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A Quantum framework for economic science: new directions

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Abstract
The current paper explores the cutting-edge applications of quantum field theory and quantum information theory modelling in different areas of economic science, namely, in the behavioural modelling of agents under market uncertainty, and mathematical modelling of asset or option prices and firm theory. The paper then provides a brief discussion into a possible extension of the extant literature of quantum-like modelling based on scattering theory and statistical field theory. A statistical theory of firm based on Feynman path integral technique is also proposed very recently. The collage of new initiatives as described in the current paper will hopefully ignite still newer ideas.

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Keywords Quantum like modelling; contextuality; QFT (quantum field theory); self-adjoint operators; uncertainty; epistemic; ontic

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1. Introduction

There are two stages of development in the basic quantum theory, one, the first quantization, which is dubbed as the ordinary quantum mechanics, and two, the second quantization, which is dubbed as the quantum field theory. The first wave of development happened in 1920s-30s, where great minds like Heisenberg, Bohr, Schrödinger, Dirac, Born, and others played critical roles. The main mathematical structures were provided by the famous mathematicians like Von Neumann and Birkhoff (1936). The main lesson from the first wave was that quantum mechanics is inherently a probabilistic theory based on a logical structure which is non-classical or non-Boolean, and certainly, the wave-particle duality nature of all particles is the fundamental truth of the universe based on a deep uncertainty principle.[1]

Certainly, the first phase of interpretations of Quantum theory also began with the founding fathers, namely the Copenhagen interpretation, which refused to accept any further description of reality other than provided by the 'epistemic' state of ψ, the wave function. Hence providing an 'information', 'belief' or 'knowledge' based interpretation of quantum theory as a whole. A more recent development on this line is the rise of Quantum – Bayesian or QUBISM (Caves et al, 2002) approach. Not surprisingly epistemic school of thought is more adaptable for recent cognitive modelling, or, contextual decision-making models (Dzafarov et al, 2016) since knowledge or belief updating is an integral feature of such models.

The other extreme of interpretation is the true ontological model (Leifer, 2014), where not only that ψ is a fundamental feature of reality, but also the probability associated through Born's rule. Hence reality is truly random. Temporal Bell-type inequalities (Bell, 1964) do play a central role in support of an ontological view. However, such inequalities also have been discovered in case of human decision making (Aerts et al, 2013), hence rendering the final interpretation (if any at all) complicated.

The second phase of development was pioneered by the next generation of great minds, like Dirac (still), Feynman, Gellman, Yukawa, Landau, and others. This phase gave rise to incredibly accurate and beautiful theories like quantum electrodynamics (QED) which is the most accurate description of the interaction between photons and electrons. However, perhaps even the founders couldn't have imagined that such strange mathematical tools could be used in social sciences too.

Let's have a few examples. The first important tool in QFT is operators (operators do play a central role in the first quantization theory also, namely, self-adjoint Hermitian operators, or positive semidefinite operators. In cognitive models such operators are used to describing observables), which then operates on a particular state of the system produces a change (there are so many examples: displacement operators, time reversal operators, parity operators etc) in the state. However most important are creation and destruction operators. In quantum field theory, creation operators when operates on a ground state of a quantum system (by ground we mean the lowest energy state which can also be a state of vacuum with 0 particles, but please note according to quantum theory even a vacuum also contains fluctuations at the Plank scale due to Uncertainty principle) it updates the system with adding a particle to the system, and destruction operators just do the opposite. Such operators are also called the raising and lowering operators. These operators are absolutely critical for interaction or scattering theory of particles.[2] Recently these operators with their mathematical commutation or anti-commutation properties (called as their c* algebras) have been used to explain price movements, and portfolio formulations in markets. Imagine a market at its ground state before trading starts, so when a trader buys or sells assets the price rise or falls which

[1] Readers can refer to Leifers (2014) comprehensive survey of different epistemic, ontic, and intermediate theories of quantum state/ the wave function.

[2] We can refer to any standard quantum field theory text, specifically the modal expansion theory.

can be captured by operating creation (for price rise) or destruction (for price fall) operators on the market ground state.

The current paper aims to present a brief yet current overview of these cutting-edge research areas developing within the social sciences as a whole, and economics and finance specifically. The applications of quantum information theory, quantum field theory tools: Path integrals or, operator formalism have caused a paradigmatic shift (Khrennikov and Haven, 2013) in social sciences. Not only the models have fundamentally changed but also the philosophy of social science or decision theory has undergone a shift. For example, with the advent of quantum-like modelling in decision theory' the concept of randomness in decision scenario/ ambiguity or uncertainty in decision theory has changed. In classical decision theory underlying is the deterministic philosophy, randomness is rather pseudo in nature, since if the full information set is available (say about the initial conditions) then randomness melts away, and a Laplace demon like researcher can predict the future ad infinitum. However, in quantum theory randomness is inherent (at least the ontic interpretation demands so), it is irreducible, and hence quantum-like decision models should also contain such inherent randomness (Yukalov and Sornette,2011).

