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Sufficient conditions of stochastic dominance for general transformations and its application in option strategy

Jianwei Gao and Feng Zhao

Abstract
A counterexample is presented to show that the sufficient condition for one transformation dominating another by the second degree stochastic dominance, proposed by Theorem 5 of Levy (Stochastic dominance and expected utility: Survey and analysis, 1992), does not hold. Then, by restricting the monotone property of the dominating transformation, a revised exact sufficient condition for one transformation dominating another is given. Next, the stochastic dominance criteria, proposed by Meyer (Stochastic Dominance and transformations of random variables, 1989) and developed by Levy (Stochastic dominance and expected utility: Survey and analysis, 1992), are extended to the most general transformations. Moreover, such criteria are further generalized to transformations on discrete random variables. Finally, the authors employ this method to analyze the transformations resulting from holding a stock with the corresponding call option.

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Keywords Stochastic dominance; transformation; utility theory; option strategy

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1 Introduction

Stochastic dominance (SD) has been proved to be a powerful tool for ranking random variables and is employed in various fields such as finance, decision analysis, economics and statistics etc. (cf., Levy, 1992, 2006; Chakravarty and Zoli, 2012; Jouini et al., 2013; Tsetlin et al., 2015; Post et al., 2015 and Post, 2016; Gao and Zhao, 2017). The SD rules indicate when one random variable is to be ranked higher than another by specifying a condition which the difference between their cumulative distribution functions (CDFs) must satisfy. However, economic and financial activities usually induce transformations of an initial risk, and the classical SD rules are inefficient in ranking such transformations. Transformations of random variables have been discussed in the early stochastic dominance literature, especially in the risk analysis portion. For example, Sandom (1971) has used a particular linear, risk altering, transformation in discussing the comparative statics of risk. Hadar and Russell (1971, 1974) have dealt with special cases of the transformation question, emphasizing its use in dealing with portfolios of random variables. Cheng, Magill, and Shafer (1987) have used the transformation approach to address the comparative statics of first degree stochastic dominance shifts in a random variable within a general decision model context. Meyer (1989) has proposed the first and second stochastic dominance (FSD and SSD) criteria for the increasing, continuous, and piecewise differentiable transformations on continuous random variables. Meyer goes on to analyze the transformation resulting from coinsurance, the transformation resulting from holding a stock with the corresponding call option, or even holding call and put options simultaneously. Gao and Zhao (2017) have developed FSD and SSD criteria for monotonic transformations on discrete random variables, and they apply these results in ranking transformations resulting from pension funds. These applications indicate that the transformation approach is useful in discussing comparatives statics of random variable changes and financial issues.

For the general transformations, Levy (1992) has given several sufficient conditions under which one transformation dominates another by FSD and SSD. Hereafter, some authors discuss the transformations of different random variables (cf., Peluso and Trannoy, 2007, 2012; Denuit et al., 2013). To the best of our knowledge, Theorem 5 of Levy (1992) is the only result on the stochastic dominance for general transformations. However, we have found that its dominance condition for SSD is not sufficient and its dominance condition for FSD can be relaxed. Then, by restricting attention to the monotone property of the dominating transformation, we present a revised exact sufficient condition for one transformation dominating another. Next, we further extend the stochastic dominance criteria to the most general transformations.

Moreover, we further generalize these stochastic dominance criteria for transformations on continuous random variables to the discrete case. Finally, we employ the SD approach to analyze the transformations resulting from holding a stock with the corresponding call option.

The paper is organized as follows. Section 2 presents a counterexample to show that Levy's theorem about SSD does not hold. By discussing the monotone property of the dominating transformation, Section 3 derives the exact sufficient condition for one transformation dominating another by SSD. Section 4 deduces the stochastic dominance criteria for the most general transformations, which further perfect and improve Levy's result and extend Meyer's result to more general case. Section 5 further provides the stochastic dominance criteria for transformations on discrete random variables. Section 6 analyzes the transformations resulting from holding a stock with the corresponding call option. Section 7 concludes the paper.

2 Levy's sufficient conditions and its counterexample

Suppose that X is a continuous random variable with support in the finite interval $[a,b]^*$. Its density function and cumulative distribution function are denoted by $f(x)$ and $F(x)$, respectively. The transformation functions $m(x)$ and $n(x)$ are assumed to be integrable in $[a,b]$, and the corresponding cumulative distribution functions of the transformed random variables $m(X)$ and $n(X)$ are denoted by $F_m(x)$ and $F_n(x)$, respectively. Then, $m(X)$ dominates $n(X)$ by FSD if $F_m(x) \le F_n(x)$ and by SSD if $\int_{-\infty}^{x} F_m(t)dt \le \int_{-\infty}^{x} F_n(t)dt$ for all $x \in [a,b]$.

