# Macroeconomic Relaxation: Adjustment Processes of Hierarchical Economic Structures

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**Abstract**

We show how time-dependent macroeconomic response follows from microeconomic dynamics using linear response theory and a time-correlation formalism. This theory provides a straightforward approach to time-dependent macroeconomic model construction that preserves the heterogeneity and complex dynamics of microeconomic agents. We illustrate this approach by examining the relationship between output and demand as mediated by changes in unemployment, or Okun's law. We also demonstrate that time dependence implies overshooting and how this formalism leads to a natural definition of economic friction.

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# 1   Introduction

In recent years a new stochastic-based macroeconomics has emerged that provides a comprehensive alternative approach to our understanding of economic policy issues[1]. Of particular importance, and unlike traditional macroeconomics, this approach deals directly with economic fluctuations and with the inherently heterogeneous nature of economic participants. It developed as a response to the direction taken by microfoundations research since it's resurgence (as seen by some) following the Lucas critique[2]: a direction characterized by the use of the representative agent (despite it's well-known shortcomings[3]) and the concomitant lack of a stochastic treatment of heterogeneous interacting agents within the real economy.[4] Drawing as it does on statistical physics, this new approach implicitly provides a framework for the understanding of how the dynamics of macroeconomic observables follow from changes in economic microstructure without resorting to the notion of the representative agent and the purpose of this paper is to demonstrate that this is so. This is of great importance for the large number of economic systems in which hierarchical structure has been found. As we shall see, hierarchical economic structure, a consequence of the heterogeneity of economic agents, has profound implications for the time dependence of macroeconomic adjustment processes. Consequently, aggregation of microeconomic dynamics into macroeconomic response that preserves hierarchical structure at the microeconomic level is crucial for economic policy design.

Macroeconomic adjustment, or relaxation, is the time-dependent modification of economic relationships often expressed as an elasticity. Elasticity is manifest in every linear relationship between macroeconomic variables. Implicit in this ubiquitous notion, however, is instantaneous response: the linear relationship holds at all times. This is, however, well known to be at odds with experience as restructuring of the economy at the microeconomic level is often required for full realization of a macroeconomic observable. Our approach to the introduction of time-dependence into macroeconomics is based on the observation that the formal assumptions underlying time-dependent elasticity in macroeconomics are identical to the assumptions underlying the formal treatment of a variety of relaxation processes in condensed-matter physics including magnetic, dielectric, and anelastic relaxation. All these physical phenomena involve time-dependent relaxation toward newly established equilibria that follow from a change in a driving force and can be described in terms of linear response theory[5]. Since these physical phenomena share a common mathematical description of relaxation/response we make the *ansatz* that macroeconomic phenomena sharing these underlying assumptions will also share this common math-

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[1]The primary references to this development are Aoki (1996, 2000) and Aoki and Yoshikawa (2007).

[2]See Lucas (1976).

[3]See, for example, Kirman (1992), Hartley (1996) and Blundell and Stoker (2005).

[4]A similar response to representative-agent based macroeconomics is the closely related literature of agent-based computational economics and finance which also uses stochastic heterogeneous agents to generate macro observables. For a discussion of the current state of agent-based computational macroeconomics see Tesfatsion (2006), Axtel (2006) and references therein: particularly Tesfatsion and Judd (2006). A similar view of financial markets can be found in LeBaron (2006) and Samanidou et al. (2007).

[5]See, for example, Dattagupta (1987).

ematical description[6]. Furthermore, relaxation is an external manifestation known to reflect the adjustment of internal variables to new equilibrium values and it is through this mechanism that microeconomics and macroeconomics can be linked[7].

We begin in Sec. 2 by considering the equilibrium relationship between output and demand, the time-dependent manner in which changes in demand are manifest in output and the assumptions that these observations entail. We derive the dynamics of output in Sec. 2.2 as a set of response functions consistent with these assumptions. To show how these response functions are consistent with dynamics at the microeconomic level, we introduce the notion of internal economic variables (in this case unemployment) in Sec. 2.3 and demonstrate that a simple exponential response of output to a demand shock can be expressed as the result of a time-dependent change in the unemployment rate. Okun's law - the relationship between output and unemployment - arises naturally in this derivation. We generalize this link between macroeconomic response and microeconomic dynamics to include heterogeneous agents in Sec. 2.4 where, through linear response theory, we find the macroeconomic solution to a microeconomic problem is reduced to the calculation of the correlation function for the macroeconomic variable. We develop this notion for the unemployment model introduced in Aoki and Yoshikawa (2005, 2007) which links the dynamics of output to the solution of a master equation for a hierarchical unemployment state space which is known to give rise to a rich collection of response functions. Response functions of this type are related to the concept of friction and in Sec. 3 we show how the time-dependent restructuring of unemployment gives rise to economic friction. We close with a discussion and summary in Sec. 4.

## 2 Output Dynamics

### 2.1 Econometric and State Space Models

Fundamental to essentially all discussions of output and demand is the notion that there exists an equilibrium relationship between output $Y$ and demand $D$ that is of the form

$$\bar{Y} = J\bar{D} \, , \tag{1}$$

where the bar indicates equilibrium. This relationship is characterized by three features: (i) a unique equilibrium output for each level of demand, (ii) instantaneous achievement of the equilibrium response and (iii) linearity of the response. We note in passing that the equilibrium output is completely recoverable.

Empirically, equilibrium response is not achieved instantaneously; a lagged response is commonly observed. To incorporate this observed lag, previous research has augmented (1) with *ad hoc* "partial adjustment" econometric models of the form

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[6]This mathematical description comes from statistical mechanics. As discussed by Bouchaud and Potters (2003), Voit (2005) and Aoki and Yoshikawa (2007), techniques from statistical mechanics have been applied with much success in financial and securities markets.