There are another growing and related literature of Econophysics (Chakraborty et, al, 2011) which draws upon statistical mechanics. Quantum-like modelling is not unrelated to statistical physics tools as will be discussed in the discussion section.

2. A brief review of extant literature

From an empirical perspective, the quantum-like modelling movement has been quite successful (Khrennikov and Haven, 2013) since the results are nearly matched by the real data, whether choice under ambiguity data or asset price movement data. However, philosophically the Importance of the movement can lie somewhere else, the main philosophical contribution seems to be the challenge to the underlying classical decision theory as a whole. Classical decision theory since the days of Boole (George Boole the originator of modern Logic theory) to the modern formalization by Kolgomorov (1933 as cited in Khrennikov and Haven, 2013, 2009), Savage (1954), Knight (1921) and others has been based on the philosophy of determinism and Boolean logic so to say.

In classical theory (whether physics or decision theory) randomness is a nuisance rather, it only arises in a model due to lack of information, or noise in the system, or in other words, it is pseudo-randomness. More specifically the need for considering probabilities in classical decision theory (as also in classical Physics, including relativity theory) is epistemic in nature. For example, Bayesian probability updating theory is based on the incompleteness of knowledge or belief of agents.

On the other hand, the philosophy of Quantum theory is based on real irreducible randomness; it is limited by nature herself to be fully deterministic, as represented by Uncertainty relations. Hence while adopting quantum theory to decision making one cannot avoid this inherent randomness and subscribe to the view that we can only compute probabilities of events.[3]

Certainly, the measurement problem is a huge and unsolved business in quantum mechanics which stems from the fact that somehow there are two 'incompatible processes' (Roger Penrose, from Oxford, has proposed on the same line since his publication of Emperor's new mind, 1989) in the theory, one the unitary time evolution of the state represented by Schrödinger equation (often called as the U process), and then a random collapse of state vector/ wave function while measuring the state (often called as the R process). Till now there have been numerous attempts to resolve this inconsistency.[4] However, while applying the set up in decision making or behavioural models since by no means we are building a physical theory of measurement, the problem of collapse of the superposed belief state into one possible Eigen-subspace may not be that daunting.

[3] Hence inherent randomness is a fundamental shift in the way of perceiving reality, a stricter version would claim that randomness is a part of ontology itself.

[4] Giving rise to many versions of the quantum theory.

Turning now to the economic theory, or the neoclassical financial economics theory, we find there are some deep underlying assumptions upon which the axiomatic formulations are based (whether it is general equilibrium theory, or efficiency market hypothesis, or social choice theory, or standard game theory). Among those assumptions, there are some behavioural assumptions like full rationality or some sort of bounded rationality of agents, which means in a competitive market scenario the choices of agents reveal that as if they are maximizing some given utility function, and then there are some mathematical assumptions like no information asymmetry and uncertainty. It is only under these strict assumptions that we are able to derive so-called Pareto optimal results (Arrow –Debreu general equilibrium model as an epitome of such axiomatic modelling based on the set-theoretical Real Analysis).

However even since 1920s economic philosophers like Keynes have warned us that economy is mired with uncertainties which are difficult to be described by standard probability models, and hence the behaviour of agents is also not faithfully described by the standard Boolean logic operations. From the 1970s onwards cognitive scientists (Kahneman, Tversky, Shaffir among the most noted)[5], and later the behavioural finance camp (Shiller, Thaler, Shleifer among the most noted) have regularly provided data on real choice behaviour which defy neoclassical predictions. The behavioural camp has provided good models, namely, prospect theory (again due to Kahneman, 1992), heuristics-based theory, and Bayesian learning models, but with limitations. For example, the 'zero prior trap' of Bayesian modelling (Basieva et al,2017), which simply says that one cannot get a significant posterior probability update if the prior is 0 or 1. However in financial markets, such big updation in belief states are not uncommon, and they are most prominent in crisis periods, so how to describe such behaviours?[6]

In standard finance or financial economics, there is acceptance of the fact that there is a diversity of opinions among investors in markets (Miller 1977 onwards) which cannot be reduced to a problem of typical information asymmetry. For example, the information asymmetry theory (mainly of adverse selection) predicts that due to some investors being better informed than the others, or insiders being better informed than the outsiders of a firm, the less informed groups will always need an information premium which will actually suppress or deflate the asset price in question.[7] However, in many cases, it is found that price bubbles are formed in the scenario of uncertainty or divergence of opinions among agents, this then cannot be easily explained by information asymmetry models. Hence, we need a different modelling approach for representing uncertainty in finance/ economics. Below are some moot points for the need of a new decision-making modelling in Economics or decision theory as a whole.