To facilitate the narrative, we will refer to transformed random variables, derived by applying transformation functions to X, as transformations on X, or shortly transformations. Obviously, the classical SD rules rely heavily on CDFs. But in most cases CDFs of transformations are difficult to compute as $F_m(x) = P(m(X) \le x)$ and $F_n(x) = P(n(X) \le x)$, and the frequently-used integration by parts is invalid in this case. Thus, the classical SD rules based on the CDFs framework lose their great charm when dealing with the transformations. In order to determine the SD relations between two general transformations, Levy (1992) gives several sufficient conditions under which one transformation dominates the other by FSD and SSD, the main result is shown as follows.

Alleged Theorem 5. (Levy, 1992) Given a random variable X with the density $f(x)$ and support in the interval $[a,b]$, the random variable $Y = m(X)$ dominates the random variable $Z = n(X)$ in the first degree if

$$\{m(x) - n(x)\} f(x) \ge 0 \quad \text{for all } x \text{ in } [a,b]. \tag{1}$$

Similarly, the dominance condition for SSD is given by

$$\int_a^x \{m(t) - n(t)\} f(t)dt \ge 0 \quad \text{for all } x \text{ in } [a,b]. \tag{2}$$

Note: For simplicity, we assume that the range of the random variable is finite. Actually, the stochastic dominance criteria can easily be extended to the infinite range by mathematical skills (see Hanoch and Levy, 1969).

Although Levy's Theorem 5 only proposes the sufficient conditions for FSD and SSD relations between two transformed random variables, its really meaningful contribution is that it tries to represent the SD rules by the transformation functions and the density function of the original random variable, rather than by CDFs of the transformed random variables. To better illustrate this meaning, Figure 1 to Figure 4 show the relationship between the method of Levy's Theorem 5 and that under the framework of CDFs for the uniformly distributed random variable.