[7]Indeed it is on this point that we extend our phenomenological theory of administered-rate dynamics (Hawkins and Arnold 2000) to the formal model of macroeconomic dynamics presented herein.

$$Y(t_n) = \sum_{i=0}^{N} \left[ a_i Y(t_{n-i}) + b_i D(t_{n-i}) \right] \ . \tag{2}$$

While these and related vector autoregression models often adequately describe observed macroeconomic dynamics, they largely lack a theoretical basis with which to interpret the resulting parameters and with which to link the model to policy.

Alternatively, we can use state space models in which one or more state variables are introduced. Later, in our example, we will introduce the employment rate as such a variable. These state variables do not respond instantaneously and allow us to generalize the ideal (i.e. instantaneous) response as expressed, for example, in (1) to allow for time-dependent response. Like previous treatments of output-demand dynamics it assumes the existence of a unique equilibrium relationship between demand and output.

## 2.2   Phenomenology

The dynamics of response and output are obtained by noting that the assumption of linearity implies a general relationship of the form

$$\left( a_0 + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} + \cdots \right) Y = \left( b_0 + b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2} + \cdots \right) D \ . \tag{3}$$

While the econometric application of this equation, like (2), requires an analysis of the number of terms needed to describe the observed dynamics, the use of (3) enables a straightforward economic interpretation of these terms and the coefficients. In practice a wide range of adjustment dynamics have been found to be described well by the comparatively simple differential relationship

$$\frac{dY}{dt} + \eta Y = J_U \frac{dD}{dt} + \eta J_R D \ , \tag{4}$$

where $\eta$ denotes the rate at which output adjusts to the equilibrium level, $J_U$ denotes that fraction of the response that occurs instantaneously, and $J_R$ denotes the ultimate extent of the response function $[= J(t = \infty)]$. The change in output with respect to time is, in this case, a function of the current output, the current demand, and the change in demand with respect to time.

Some intuition for the interpretation of this relationship between output and demand can be obtained for the case of a simple demand shock cycle. Given a sudden step change in demand, that is subsequently held constant at $D$, and the equilibrium relationship given by (1), (4) can be integrated to yield the time-dependent output

$$Y(t) = \left( J_U + \delta J \left[ 1 - e^{-\eta t} \right] \right) D \ , \tag{5}$$

where $\delta J \equiv [J_R - J_U]$, whence

$$J(t) = J_U + \delta J \left[ 1 - e^{-\eta t} \right] \ ; \tag{6}$$

illustrating the decomposition of the response $J(t)$ into an instantaneous contribution $J_U$ and a time-dependent portion (proportional to $\delta J$) mentioned above. The response of output to a step change in demand is illustrated in the upper panel of Fig. 1 where we show the response of output to a unit step change in demand where 67% of the response is instantaneous ($J_U$=0.67), 33% of the response is time dependent ($\delta J = 0.33$) and the response time $1/\eta$ is 1. Output tracks the demand change instantaneously over a range defined by $J_U$; in this case to 67%. Output then adjusts to equilibrium with demand. When the demand shock is released we see the initial elastic decrease of output followed by a time-dependent adjustment calculated using Boltzmann superposition:

$$Y(t) = \sum_{i=1}^{M} J(t - t_i)D(t_i) \ . \tag{7}$$

Varying $J_U$ and $J_R$ (or, equivalently $\delta J$) one can span the range of response from completely instantaneous, $J_U = J_R > 0$, to completely time dependent, $J_U = 0$.[8]

While (4) does describe many observed adjustment processes, deviations from this expression have also been observed and to describe these processes two popular approaches have emerged. First, one can expand (4) to include the higher order derivatives in (3) which results in the response function

$$J(t) = J_U + \sum_{i=1}^{N} \delta J_i \left[ 1 - e^{-\eta_i t} \right] \ , \tag{8}$$

where $N$ represents the highest order of derivative included in (3). With this expansion the single relaxation time for the system is replaced by a collection, or spectrum, of relaxation times reflecting more complex relaxation dynamics, and it is a relatively straightforward matter to represent empirically observed response functions. The second approach has been to replace the exponential response function that appears in (6) with a more general form. Of the response functions that have been used to

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[8]The undershooting response $J(t)$ that follows from a demand shock implies an overshooting response that follows from an output shock. With instantaneous response we can write $Y = JD$ and $D = MY$ and the response functions $J$ and $M$ are related by $M = 1/J$. When there is time-dependence this relationship changes and $M(t)$ is not equal to $1/J(t)$. Solving (4) for the case of a step output shock $Y$ yields