Recently Khrennikov Polina and Patra Sudip (2019, to be published) have developed a quantum framework-based model of agents' belief updating under uncertainty in financial markets, which provides an alternative formulation of the law of total probability, which may hold the key to describe asset price movements under uncertainty. Some important features of limitations of standard decision theory are as below:

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[5]These authors are widely known, and their works are cited in any standard behavioural economics text. Models of Behavioral economics is rather based on heuristics and at times on Bayesian learning models, but Quantum-like modelling is based on non –Boolean logic.

[6] In standard economic literature such rare events are termed as fat tailed events, or black swans.

[7] There are many applications of such diversity of opinions in finance, for example in mergers and acquisition announcement impacts on stock returns.

1. Experimental data on real decision making started challenging EUT (expected utility theory) predictions, this is again related to the works of Kahneman and others as cited earlier.

2. Ellsberg (1961) in a seminal work showed that EUT tenets are violated under uncertainty situations (these situations are fundamentally different from risky situations with known or inferred probability distributions), specifically that agents exhibit ambiguity aversion rather than utility maximization under such scenarios.[8]

3. Failure of sure thing principle: *'Decision-making errors. People prefer action A over action B if they know that an event E occurs, and also if they know that E does not occur, but they prefer B over A if they do not know whether E occurs or not, which entails a violation of the total law of Kolmogorovian probability*

4. Order effects P(A&B) not = P(B&A), where P(.) stands for probability and A and B are mutually exclusive events. This equality is obvious in classical decision theory but based on the Hilbert space formulation of quantum-like decision theory such equalities can be violated, which also corroborates with experimental data on real behaviour.

5. Conjunction and disjunction fallacies*:'Probability judgment errors'. People estimate the conjunction event 'A and B' (disjunction event 'A or B') as more (less) likely that the events A or/and B separately, which entails a violation of the monotonicity law of Kolmogorovian probability.*

6. Failure of the law of total probability: Khrennikov and Haven (2009, 2013) have shown multiple times how data from real choice behaviour under uncertainty violates the law of total probability (LTP).

7. Failure to explain decision under uncertainty (Gilboa, Postlewaite & Schmeidler, 2008) Hence we who are involved in quantum-like modelling do believe that a different axiomatic and logical set up of decision making or belief state representation will help resolve such anomalies.

The below section provides a brief overview of Hilbert space modelling implication in decision problems, and then the sections to follow delves deeper into QFT and quantum information theoretic applications.

Application of Hilbert space modelling, & failure of the law of total probability

Below is a brief comparison between the Classical set theory of decision making (often known as the Kolgomorovian set theory after Kolgomorov formalized probability theory based on set-theoretic measures in 1933) and the Hilbert space formulation (Von Neumann,1932).

Classical decision or probability theory has the following fundamental features:

a. Events are represented by sets, which are subsets of Ω, the universal set.

b. Sample space, sigma algebra, measure (probability)*[9]

c. Only Boolean logic is permissible over such sets.

d. Conditional probability: P (A givenB) = p(A and B)/p(B); p(B)>0

[8] We will see later briefly how Quantum-like modelling can be used to depict ambiguity aversion but as a logical result rather than so-called 'irrational' behaviour.

[9] Though probabilities are defined as measures here, in practice we can use the so-called frequencies imagining that such frequencies will converge with the real probability measures as no of observations tends to infinity. Measure theory of probability also makes use of frequency description thus.

We see conditional probability is a direct consequence of Boolean operations.

Generally, in Quantum theory state space refers to sesquilinear, complex normed vector space called Hilbert space H, which can be infinite dimensional. However, for cognitive modelling we can work with finite dimensional H space, following are the specific features of the state space.

a. H is endowed with a scalar product (positive definite), norm, and an orthonormal basis, non-degenerate

b. Any state can be visualized as a ray in this space

c. **Superposition principle:** perhaps the most important feature which sets the description of states apart from any familiar description in classical theories. This postulate says before any measurement is performed on the isolated state, the state itself can be in a superposition of basis states, this is the so-called pure state. An ensemble of pure states with probability weights is known to be a mixed state.[10]

