_1)$ and

$$\sum_{i=l_1+1}^{n}(m_i - n_i)p_{i1} = (n_{l_1} - m_{l_1})p_{l_1}. \tag{B11}$$

Similarly, from $\sum_{i=l_2}^{n}(m_i - n_i)p_i \geq 0$, we get

$$\sum_{i=l_2+1}^{l_1-1}(m_i - n_i)p_i + (m_{l_1} - n_{l_1})p_{l_1} + \sum_{i=l_1+1}^{n}(m_i - n_i)p_i \geq (n_{l_2} - m_{l_2})p_{l_2},$$

or

$$\sum_{i=l_2+1}^{l_1-1}(m_i - n_i)(p_i - p_{i1}) + \sum_{i=l_1+1}^{n}(m_i - n_i)(p_i - p_{i1}) \geq (n_{l_2} - m_{l_2})p_{l_2}.$$

Hence, there exist $n - l_r$ numbers $p_{12}, p_{22}, \cdots, p_{(n-l_r)2}$, such that $0 \leq p_{i2} \leq p_i - p_{i1}(l_2 < i \leq n)$, $p_{l_1 2} = 0$, $p_{i2} = 0(l_r < i \leq l_2)$ and

$$\sum_{i=l_2+1}^{n}(m_i - n_i)p_{i2} = (n_{j_2} - m_{j_2})p_{j_2}. \tag{B12}$$

After repeating this process for r times, we can draw the conclusion that there exist $n - l_r$ numbers $p_{1r}, p_{2r}, \cdots, p_{(n-l_r)r}$ such that $0 \leq p_{ir} \leq p_i - p_{i(r-1)}(l_r < i \leq n)$, $p_{l_{r-1}r} = 0$ and

$$\sum_{i=l_r+1}^{n}(m_i - n_i)p_{ir} = (n_{l_r} - m_{l_r})p_{l_r}. \tag{B13}$$

For any utility function $u(x) \in U_2$, by using the differential mean value theorem, we have

$$E[u(m(X)] - E[u(n(X)] = \sum_{i=1}^{n}[u(m_i) - u(n_i)]p_i$$

$$= \sum_{i=1}^{n}u'(\xi_i)(m_i - n_i)p_i \ (\xi_i \text{ is among } m_i \text{ and } n_i)$$

$$= \sum_{i=1}^{l_r-1}u'(\xi_i)(m_i - n_i)p_i + \sum_{i=l_r}^{n}u'(\xi_i)(m_i - n_i)p_i. \tag{B14}$$

It is obvious that $u'(\xi_i) \geq 0$ and $(m_i - n_i)p_i \geq 0$ for all $1 \leq i \leq l_r - 1$. Then

$$E[u(m(X)] - E[u(n(X)] \geq \sum_{i=l_r}^{n}u'(\xi_i)(m_i - n_i)p_i$$

$$= \sum_{i=l_r+1}^{n}u'(\xi_i)(m_i - n_i)p_i + u'(\xi_{l_r})(m_{l_r} - n_{l_r})p_{l_r}$$

$$= \sum_{i=l_r+1}^{n} u'(\xi_i)(m_i - n_i) p_i - \sum_{i=l_r+1}^{n} u'(\xi_{l_r})(m_i - n_i) p_{ir} . \quad \text{(B15)}$$

Since $u'(x)$ is decreasing, $\xi_i \le \max\{m_{l_r+1}, n_{l_r+1}\} = m_{l_r+1} \le m_{l_r}$ for any $l_r + 1 \le i \le n$ and $m_{l_r} \le \xi_{l_r}$, we get that

$$E[u(m(X)] - E[u(n(X)] \ge [u'(m_{l_r}) - u'(\xi_{l_r})] \sum_{i=l_r+1}^{n} (m_i - n_i)(p_i - p_{ir}) \ge 0 .\square \quad \text{(B16)}$$

Proof of Theorem 6. The sufficiency of this theorem can be immediately obtained from Theorem 5. We then only need to prove the necessity.

Suppose that the condition of $\sum_{i=k}^{n}(m_i - n_i) p_i \ge 0$ (for $k = 1, 2, \cdots, n$) is invalid, we then let T denote the set of all subscripts violating this condition, i.e., $T = \{k \mid \sum_{i=k}^{n}(m_i - n_i) p_i < 0, 1 \le k \le n\}$. In addition, we denote l as the maximum of T. Let $u(x) = \begin{cases} x, & x < n_l \\ n_l, & x \ge n_l \end{cases}$, we then have $u(x) \in U_2$. According to the definition of $u(x)$ and the monotonicity of $m(x), n(x)$ and $u(x)$, we conclude that

(a) if $1 \le i < l$, we get $u(n_i) = n_l$ and $u(m_i) \le u(n_i)$ since n_l is the maximum of $u(x)$;

(b) if $l \le i \le n$, we have $u(m_i) = m_i$, $u(n_i) = n_i$.