$$D(t) = \left( M_R + \delta M e^{-\eta t} \right) Y \ ,$$

where $\delta M \equiv [M_U - M_R]$, and

$$M(t) = M_R + \delta M e^{-\eta t} \ ,$$

where $M_U = 1/J_U$ and $M_R = 1/J_R$. As illustrated in the lower panel of Fig. 1 a step change in output results in an immediate change in demand that overshoots the equilibrium level followed by an adjustment back to that level.
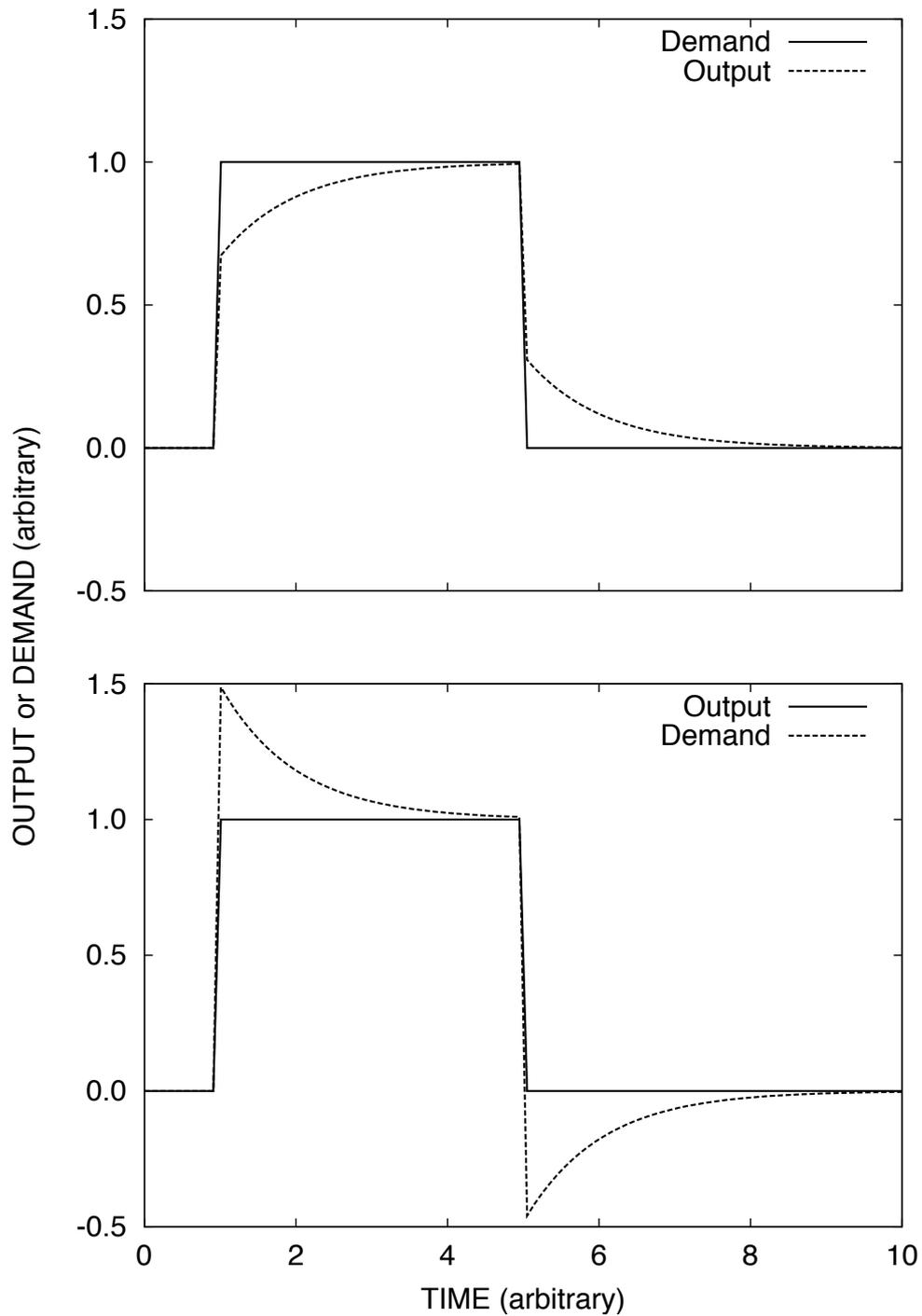
Figure 1: The response of output to a step change in demand in the upper panel and the response of demand to a step change in output in the lower panel.

represent complex systems, the stretched exponential function or Kohlrausch law[9]

$$J(t) = J_U + \delta J \left[1 - e^{-(\eta t)^\alpha}\right] \ , \ 0 \leq \alpha \leq 1 \ , \tag{9}$$

has been exceptionally successful across a wide range of systems[10] and is, as we shall see below, consistent with the hierarchical dynamics of unemployment.

While (6), (8) and (9) are remarkably successful in representing the response of many complex systems, the microeconomic origin of the response lies in a description of microeconomic change in response to demand and it is to this issue in general, illustrated by the dynamics of employment in particular, that we now turn.

## 2.3 State Space Model

That macroeconomic adjustment is a manifestation of changes in state variables can be seen by considering the case of a single state variable $\xi$ which we will take to be the employment rate. Since this is a linear theory, both $D$ and $\xi$ are treated as independent and appear to first degree. The output is a linear function of demand $D$ and the state variable $\xi$.

$$Y(D, \xi) = J_U D + \kappa \xi \ , \tag{10}$$

where $\kappa$ measures the coupling between the employment rate $\xi$ and output $Y$. In the second term of this equation we see the common linear representation of Okun's law[11]. We also recall that there is a unique equilibrium output corresponding to demand. Consequently, there exists an equilibrium value of $\xi$ (denoted by $\bar{\xi}$) for each value of demand and since $\bar{\xi} = 0$ for $D = 0$ we have that

$$\bar{\xi} = \mu D \ , \tag{11}$$

since, for our step shock in demand, $D = \bar{D}$. Finally, in response to a change in demand the state variable $\xi$ approaches equilibrium in a time-dependent manner

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[9]See, for example, Kohlrausch (1863) or Dattagupta (1987). This response function was discovered independently for dielectric relaxation process by Williams and Watts (1970). Consequently, the stretched exponential is often referred to in the literature as the Kohlrausch-Williams-Watts function.

[10]While the Kohlrausch law presumes $0 \leq \alpha \leq 1$, it is possible to have $\alpha > 1$ as discussed by Bouchaud (2008) and references therein. As the dates of Bouchaud (2008) and Kohlrausch (1863) indicate, the theory of anomalous relaxation has been and remains a topic of active research across a variety of disciplines.