Let the state space of some system (physical or cognitive) be represented by finite-dimensional Hilbert space H. Consider the pure state ψ and the observable A, denote its eigenvalues by a1, .., am and the corresponding eigenvectors by e1, ..., em. This is an orthonormal basis in H. (We again proceed under the assumption that spectrum is nondegenerate.) We expand the vector ψ with respect to this basis: ψ = c1e1 + ... + cm em, (3) where (cj) are complex numbers such that the sum of their squared absolute values equals to one (this is the coordinate expression of the normalization by one of a pure state-vector): |c1|^2+...+|cm|^ 2 = 1. By using the terminology of linear algebra, we say that the pure state ψ is a superposition of pure states ej. The density matrix corresponding to ψ has the elements ρij = cic*j. Hence, for the pure state ψ, the basic probabilistic postulate of quantum mechanics, (2), has the form: p(ej) = ρjj = cj c*j = |cj|^2.

d. Measurement: most of the times' projection postulate**

e. Measurement implies projection onto a specific Eigen subspace

f. Probability computation and failure of LTP(law of total probability):

g. The underlying logic is Non –Boolean in nature, which also enables decision theorists to resolve various anomalies. Algebra of events is prescribed by quantum logic. Events form an event ring R, possessing two binary operations, addition and conjunction, following are the allowed operations:

h. P(A U B) = P(BUA) (Boolean logic)

i. P{AU(BUC) }= P{(AUB)U(AUC)} (associative)

j. AUA = A (idempotency)

k. **P(A and B) NOT = P(B and A) (non-commutative, incompatible variables)**

l. **A and (B UC) NOT = (A and B) U (A and C) (no distributivity)**

[10] States are mathematically represented by density matrices, n x n positive matrices with the sum of the diagonal terms equal to unity. There are many beautiful properties of density matrices which help us to detect the purity of states, the entropy of systems, compute probabilities (Susskind,2014).

m. The fact that distributive property is absent in quantum logic was emphasized by Birkhoff and von Von Neumann. Suppose there are two events B1 and B2 that, when combined, form unity, B1 ∪ B2 = 1. Moreover, B1 and B2 are such that each of them is orthogonal to a nontrivial event A 6= 0, hence A ∩ B1 = A ∩ B2 = 0. According to this definition, A ∩ (B1 ∪ B2) = A ∩ 1 = A. But if the property of distributivity were true, then one would get (A ∩ B1) ∪ (A ∩ B2) = 0. This implies that A = 0, which is self-contradictory.

There are obviously many interpretational problems here, one of the most critical being the or the interference terms in the modified LTP, there is no doubt that these interference terms will set out the difference between set-theoretic modelling and Hilbert space-based modelling of behaviour (a good exploration of the topic can be found in Khrennikov and Haven, 2013). However, it is also found that in human cognition models s can also be in hyperbolic rather than standard trigonometric forms as we get in ordinary quantum mechanics (Khrennikov and Haven, 2013).

Let us now briefly cover the attempts to model agent's belief states based on quantum theory set as discussed above. Bruza et, al (2015) has extensively used such representations in cognitive modelling based on the quantum probabilistic framework, where the main objective is assigning probabilities to events. Space of belief is a finite-dimensional Hilbert space H, which is spanned by an appropriate set of basis vectors Observables are represented by operators (positive operators / Hermitian operators) which need not commute, i.e, [A, B] = AB –BA NOT= 0. Generally, any initial belief state is represented by density matrix/ operator, outer product of ψ with itself ρ =Iψ ⟩⟨ ψI,[11] this is a more effective representation since it captures the ensemble of beliefs. Mixed states are represented by ∑w Iψ ⟩⟨ ψI, where the sum is over all w's which can be treated as probabilities for each pure state to occur.

Some properties of ρ which become very useful in such modelling are: ρ = (ρ*) which means the transposed conjugate, for pure states

ρ = ρ^2, (ψ, ρ, ψ)>0: positivity, Trace ρ = 1.

The most important tenet is the measurement process, meaning the measurement of probabilities of say choosing one alternative from the available choice set, whose results may well differ from classical decision theory results. Here Measuring the probability of choosing one of the given alternatives, is represented by the action of an operator on the initial belief state. Here we may think of the action of this Hermitian (or a Positive semidefinite operator generally)[12] the operator on the initial belief state as a question asked of the agent regarding the choice, say in the case of financial decision making whether the price of an asset will go up or down.