So,

$$E[u(m(X)] - E[u(n(X)] = \sum_{i=1}^{n} [u(m_i) - u(n_i)] p_i$$

$$= \sum_{i=1}^{l-1} [u(m_i) - u(n_i)] p_i + \sum_{i=l}^{n} [u(m_i) - u(n_i)] p_i$$

$$\le \sum_{i=l}^{n} (m_i - n_i) p_i < 0 ,$$

which is a contradiction with the assumption that $m(X)$ dominates $n(X)$ by SSD.□

References

Bibi S., Duclos J.Y., and Audrey V.C. (2012). Assessing Absolute and Relative Pro-Poor Growth, with an Application to Selected African Countries. *Economics: The Open-Access, Open-Assessment E-Journal*, 6: 1−43. http://dx.doi.org/10.5018/economics-ejournal.ja.2012-7

Blavaskyy, P.R. (2010). Modifying the mean-variance approach to avoid violations of stochastic dominance. *Management Science* 56(11): 2050−2057. http://dx.doi.org/10.1287/mnsc.1100.1224

Blavaskyy, P.R. (2011). A model of probabilistic choice satisfying first-order stochastic dominance. *Management Science* 57(3): 542−548. http://dx.doi.org/10.1287/mnsc.1100.1285

Chiu, W.H. (2005). Degree of downside risk aversion and self-protection. *Insurance: Mathematics and Economics* 36(1): 93−101. http://dx.doi.org/10.1016/j.insmatheco.2004.10.005

Deutsch J. and Silber J. (2011). On Various Ways of Measuring Pro-Poor Growth. *Economics: The Open-Access, Open-Assessment E-Journal* 5: 1−57. http://dx.doi.org/10.5018/economics-ejournal.ja.2011-13

Levy, H. (1992). Stochastic dominance and expected utility: Survey and analysis. *Management Science* 38(4): 555−593. http://www.jstor.org/stable/2632436

Levy, H. (2006). Stochastic dominance: Investment decision making under uncertainty 2nd edition. Kluwer Academic, Boston. http://www.springer.com/in/book/9781441939838

Li, J. (2009). Comparative higher-degree Ross risk aversion. *Insurance: Mathematics and Economics* 45(3): 333−336. http://dx.doi.org/10.1016/j.insmatheco.2009.07.012

Loomes, G., Inmaculada, R.P., and Pinto-Prades, J.L. (2014). Comment on "A model of probabilistic choice satisfying first-order stochastic dominance". *Management Science* 60(5): 1346−1350. http://dx.doi.org/10.1287/mnsc.2013.1810

Meyer, J. (1989). Stochastic dominance and transformations of random variables. In Thomas B. Fomby and Tae Kun Seo (Eds.), *Studies in the economics of uncertainty*. In Honor of Josef Hadar, Springer Verlag, New York.

Shaked, M., and Shanthikumar, G. (2007) Stochastic orders. In: Springer Series in Statistics. http://www.springer.com/us/book/9780387329154

Tzeng, L.Y., Huang, R.J., and Shih, P.T. (2013). Revisiting almost second-degree stochastic dominance. *Management Science* 59(5): 1250−1254. http://dx.doi.org/10.1287/mnsc.1120.1616

Tsetlin, I., Winkler, R.L., Huang, R.J., and Tzeng L.Y. (2015). Generalized almost stochastic dominance. Operations Research 63(2): 363−377. http://dx.doi.org/10.1287/opre.2014.1340

Valentini E. (2015). Indirect Taxation, Public Pricing and Price Cap Regulation: A Synthesis. *Economics: The Open-Access, Open-Assessment E-Journal*, 9: 1−39. http://dx.doi.org/10.5018/economics-ejournal.ja.2015-2

Wang, L., Daniel, B., and Zhang, S. (2014). Pension financing in China: Is there a looming crisis? *China Economic Review* 30:143−154. http://dx.doi.org/10.1016/j.chieco.2014.05.014

Yalonetzky G. (2012). Measuring Group Disadvantage with Inter-distributional Inequality Indices: A Critical Review and Some Amendments to Existing Indices. *Economics: The Open-Access, Open-Assessment E-Journal*, 6: 1−32.
http://dx.doi.org/10.5018/economics-ejournal.ja.2012-9

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