[11]While the existence of a stable relationship between output and unemployment was first noted by Okun (1962), a sound microeconomic basis for Okun's law that preserved the heterogeneity of economic agents emerged only recently as a result of the introduction of the notion of hierarchical structure into economics. For a recent discussion of Okun's law see Knotek, II (2007) and references therein; particularly Moosa (1997), Lee (2000) and Schnabel (2002) that deal with the international robustness of this relationship. A discussion of hierarchical structure in economics can be found in Aoki (1996, 2000) and Aoki and Yoshikawa (2005, 2007); the implications of ultrametric hierarchical dynamics for unemployment being covered in the latter two works.

involving first-order kinetics[12], given by

$$\frac{d\xi}{dt} = -\eta \left( \xi - \bar{\xi} \right) \ . \tag{12}$$

Equations (10), (11) and (12) are a state-space model of the demand-output relation. Solving them for a step shock in demand, (10) becomes

$$Y(D, \xi) = J_U D + \kappa\mu \left( 1 - e^{-\eta t} \right) D \ , \tag{13}$$

where, as with (5), we see the output response to demand is the sum of an instantaneous elastic component (the first term) and a time-dependent, or anelastic, component (the second term). The anelastic component arises from the kinetic nature of the response of employment to demand and, thus, reflects a time-dependent form of Okun's law[13]. In this way we see how the coefficients of the phenomenological theory discussed in Sec. 2.2 can be related directly to the dynamical representation of the state variables (e.g. $\delta J = \kappa\mu$) and, thus, provide a direct link between microeconomic phenomena and macroeconomic observables.

## 2.4    Fluctuations and Response

While our discussion so far has shown how changes in internal variables within an economy result in relaxation at the macroeconomic level, a complete treatment of the heterogeneity of economic agents and a deeper understanding of the microeconomic origin of macroeconomic relaxation can be had through the use of linear response theory[14]. In this approach, (7) and (8) generalize to

$$Y(D, \xi) = \int_{-\infty}^{t} J(t - \tau) \frac{dD(\tau)}{d\tau} d\tau \ , \tag{14}$$

and the response is expressed in terms of the time-correlation function as[15]

$$J(t) = \beta \left[ \langle Y(0)^2 \rangle - \langle Y(t)Y(0) \rangle \right] \ , \tag{15}$$

where $\langle \ \rangle$ is the equilibrium average in the absence of a demand shock and $\beta^{-1}$ is the normalized economic temperature. This generalization of (7) and (8) shows that

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[12]The Kohlrausch stretched exponential relaxation can be obtained in a similar manner by replacing $\eta$ in (12) with the time-dependent relaxation rate $\eta(t) = \beta\eta(t\eta)^{\beta-1}$ as discussed in Kohlrausch (1863) and Dattagupta (1987).

[13] This can be generalized in a straightforward manner (cf. pp. 117-120 of Nowick and Berry (1972)) to include multiple channels linking demand and output: $Y(D, \xi) = J_U D + \sum_{p=1}^{n} \kappa_p \xi_p$. In this case, however, the dynamics are coupled and (12) generalizes to $d\xi_p/dt = -\sum_{q=1}^{n} \eta_{pq} \left( \xi_q - \bar{\xi}_q \right)$. With a suitably chosen linear transformation, however, one can obtain decoupled variables $d\xi_p/dt = -\eta'_p \left( \xi_p - \bar{\xi}_p \right)$, and an anelastic output response represented by the second term on the right-hand side of (13) generalizes to $\sum_{p=1}^{n} \kappa'_p \mu'_p \left[ 1 - \exp\left( -\eta'_p t \right) \right] D$ . Our discussion in this footnote and Sec 2.3 follows a related presentation in Nowick and Berry (1972: 115-117) closely.

[14]The generalization of the single internal variable discussed in footnote 13, is a step in this direction but is a less general approach than that of linear response theory.

[15]General discussions of linear response theory and the time correlation formalism appears in Kubo (1966), Agarwal (1972), Lucarini (2008) and references therein.

the macroeconomic solution of a particular microeconomic problem reduces to the evaluation of the autocorrelation

$$\langle Y(0)Y(t) \rangle = \int \int Y_0 Y_1 \; p(Y_1, t|Y_0, 0) \; p(Y_0) \; dY_0 dY_1 \; , \tag{16}$$

where $p(Y_1, t|Y_0, 0)$ is the conditional probability of transitioning from level of output $Y_0$ to level of output $Y_1$ during the time interval $t$. Given an economy with the initial distribution of output $p(Y_0)$, we have reduced this problem to the evaluation of the conditional probability $p(Y_1, t|Y_0, 0)$. Furthermore, given a relationship between output, $Y$, and the productivity, $n$, of a given sector of the economy, $\mathbf{r}$, as discussed in Aoki and Yoshikawa (2007), one can recast this conditional probability as $p(n_t, \mathbf{r}_1, t|n_0, \mathbf{r}_0, 0)$, representing the the conditional probability that during the time interval $t$ a person moves from the state $|n_0, \mathbf{r}_0)$ to the state $|n_t, \mathbf{r}_1)$ as discussed below in the Appendix, and it is to the structure of these dynamics that we now turn.

## 2.5    Hierarchical Dynamics

The response of economies to demand appears to be hierarchical in general and ultrametric in particular[16]. The basis for a hierarchical representation of unemployment dynamics begins with the economy as composed of heterogeneous sectors and people. Sectors are differentiated, as mentioned above, by factors such as geographic location, technology and educational qualifications. People similarly differ in factors such as job experience and human capital. If we view the economy as consisting of the sectors described above and use ultrametric distance to measure the distance between these sectors, the dynamics of unemployment can be seen as the random hiring or firing by a sector of a person from a pool of unemployed composed of different sectors weighted by the ultrametric distance. In response to an increase in demand the probability of being hired will differ across people and this probability is a function of the ultrametric distance between sectors: the transition probability depends on this distance. These sectors form a tree structure and the autocorrelation of output is a function of the product of the labor productivity of a sector and the probabilistic size of a given sector.