While making decision superposition state collapses to one single state (can be captured by the Eigenvalue equation). Observables in QPT represented by Hermitian operators: A = (A*) ^T, where * represents complex conjugation, and T represents transpose operation. Hence expected value of A, E(A) = Trace (A ρ), every time measurement is done one of the Eigenvalues of the A is realized. A=∑aP spectral decomposition rule: a's are the Eigenvalues and P's are the respective projectors which project the initial state to the Eigen subspace with a specific a Trace formula is used here to compute that probability: p(ai)= Tr(Pi ρ), this is another form of the celebrated Born's rule. As soon as the measurement is done the state updates to ρ': Pi ρPi/ Tr(Pi ρ), which captures Simultaneously updation of the agents' belief state.

This updating formula can be considered as to be analogical to Bayesian updating rule. Again, if immediately after another observable is measured, which is represented by B, another Hermitian operator, then the state ρ' is further updated to a new state, such updating is analogical to conditional probability computation in Bayesian framework.

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[11] This is also called as the direct product or Tensor product.

[12] Referring to POVM methods in quantum theory, a brief note on POVM and its need in cognitive models is discussed in the appendix.

3. Applications of QFT and quantum information theory

The use of QFT or Quantum information theory set up in Economic and Social sciences have been manifold, here let us classify them into two broader types: behavioural modelling and mathematical modelling. Behavioural modelling is in keeping with the quantum decision theory as briefly explored earlier (with its measurement problem issues), and mathematical modelling is free of measurement theory approach since the main objective here is to model asset or option prices in financial markets, or model firm theory with empirical simulation results.

Behavioural models

Use of quantum information theory set up

Recently quantum information theoretic modelling has been used in financial market behaviour modelling (Bagarello and Haven,2017, Khrennikov,2016). In such models, we note two important features, one, the basic belief state of the agents is represented as a superposition of basis states in a finite-dimensional Hilbert space, as discussed earlier. In the parlance of quantum information theory one can say that the belief states can be represented by QUBIT: superposition of 0 and 1 states, where 0 and 1 are the basis states which spans the given Hilbert space: ψ> =1/ √2(0> +1>) can be a description of such a QUBIT, again the 0 and 1 states can be given matrix representations.

However, we can expand the description by direct or tensor products of the individual belief states of the agents, this can be done when there is n no of interacting agents in the market. For example, we consider that two agents say Alice and Bob's belief spaces are represented by 2 finite-dimensional Hilbert spaces, hence their initial belief states are superposition states in these two separable H spaces, and thus the composite belief state is the tensor product of them. Certainly, here there can be a scope of introducing the Entanglement state, where the composite state is pure, but component states are totally mixed.[13] However, we don't want to build a physical theory of entanglement between cognitive states here, which may be a possibility for future research.

The second important feature is the concept of Bath (with many degrees of freedom, ideally infinite no) which may be an information bath (Basieva and Khrennikov, 2017) which contains various types of information (in finance this can be a composite of soft and hard information).[14] Here the concept of decoherence is called for, where the interaction between the isolated pure belief state of the agents and the information bath causes the pure state to decohere and stabilize into an observable mixed state.

Certainly, there are challenges to overcome here since the widely established physical theory of atmospheric decoherence (where the quantum entanglement effect is destroyed once the isolated quantum system in a pure state comes in to interaction with atmosphere, which can well be a single photon) may not be directly applicable to such belief state models, one, reason is Obviously (as mentioned earlier too) that we are not building any physical model here, another reason can be difficulties with interpretations of parameters, for example, the decoherence time has to be finite and calculable (Basieva and Khrennikov, 2017).[15] In other applications Linblad / super operator has been used to capture the interaction of the system with information bath.[16]

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[13] Entanglement being the enigma in quantum information theory, provoking the idea of non-locality.

[14] Also Khrennikov Polina and Patra Sudip a conference proceeding (Ultimate Quantum Theory Conference, Sweden, 2018).

[15] Where hard information is easily verifiable information, and soft information is fuzzy and more costly to verify, there is strong literature on this line, however, quantum-like modelling can describe such an ambience with more mathematical precision and elegance.

[16] Polina Khrennikova has used such approach in financial market applications.