●The comparative diagrams for FSD

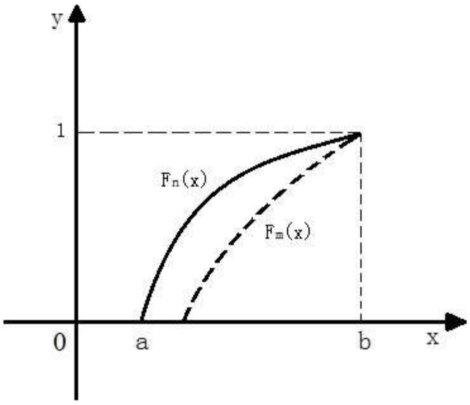

Figure 1: The CDFs of $m(x)$ and $n(x)$

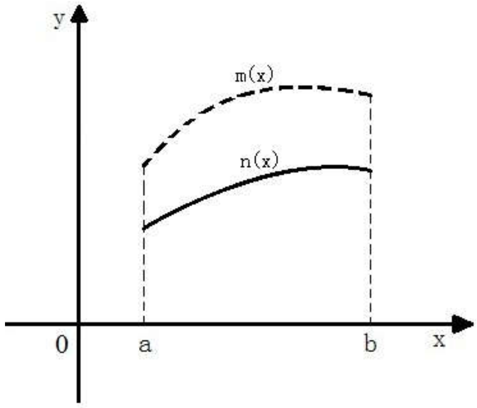

Figure 2: The transformations of $m(x)$ and $n(x)$

●The comparative diagrams for SSD

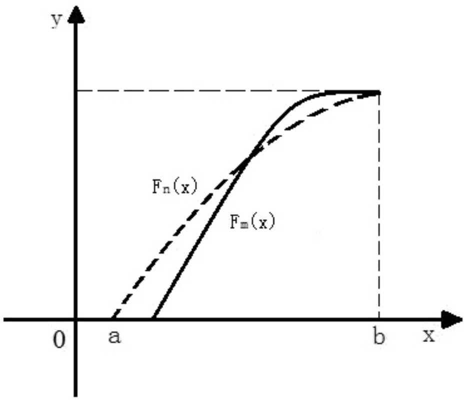

Figure 3: The CDFs of $m(x)$ and $n(x)$

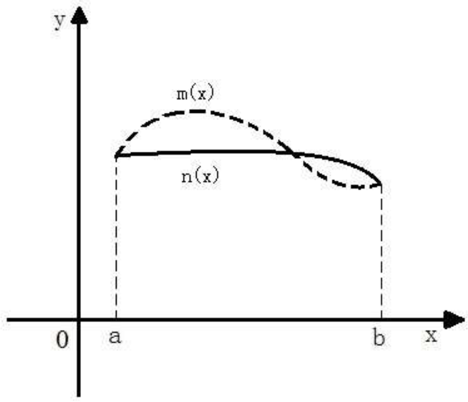

Figure 4: The transformations of $m(x)$ and $n(x)$

Figure 1 and Figure 3 respectively illustrate the classical SD rules of $m(x)$ dominating $n(x)$ by FSD and SSD under the framework of CDFs. Figure 2 and Figure 4 respectively describe the SD relations of $m(x)$ dominating $n(x)$ by FSD and SSD under the framework of Alleged Theorem 5.

From Figure 1 to Figure 4 we find that $F_m(x)$ and $F_n(x)$ under the framework of CDFs are respectively replaced by $m(x)$ and $n(x)$ under Alleged Theorem 5, while they have the relatively reverse position relation in the cases of FSD and SSD. Obviously, it is much more convenient to verify the dominance relations by condition (1) and condition (2) than by the framework of CDFs which needs to justify $F_m(x) \le F_n(x)$ and $\int_{-\infty}^{x} F_m(t)dt \le \int_{-\infty}^{x} F_n(t)dt$.

Notice that the cumulative distribution functions $F_m(x)$ and $F_n(x)$ are both increasing and right continuous while the transformation functions $m(x)$ and $n(x)$ are only assumed to be integrable, and the monotonicity is not required in Alleged Theorem 5, we have adequate reasons to question the correctness of this theorem. We will first provide a counterexample to the second part of Alleged Theorem 5.

Example 1. Let X be a random variable with the uniform distribution in the interval $[-1,1]$. Define

$$m(x)=\begin{cases}-4x, & -1\le x\le 0\\ x, & 0\le x\le 1\end{cases} \quad \text{and} \quad n(x)=\begin{cases}-3x, & -1\le x\le 0\\ 2x, & 0\le x\le 1\end{cases}.$$

Then $m(x)-n(x)=-x$ and $\int_{-1}^{x}\{m(t)-n(t)\}f(t)dt=\frac{1-x^2}{4}\ge 0$ for all x in $[-1,1]$, which means that $m(x)$ and $n(x)$ satisfy condition (2) in Alleged Theorem 5. But for the increasing and concave utility function $u(x)=-e^{-x}$, we have

$$\begin{aligned}&E[u(m(X)]-E[u(n(X)]\\ &=\int_{-1}^{0}(e^{3x}-e^{4x})\tfrac{1}{2}dx+\int_{0}^{1}(e^{-2x}-e^{-x})\tfrac{1}{2}dx\\ &=\frac{1}{2}[(\frac{1}{e}-\frac{5}{12})-(\frac{1}{2e^2}+\frac{1}{3e^3}-\frac{1}{4e^4})]\\ &<\frac{1}{2}[\frac{1}{e}-\frac{5}{12}]=\frac{12-5e}{24e}\\ &<0.\end{aligned}$$

Thus, $m(X)$ does not dominate $n(X)$ by SSD.□

Example 1 shows that the condition $\int_{a}^{x}\{m(t)-n(t)\}f(t)dt\ge 0$ for all x in $[a,b]$ is not sufficient for $m(X)$ dominating $n(X)$ by SSD. By carefully analyzing Theorem 5 of Levy (1992), we find that the monotone property of the dominating transformation is inevitable for stochastic dominance of transformations. Actually, we have proved the following conclusion.

3 A revised sufficient condition for SSD

In this part, we will revise Theorem 5 of Levy (1992) and derive the exact sufficient condition for one transformation dominating another by SSD.

Theorem 1. Given a random variable X with the density $f(x)$ and support in the interval $[a,b]$, $m(x)$ and $n(x)$ are transformations defined on $[a,b]$. If $m(x)$ is increasing, continuous, and piecewise differentiable in $[a,b]$, then the random variable $Y=m(X)$ dominates $Z=n(X)$ by SSD if $\int_{a}^{x}\{m(t)-n(t)\}f(t)dt\ge 0$ for all x in $[a,b]$.