A sense of the hierarchical structure in employment dynamics is seen in Fig. 2 where we see the minimum spanning tree (upper panel) and ultrametric hierarchical tree (lower panel) for a comparatively simple case of employment-level changes in the Unites States. Minimum spanning and ultrametric hierarchical trees were introduced by Aoki (1993, 1994, 1996) to the study of economic dynamics and by Mantegna (1998, 1999) to the study of financial market dynamics. Subsequent research using

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[16]Dynamics on hierarchical spaces in general and ultrametric spaces in particular has been studied extensively in the condensed-matter physics and complexity literature (Palmer et al. 1984; Grossmann et al. 1985; Huberman and Kerszberg 1985; Ogielski and Stein 1985; Paladin et al. 1985; Schreckenberg 1985; Blumen et al. 1986; Kumar and Shenoy 1986a,b; Bachas and Huberman 1986, 1987; Hoffmann and Sibani 1988; Uhlig et al. 1995: and references therein). This work, introduced into economics by Aoki (1993, 1994) and Yang (1994), is discussed in Aoki and Yoshikawa (2007) and references therein.

this approach revealed hierarchical structure in all securities markets[17]. The ubiquity of hierarchical structure revealed through the use of tree methods inspired our use in the present study[18].

The employment data used to construct Fig. 2 were monthly employment totals and unemployment rates for each of the United States Data from the Current Population Survey (CPS) conducted by the Bureau of Census for the Bureau of Labor Statistics. The data used spanned the time period from January of 1995 to July of 2008. With these observations we constructed a simple proxy for the employment rate[19] in sector $\mathbf{r}$ with level of productivity $n$ as the level[20] of employment in a particular state with each state denoted by $\mathbf{r}$ (e.g. California) and from this a time-series of changes in employment level for each of the United States.

Examination of the minimum spanning tree in the upper panel of Fig. 2 reveals some expected regional clustering. In the upper-left section of this tree, for example, we see a cluster associated with New England. In general, however, clusters are not strictly associated with geographical location. This is somewhat easier to see in the hierarchical tree shown in the lower panel of Fig. 2. In the lower portion of the tree a number of expected regional clusters are readily apparent such as the AR-IL-KS-MO-OK midwest cluster, the FL-GA-MS southeast cluster, the AZ-NV-CO-UT southwest cluster and the MA-NJ-CT-NH-NY northeast cluster. The basis for other clustering, however, is less spatially apparent such as SD and ND linking with ME and VT, or AL and TN linking with CA and TX. These imply employment dynamics that transcend geography, that suggest the ultrametric interpretation of these dynamics advanced in prior work[11] and that require the introduction of hierarchical structure into the time-evolution of the probability to which we now turn.

The first-order kinetics that we have discussed previously is governed by an

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[17]Hierarchical structure has been found in equity markets by Mantegna (1998, 1999), Bonanno et al. (2001, 2003, 2004), Onnela et al. (2002, 2003a), Onnela et al. (2003b) and Miccichè et al. (2003), in equity-index markets by Bonanno et al. (2000), in fixed-income markets by Bernaschi et al. (2002) and Di Matteo et al. (2004), in foreign-exchange markets by McDonald et al. (2005, 2008) and Naylor et al. (2007) and in macroeconomics by Aoki (1993, 1994, 1996, 2000), Yang (1994) and Aoki and Yoshikawa (2005, 2007)

[18]The construction of these trees is straightforward. Given a collection of time series one first calculates the Pearson product-moment correlation matrix with elements $\rho_{ij}$. This is transformed into a distance matrix with elements $d_{ij} = \sqrt{2(1 - \rho_{ij})}$ that, unlike the correlation matrix, satisfy the three axioms of a metric distance: (i) $d_{ij} = 0$ if and only if $i = j$, (ii) $d_{ij} = d_{ji}$ and (iii) $d_{ij} \leq d_{ik} + d_{kj}$. From the distance matrix the minimum spanning tree can be calculated using the `vegan` package of the R programming environment; the hierarchical tree was calculated using the single-linkage clustering option of the R routine `hclust`. The distances $d_{ij}^<$ in the hierarchical tree shown in the lower panel of Fig. 2 are elements of the subdominant ultrametric distance matrix defined by replacing the third axiom of metric distance given above with the ultrametric inequality $d_{ij}^< \leq \max\{d_{ik}^<, d_{kj}^<\}$.

[19]c.f. the variable $X_n(\mathbf{r}, t)$ in the Appendix.