b. Use of interaction picture/ Hamiltonian based on quantum field theory

Bagarello(2015), Bagarello and Haven(2017), Basieva and Khrennikov (2017), among others have formalised the dynamics of a behavioural model based on the QFT set up of interaction Hamiltonian[17]. Once the full Hamiltonian of the system is formalised, either Schrodinger or Heisenberg technique is used for describing the dynamics of the system (which can be either time evolution of the belief state of agents in say uncertainty conditions in a financial market, or say the time evolution of the market as a whole with interactions or trading between agents).[18]

c. Use of QFT operators in financial markets

Khrennikov (2017), Khrennikov Polina and Patra Sudip (op cit)[19] among others have used operator formalism of QFT to model interesting behavioural scenarios, for example in political sciences and financial market interactions. The most applicable are the creation and destruction operators, or simply, raising and lowering operators. These operators are the most basic tools in QFT, whose main function is to upgrade the vacuum or the ground state of the system by adding or destroying particles, a' (here ' is to be understood as the dagger or complex conjugation of the transpose of an operator) is known as the creation operator which when acted upon the I0> or the vacuum of the system upgrade the system with one extra particle, and a is the destruction operator with the opposite function. The operators a', and a follow certain commutation (for Bosonic particles) / anti-commutation properties (for Fermionic particles) known as their algebras. Again a' and a can be given square positive matrix representations.

Now such operators are used for various purposes in financial market modelling. For example, we can imagine a scenario with initial belief state of an agent being I0> regarding say the asset price in question, but that state can be updated to I1> when there is a signal in the market about the price, and such an operation can be done by operating a' or a creation operator on I0>. Again, if the creation and destruction operators are operated subsequently then a new operator called as price operator can be created. The idea of a Number operator which denotes the total no of particles in a system can also be used in the financial market, say to compute the total no of stocks traded in the market.

Recently Haven and Bagarello (op cit) have prescribed certain steps for building a theory of trading stocks based on number operator representation. First, the Hamiltonian of the system can be formulated, which may simply represent the total no of stocks, and it may also consist of an interaction part which may describe the no of stocks or cash exchanged between two or more traders. Second to describe the dynamics of the system Heisenberg equations of motions can be constructed out of the Hamiltonian, for example the rate of change of no of stocks is nothing but the commutation of the number operator with the Hamiltonian operator. Third, such equations of motions have to be solved and can be geometrically described.

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[17] Alternatively, interaction Lagrangian can also be used if the formulation is based on the path integral, which will be discussed later in the paper.

[18] We recall here that Schrodinger picture is where the state of the system evolves while operators remaining invariant with time, and Heisenberg picture is where the state is taken as to be fixed and operators undergo unitary evolution. Heisenberg picture has been hugely successful in QFT due to its mathematical beauty.

[19] Working paper presented at the ultimate quantum theory conference in Sweden, 2018. Under review please don't quote.

4. Mathematical models

These models are free from any cognitive element so to say, mainly focused on alternative descriptions of asset pricing and firm theory, however with a strong empirical strand of numerical simulations.

a. Use of Schrödinger equation in Black-Scholes option pricing

Haven (2002, 2003) and many other scholars (summarized in Khrennikov and Haven, 2013) have attempted to use the Schrödinger equation for an alternative Black-Scholes option pricing model. Here the main attempt is to construct an option price as a state function and find a suitable potential function for solving the Eigenvalues of the equation. Schrödinger equation can be transformed into an option pricing equation via suitable variable transformations, and then two types of potentials are considered, one, arbitrage-free potential, and two, with arbitrage.[20] Haven (2002) has shown that such potential functions are vital to option pricing theory, and arbitrage-free pricing is obtained only when the function converges to one.

The solutions to the modified equation can be obtained by WKB approximation method or numerical simulation methods. Another interesting extension has been done again by Haven (2003), where the uncertainty in the original option pricing set up is replaced by the Quantum information set up of Qubit, which is the superposition state of 0 and 1 states as described earlier.

b. Path integrals and possible extensions

Certainly, the path integral formulation has been a revolution in QFT, earlier to the advent of path or field integrals propagators, or Green Functions, or transition probability amplitudes or scattering amplitudes were computed mainly with the standard techniques like Dyson expansions.[21] However, the path integral technique developed later made computations easier (and also crazy!).[22]

Applications of path integral techniques in finance, championed by Belal Baaquie (1997 onwards) can be summarised briefly here.[23] Based on the mathematical symmetry between the Brownian-like motion of stock prices and the path integral technique in QFT (for example both being continuous and non-differentiable) asset pricing or option pricing theory has been proposed based on path integral techniques.

Belal Baaquie (1997 onwards) has generalized many earlier results in form of Feynman path integral formulation, for example, Merton and German (as in Belal, 1997) stochastic modelling has been recast in the form of the path integral. Here there are different types of path integrals proposed, for example if the need is to understand how the asset price, as well as the underlying volatility which is stochastic, evolves over the time, then correlates have to be found which can be provided by a modified discrete time path integral, however, if simply we need to study the propagations of prices over space-time points than a standard propagator (as widely used in QFT) based on a simple discrete time path integral is proposed.