[20]Note that the level of employment in this example is not the employment rate for a given state, rather, it is the number of employed persons in a given state divided by the total population (employed plus unemployed) across all states. Note also that differences in the productivity variable $n$ were not considered.
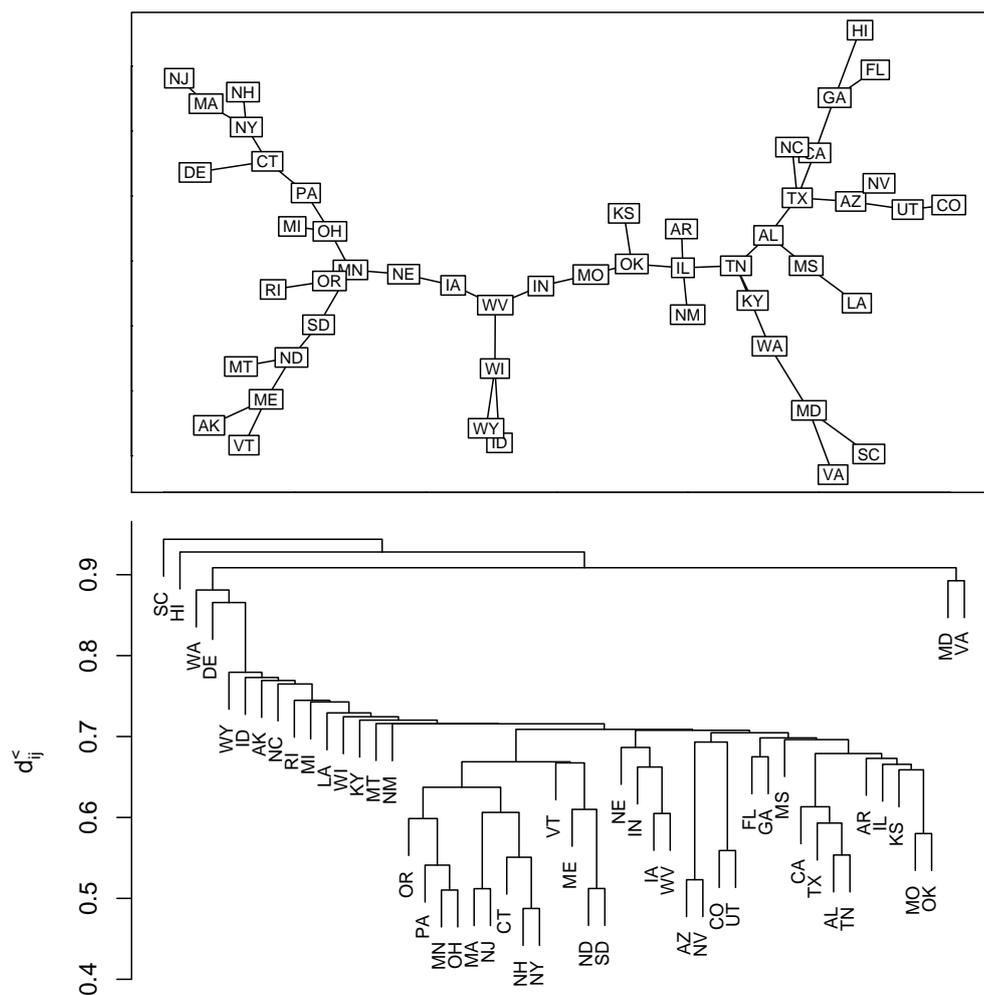
Figure 2: The minimum spanning tree (upper panel) and the hierarchical tree (lower panel) for changes in employment levels.

equation of the general form

$$\frac{dp(x_i, t)}{dt} = \sum_{j=1}^{N} \epsilon_{ij} p(x_j) \; , \tag{17}$$

where $p(x_i)$ is the probability of finding a person in state $x_i$ ($i = 1, \ldots, N$) and $\epsilon_{ij}$ are the transition probabilities per unit time from state $x_i$ to state $x_j$. There are comparatively few *a priori* restrictions on the transition probabilities beyond positivity ($\epsilon_{ij} \geq 0$ for $i \neq j$), that total probability be conserved ($\sum_{i=1}^{N} \epsilon_{ij} = 0$) and detailed balance ($\epsilon_{ij} p_{eq}(x_j) = \epsilon_{ji} p_{eq}(x_i)$ where $p_{eq}(x_i)$ is the equilibrium probability). Constraints that have been found to yield commonly observed response functions are hierarchical models through which a number of mathematically tractable and nontrivial response functions including exponential, Kohlrausch and algebraic have been derived. A particularly popular form of hierarchical structure is ultrametricity, or the constraint that

$$\epsilon_{ij} \geq \min\left(\epsilon_{ik}, \epsilon_{jk}\right) \; . \tag{18}$$

This imposes a tree-like structure on the space, transforming it into the sector landscape described above. This also leads naturally to a variety of non-exponential response functions consistent with the observed dynamics of a number of complex systems.

The conditional probability corresponding to (17) can be written[21]

$$p(x_1, t | x_0, 0) = \sqrt{\frac{p_{eq}(x_1)}{p_{eq}(x_0)}} \sum_{i=1}^{N} a(x_1, i) a(x_0, i) e^{\lambda_i t} \; , \tag{19}$$

which, when combined with (16) expresses the output response in a form that reveals the source of the exponential expansion that we saw above in (8).

A key aspect of the time-correlation approach to the mapping of a stochastic microeconomic dynamics to macroeconomic observables is that there is no representative agent. The often complex dynamical interrelationships of the heterogeneous economic agents are aggregated into a macroeconomic response through the time-correlation function, making the notion of a representative agent unsuitable for proper analysis of economic policy. The time-dependence inherent in this approach does, however, lend itself quite naturally to addressing the problem of economic friction and it is to this that we now turn.

## 3   Economic Friction

In a mechanical system the time dependent stress-strain behavior is "an external manifestation of internal relaxation behavior that arises from a coupling between

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[21]Here $a(x, j)$ is the $x$th component of the $j$th normalized eigenvector of the symmetric matrix corresponding to the solution of the master equation for the variable $u_i(t) = p(x_i, t)/\sqrt{p_{eq}(x_i)}$ and $\lambda_j$ is the corresponding eigenvalue. This is discussed in Uhlig et al. (1995) and references therein.

stress and strain through internal variables that change to new equilibrium values only through kinetic processes such as diffusion"[22]. Similarly, time dependent demand-output behavior is an external manifestation of internal relaxation behavior that arises from a coupling between demand and output through internal variables such as unemployment that change to new equilibrium values only after the passage of time. In both mechanical and economic systems this temporal lag in response to an applied force is a manifestation of friction.

Our identification of demand-output dynamics as relaxations provides a way of quantifying economic friction. An expression for this dissipation, or "internal friction", can be obtained by considering the case of a periodic demand $D(t)$

$$D(t) = D(0)e^{i\omega t} \; , \tag{20}$$

where $D(0)$ is the demand at time $t = 0$, $i = \sqrt{-1}$, and $\omega$ is the cyclic frequency of the demand. Output will track demand with a lag that can be represented by a loss angle $\phi$:

$$Y(t) = Y(0)e^{i(\omega t - \phi)} \; . \tag{21}$$

These expressions for demand and output imply a frequency dependent proportionality factor $J(\omega)$ (the Fourier transform of $J(t)$) that is complex $J(\omega) = J_1(\omega) - iJ_2(\omega)$ and a loss angle related to the components of $J(\omega)$ by $\tan(\phi) = J_2(\omega)/J_1(\omega)$ which, for (4), is

$$\tan \phi = \delta J \frac{\omega/\eta}{J_R + J_U \omega^2/\eta^2}. \tag{22}$$

Thus we see that the existence of an anelastic response ($\delta J \neq 0$) in an economy implies dissipation and provides a formal definition of economic friction. The existence of this loss angle is due to the restructuring within the economy needed to reestablish equilibrium: $Y \to \bar{Y}$ .

## 4 Discussion and Summary

Our approach to the microeconomic basis of macroeconomic dynamics consistent with the existence of heterogeneous economic agents is relatively straightforward: maintain commonly assumed and/or observed linear relationships between macroeconomic variables but allow for a time delay in reestablishing that relationship after one of the variables has been shocked. In the case of output and demand this can be expressed in terms of three postulates: (i) a unique equilibrium relationship between output and demand, (ii) time is required to establish the equilibrium relationship and (iii) the equilibrium relationship is linear. These assumptions are, however, identical to those in a variety of dynamical systems and with that observation we could leverage the common theoretical framework of linear response theory and time-correlation formalism used to describe the dynamics of these often complex systems

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[22]Paraphrasing the discussion on page 5 of Nowick and Berry (1972).

to the treatment of macroeconomic dynamics.

As we have seen, Okun's law is a natural consequence of this approach. The common linear form follows directly from the requirement of anelasticity that the equilibrium relationship between output and demand be linear. The time dependence of the unemployment response is, as expected, picked up in econometric partial adjustment analysis as lagged variables. Furthermore, econometric analysis of first-difference versions of Okun's law are clearly expected to work given the relationship between differential representations of these dynamics and response functions discussed in Sec. 2.2. Finally, response functions as seen in econometric analysis using vector autoregression are expected given the response function representation of these dynamics that follows from linear response theory as expressed in (14). While consistent with current econometric analysis, our approach differs from current practice in that the number of parameters in the model is determined by the nature of the differential relationships (whether using the phenomenological theory or the master equation) and not on the number of lag terms maintained in a statistical analysis.

The time-correlation formalism used in our development also highlights the limited reach of the representative agent concept. Specifically, we saw in (15) that the macroeconomic output solution to a particular microeconomic problem can be reduced to the evaluation of the autocorrelation $\langle Y(0)Y(t) \rangle$. The notion that the autocorrelation can be expressed in terms of a representative agent is equivalent to writing $\langle Y(0)Y(t) \rangle = N \langle y(0)y(t) \rangle$ where $N$ is the number of representative agents in the economy and $y(t)$ represents the output of the representative agent. The implications of this assumption, however, are rather dramatic as indicated by our expression for $\langle Y(0)Y(t) \rangle$ in (A-5) and (A-6). To reduce our model to that of a single representative agent requires two simplifications. First, all productivity coefficients $\lambda_n$ would need to be the same: a complete loss of heterogeneity. Second, the cross terms (e.g. those in (A-6)) involving the product $\lambda_n \lambda_m$ when $n \neq m$) would need to be negligible. This corresponds to an economy where there is no interaction between homogeneous agents: all agents respond to demand as if in isolation. While this does represent the expected response of the limiting case of an economy with a dilute arrangement of identical agents, it is bereft of heterogeneity and far removed from the general case of an economy with heterogeneous interacting agents. Furthermore, as noted by Blundell and Stoker (2005) this approach simply doesn't work: "Even with great statistical fit, there was too much uncertainty as to what drove the aggregate data, and for policy prescriptions it is crucial to know something about those processes". This echos the second objection of Kirman (1992) that "[t]he reaction of the representative to some change in a parameter of the original model - a change in government policy for example - may not be the same as the aggregate reaction of the individual he 'represents'. Hence using such a model to analyze the consequences of policy changes may not be valid". Finally, we see that our analysis, like that of the other approaches to macroeconomics that preserve heterogeneous interaction, is consistent with the Lucas critique in insisting that the impact of economic policy on the macroeconomy can be assessed properly only if the the relationship of said policy and microeconomic parameters, such as inter-sector transition rates, can be determined and the resulting policy-induced changes in microeconomic dynamics aggregated into macro observables.

While microeconomic restructuring gives rise to time-dependent macroeconomic response, the specific temporal signature of that response is a function of the constraints faced by the microeconomic agents which can often be represented as a topology imposed on the economic space. Hierarchical economic structure is a straightforward explanation for slow (i.e. non-exponential) macroeconomic response and ultrametric hierarchical structure is a simple topology that is empirically ubiquitous in economic systems, consistent with theoretical descriptions of the world encountered by heterogeneous economic agents and known to yield the rich set of response functions observed in complex systems.

In summary, we have shown that a description of macroeconomic response consistent with the microeconomic dynamics of heterogeneous economic agents can be had without resorting to the notion of a representative agent. The time dependence of macroeconomic adjustment is expressed as a direct consequence of stochastic restructuring at the microeconomic level. Our approach to the aggregation of the micro into the macro preserves observed topological constraints such as ultrametric hierarchical structure, and in so doing ensures the fidelity between the micro and macro perspectives essential to economic policy design. We illustrated this approach using the relationship between output and demand as mediated by changes in unemployment as an example: Okun's law in all its forms was found to be a natural consequence. We were able to show how the time-dependence of Okun's law is related to the hierarchical nature of the economic landscape negotiated by the unemployed. We also saw how the introduction of time dependence implies overshooting and how economic friction arises naturally as a result of the relationship between microeconomic dynamics and macroeconomic response.