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[20] However, the price to pay is that the Hamiltonian of the system (at least in case of a call option pricing equation) is non-Hermitian, which means in the original form the Eigenvalues need not be real! More variable transformations are required to avoid this problem, but the pseudo-Hermitian property still remains an awkward issue which needs deeper interpretations.

[21] Freeman Dyson, another pioneer in QFT also showed later that there is a beautiful equivalence between S matrix or Dyson expansion techniques and the celebrated Feynman diagrams.

[22] Dyson told Feynman 'you must be crazy!' but later found out 'he was not'.

[23] There is a recent publication by Baquiee Belal (2018), which summarizes the path-breaking models for economics and finance.

5. Discussion: *possible extensions using scattering picture and statistical field theory*

We have seen briefly that QFT apparatus can be used in both cognitive and mathematical economics modelling, however, there are some further areas which can be quite promising, namely, analogies with scattering theory and statistical field theory.

Particle interaction pictures (analogy to Feynman diagrams) have various so-called 'channels' in interaction theories like in QED[24], where the main job is to compute such probabilities of particle-antiparticle interactions (such probabilities can be called transition amplitudes[25], Green Functions[26] which can be found out from so-called generating functional) with the help of either S matrix theory or Dyson expansion technique[27], or using the path integral technique. Now, these pictures can be used in financial models, for example in different types of portfolio formations.

Last but not the least let's also see the link between the partition function in statistical mechanics and the path integral and see if such connections can be useful in Finance models. The usefulness of a partition function recast in a path integral (Lancaster and Blundell, op cit) is that the formula we get is an integral with-in a summation, and the summation is over all possible configurations of the system which have the same boundary conditions (known as periodic boundary conditions). Such a representation will help us to capture many more Market configurations with similar movements of asset prices.

Another very recent yet not thoroughly explored area is 'statistical theory of the firm', as proposed by Baaquie (2018). Here a firm is perceived as facing uncertainty over time, hence the configuration of the firm is evolving dynamically. Hence a profit maximizing firm with labour and capital endowments can be described by a specific action functional (where potential is a function of labour employed and capital invested, which are again governed by standard production functions and growth models), and thus the dynamic evolution of the firm can be described by Feynman path integral. However, this nascent theory can be expanded based on various frictions firms face along with decisions to hire labour and invest capital, for example information asymmetry problems and agency conflict problems, which further impacts capital and employment decisions, as well as pay out decisions like dividend pay outs.

Dividend considerations might be important since if one defines an option on the profitability of the firm, and compute pricing based on the path integral approach, then pay outs should be considered in the pay off functions of such options. This is a budding field to be explored deeply.

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[24] QED: quantum electrodynamics, the most accurate theory of interactions of electrons and photons to put it very simply, is replete with many interesting pictures or channels describing particle-antiparticle interactions:

[25] There are many such fascinating propagators so to say, like Feynman propagators etc which mainly depicts the creation and annihilation of particles/ anti-particles in different points in space-time: a good reference is: 'Quantum Field Theory for the Gifted Amateur' by Lancaster and Blundell (2014).

[26] Green functions are vital to the QFT and theories onwards, they can be defined as such functions which when operated on by any linear operator generates Dirac –Delta functions, for example in simple scalar field theories when Klein –Gordon operator is operated on greens function of the field delta functions are generated, such Green functions are also the transition amplitudes in QFT

[27] Lancaster and Blundell (2014).

Appendix

Positive operator value method (POVM): recently (Basieva and Krennikov, op cit) POVM has been used in cognitive modeling related to describing choice behavior of agents under uncertainty, this is a very helpful tool in describing agent's behaviour in case of uncertainty in financial markets, since many interesting results like order effects can be explained.

Positive operators are a class of projection operators which have more general properties, for example, if E is one positive operator then it can be conceived of as E = M'M, where M is a self-adjoint operator and M' is the transpose conjugate of M, such that for all such observations ∑M'M= I where I is the identity operator. Again, M can be given a square matrix, of the form $\begin{bmatrix} \sqrt{1-\varepsilon} & 0 \\ 0 & \sqrt{\varepsilon} \end{bmatrix}$ where the sign epsilon denotes any noise in the system.

Noise in the system has an important interpretation in the decision theory literature, for example, say due to some noise in the final choice action, or due to some error, the agent rather choosing the optimal chooses a wrong option, now such actions can be represented by positive operators, rather than more stringent projection operators as described earlier.

There are several interesting properties of positive operators (Yearsley, 2017), such as: they are non-repeatable (E^2NOT=E), they are not unique, they are used when the basic elements in the Hilbert space of the model need not be orthogonal, they are used when there are more responses than there are basis states, this last property can be used in the decision making models with noise in the system.