# Appendix

In this section we discuss in greater detail the time-dependent response of output to a demand shock mediated by the time-dependent reestablishment of equilibrium of agents across levels of productivity and sectors. To proceed we need to develop the notion of heterogeneity for economic agents. We take the economy to be composed of sectors and that these sectors adjust their output by hiring or firing people in response to changes in demand. Sectors are differentiated with respect to the distance between each other. These distances reflect such factors as geographical differences, differences in technology and educational qualifications. We represent the location of these sectors by the variable $\mathbf{r}$.

We let $C_n(\mathbf{r})$ be the number of people in state of productivity $n$ (denoted by productivity coefficient $\lambda_n$, $n = 1, 2, \ldots, N$) within the infinitesimal volume element $\mathbf{r}$ in the space of sectors. Allowing for spatial variation of the step change in demand that we used in our previous example, (14) and (15) generalize to[23]

$$\langle Y(\mathbf{r}, t) \rangle = \beta \int d\mathbf{r}' D(\mathbf{r}') \left[ \langle Y(\mathbf{r}, 0) Y(\mathbf{r}', 0) \rangle - \langle Y(\mathbf{r}, t) Y(\mathbf{r}', 0) \rangle \right] . \tag{A-1}$$

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[23]We have included the factor of $V$, a volume element of the economy, at this point to preserve notational consistency with Balakrishnan (1978).

Taking $C$ to be the total number of people in the economy, the employment rate in state of productivity $n$, $X_n(\mathbf{r}) = C_n(\mathbf{r})/C$, will satisfy the normalization condition

$$V^{-1} \int d\mathbf{r} \sum_{n=1}^{N} X_n(\mathbf{r}) = 1 \ , \tag{A-2}$$

at all times. Given the employment rate together with the productivity coefficient we can write output in terms of employment as

$$Y(\mathbf{r}) = C \sum_{n=1}^{N} \lambda_n \left[ X_n(\mathbf{r}) - \frac{1}{N} \right] \ , \tag{A-3}$$

where the subtraction has been included so that there is no excess output in equilibrium. We take the employment rate $X_n(\mathbf{r})$ to be a stochastic variable that gives rise to output fluctuations. These fluctuations exist in all states (including the absence) of a demand shock as they follow from people changing both the sector to which they belong and their level of productivity. To compute the associated output autocorrelation we consider a set of stochastic states $\{|n, \mathbf{r})\}$ ($n = 1, 2, \ldots, N$, $\mathbf{r} \in V$). We begin with the assumption that the *a priori* occupation of state $|n, \mathbf{r})$ in the absence of a demand shock demand is $p(n, \mathbf{r})d\mathbf{r} = d\mathbf{r}/(VN)$. We further note that in the absence of a demand shock the time dependence of $X_n(\mathbf{r})$ can be expressed in terms of a time-evolution operator $P^{eq}(t)$ where

$$Y(\mathbf{r}, t) = P^{eq}(t)Y(\mathbf{r}, 0) \ ; \tag{A-4}$$

the matrix element $(n_1, \mathbf{r}_1|P^{eq}(t)|n_2, \mathbf{r}_2)$ being the conditional probability that during the time interval $t$ a person moves from the state $|n_1, \mathbf{r}_1)$ to the state $|n_2, \mathbf{r}_2)$.

The output autocorrelation for the demand response of the economy is

$$\langle Y(\mathbf{r}, t)Y(\mathbf{r}', 0) \rangle = \sum_{n_1} \sum_{n_2} \int d\mathbf{r}_1 \int d\mathbf{r}_2 \ p(n_1, \mathbf{r}_1)$$

$$\times \frac{(n_1, \mathbf{r}_1|Y(\mathbf{r}', 0)|n_1, \mathbf{r}_1)}{(n_1, \mathbf{r}_1|n_1, \mathbf{r}_1)}(n_1, \mathbf{r}_1|P^{eq}(t)|n_2, \mathbf{r}_2)\frac{(n_2, \mathbf{r}_2|Y(\mathbf{r}, 0)|n_2, \mathbf{r}_2)}{(n_2, \mathbf{r}_2|n_2, \mathbf{r}_2)} : \tag{A-5}$$

the initial state weight factor $p(n_1, \mathbf{r}_1)$ multiplying the expectation values of output in the initial and final states together with the probability of evolving between these states. Substituting (A-3), the constitutive expression relating employment and output, into (A-5) and applying the properties of the states $\{|n, \mathbf{r})\}$ the correlation function reduces to[24]

$$\langle Y(\mathbf{r}, t)Y(\mathbf{r}', 0) \rangle = \frac{Cv_0}{N} \sum_n \sum_m \lambda_n \lambda_m \left[ (m, \mathbf{r}'|P^{eq}(t)|n, \mathbf{r}) - \frac{1}{VN} \right] \ , \tag{A-6}$$

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[24]The details of this derivation are given in the Appendix of Balakrishnan (1978). Our presentation follows that of Balakrishnan et al. (1978) and Balakrishnan (1978) closely.

which, when substituted into (A-1), yields

$$\langle Y(\mathbf{r},t) \rangle = \frac{\beta C v_0}{N} \sum_n \sum_m \lambda_n \lambda_m \int d\mathbf{r}' D(\mathbf{r}')(m,\mathbf{r}'|\mathbf{1} - P^{eq}(t)|n,\mathbf{r}) \ , \qquad \text{(A-7)}$$

where $v_0$ is the volume per person in the economy and $\mathbf{1}$ is the unit operator.

From this we see that the central problem in understanding the time-dependent response of output in an economy is the evaluation of $(m,\mathbf{r}'|P^{eq}(t)|n,\mathbf{r})$: the matrix element describing microeconomic dynamics. While it is possible that a single adjustment process may dominate in some economic systems, adjustment can be a more general process and in complex systems richer probability dynamics are generally expected. Indeed, in physical and economic systems the concept of hierarchical dynamics provides a natural framework for expressing observed dynamics and provided a microeconomic basis for the response functions introduced in (8) and (9) above.

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