Hence, A positive operator valued measure (POVM) is a family of positive operators {Mj}

such that Pm j=1 ∑Mj = I, where I is the unit operator. It is convenient to use the following representation of POVMs: **Mj = V* j Vj,** where **Vj: H → H** are linear operators. A POVM can be considered as a random observable. Take any set of labels α1,...,αm, e.g., for m = 2,α1 = yes,α2 = no. Then the corresponding observable takes these values (for systems in the state ρ) with the probabilities p(αj) ≡ pρ(αj) = TrρMj = **TrVjρV* j.** We are also interested in the post-measurement states. Let the state ρ was given, a generalized observable was measured and the value αj was obtained. Then the output state after this measurement has the form

ρj =VjρV * j /(TrVjρV* j).

Most importantly, both order effects and the violation of LTP can be explained with the help of POVM modelling. Consider two generalized observables a and b corresponding to POVMs Ma = {V * j Vj} and Mb = {W* j Wj}, where Vj ≡ V (αj) and Wj = W(βj) correspond to the values αj and βj. If there is given the state ρ the probabilities of observations of values αj and βj have the form

pa(α) = TrρMa(α) = TrV(α)ρV*(α), p(β) = TrρMb(β) = TrW(β)ρW*(β).

Now we consider two consecutive measurements: first the measurement and then the b-measurement. If in the first measurement the value a = α was obtained, then the initial state ρ was transformed into the state

ρa α =V (α)ρV *(α) /(TrV(α)ρV*(α))

For the consecutive b-measurement, the probability to obtain the value b = β is given by p(β|α) = Trρa(α)Mb(β) =

TrW(β)V(α)ρV*(α)W*(β)/(TrV(α)ρV*(α))

This is the conditional probability to obtain the result b = β under the condition of the result a = α. We set p(α,β) = pa(α)p(β|α).

Now since operators need not commute p(α,β) NOT= p(β, α),

We recall that for two classical random variables a and b which can be represented in the Kolmogorov measure-theoretic approach, the formula of total probability (FTP) has the form $p_b(\beta) = \sum p_a(\alpha)p(\beta|\alpha)$.

Further, we restrict our consideration to the case of dichotomous variables, $\alpha = \alpha_1, \alpha_2$ and $\beta = \beta_1, \beta_2$.

FTP with the interference term for in general non-pure states given by density operators and generalized quantum observables given by two (dichotomous) PVOMs:

$$p_b(\beta) = p_a(\alpha_1)p(\beta|\alpha_1) + p_a(\alpha_2)p(\beta|\alpha_2) + 2\lambda\sqrt{p_a(\alpha_1)p(\beta|\alpha_1)p_a(\alpha_2)p(\beta|\alpha_2)},$$

or by using ordered joint probabilities $p_b(\beta) = p(\alpha_1,\beta) + p(\alpha_2,\beta) + 2\lambda_\beta\sqrt{p(\alpha_1,\beta)p(\alpha_2,\beta)}$. Here the coefficient of interference λ has the form: $\lambda=$

$$\mathrm{Tr}\rho\{W^*(\beta)V^*(\alpha_i)V(\alpha_i)W(\beta) - V^*(\alpha_i)W^*(\beta)W(\beta)V(\alpha_i)\} / 2\sqrt{p_a(\alpha_1)p(\beta|\alpha_1)p_a(\alpha_2)p(\beta|\alpha_2)}$$ Introduce the parameters

$$\gamma_{\alpha\beta} = \mathrm{Tr}\rho W^*(\beta)V^*(\alpha)V(\alpha)W(\beta) / (\mathrm{Tr}\rho V^*(\alpha)W^*(\beta)W(\beta)V(\alpha))$$

$$= p(\beta,\alpha) / p(\alpha,\beta)$$

This parameter is equal to the ratio of the ordered joint probabilities of the same outcome, but in the diff recent order, namely, "b then a" or "a then b". Then,

Interference term $\lambda = \frac{1}{2}\{\sqrt{p(\alpha_1,\beta)/p(\alpha_2,\beta)} * (\gamma_{\alpha_1\beta} - 1) + \sqrt{p(\alpha_2,\beta)/p(\alpha_1,\beta)} * (\gamma_{\alpha_2\beta} - 1)$, In principle, this coefficient can be larger than one. Hence, it cannot be represented as $\lambda = \cos\theta$ for some angle ("phase") θ, cf.. However, if POVMs M_a and M_b are, in fact,spectral decompositions of Hermitian operators, then the coefficients of interference are always less than one, i.e., one can find phases θ. A more elaborate treatment can be found in works of Andrei Khrennikov (2017).

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