Proof. See Appendix A.

By restricting the monotonicity and differentiability of the dominating transformation, Theorem 1 provides the exact sufficient condition for one transformation dominating another by SSD. Compared with Theorem 5 of Levy (1992), Theorem 1 gives a revised dominance condition concerning SSD, so it can be viewed as an primary improvement of Theorem 5 in Levy (1992).

Furthermore, in the next paragraph we will prove that the FSD condition listed in Alleged Theorem 5 and the SSD condition listed in Theorem 1 can be weakened via complicated mathematics skill.

4 Stochastic dominance criteria for general transformations

Theorem 5 in Levy(1992) and Theorem 1 of this paper give the dominance condition under which one transformation dominates another by FSD and SSD. In the following, we will prove that these conditions can be relaxed to a more general case. That is, the restrictions to the dominating transformation in Theorem 1 can be further relaxed.

Theorem 2. Given a random variable X with the density $f(x)$ and support in the interval $[a,b]$, $m(x)$ and $n(x)$ are transformations defined on $[a,b]$. Then we have

(1) $m(X)$ dominates $n(X)$ by FSD if $(m(x)-n(x))f(x)\geq 0$ holds almost everywhere in $[a,b]$, i.e., $S=\{x\in[a,b]\,|\,(m(x)-n(x))f(x)<0\}$ is a set of measure zero.

(2) If $m(x)$ is increasing in $[a,b]$, then $m(X)$ dominates $n(X)$ by SSD if $\int_a^x\{m(t)-n(t)\}f(t)dt\geq 0$ for all x in $[a,b]$.

Proof. See Appendix B.

Theorem 2 provides two dominance conditions under which one transformation dominates another by FSD or SSD for the most general transformations. Compared with Theorem 5 of Levy (1992), in Theorem 2(1) points are permitted to violate the dominance condition (1) only if they constitute a set of measure zero. So, Theorem 2(1) reduces the dominance condition for FSD in Theorem 5 of Levy (1992). Compared with Theorem 1, Theorem 2(2) only requires the dominating transformation to be increasing, and the property of differentiability is not necessary.

Moreover, only the increasing property is considered in Theorem 2, and we can derive a similar conclusion if the dominating transformation is decreasing.

Theorem 3. Given a random variable X with the density $f(x)$ and support in the interval $[a,b]$, $m(x)$ and $n(x)$ are transformations defined on $[a,b]$. Suppose $m(x)$ is decreasing in $[a,b]$, then we have $m(X)$ dominates $n(X)$ by SSD if $\int_x^b\{m(t)-n(t)\}f(t)dt\geq 0$ for all x in $[a,b]$.

Proof. See Appendix C.

Remark 1. Meyer (1989) proposes the FSD and SSD criteria for transformations that if $m(x)$ and $n(x)$ are increasing, continuous and piecewise differential functions, then $m(X)$ dominates $n(X)$ by FSD if and only if $\{m(x)-n(x)\}f(x)\geq 0$ for all x in $[a,b]$, and $m(X)$ dominates $n(X)$ by SSD if and only if

$\int_a^x \{m(t)-n(t)\}f(t)dt \geq 0$ for all x in $[a,b]$.

Obviously, Theorem 2 and Theorem 3 extend Meyer's result to a more general case. Either the dominating or the dominated transformation in Meyer (1989) is assumed to be increasing, continuous, and piecewise differentiable. However, in Theorem 2 and Theorem 3, only the dominating transformation is assumed to be monotonous, and there are no any other restrictions to both the dominating and the dominated transformation. Apparently, the differentiability is redundant. Furthermore, Theorem 3 considers the decreasing transformation that is absence in Meyer's result.

Remark 2. The concept of increasing risk and increasing n th degree risk, introduced by Rothschild and Stiglitz (1970) and Ekern (1980), play an important role in risk analysis. It requires that all the random variables to be compared have the same mathematical expectations. Given this supposition, we can easily induce the following conclusion from Theorem 2 and Theorem 3.

Corollary. Given a random variable X with the density $f(x)$ and support in the interval $[a,b]$, $m(x)$ and $n(x)$ are transformations defined on $[a,b]$. Suppose that $E[m(X)] = E[n(X)]$, then we have

(1) Supposing that $m(x)$ is increasing in $[a,b]$, $m(X)$ has more increasing risk than $n(X)$ if $\int_a^x \{m(t)-n(t)\}f(t)dt \geq 0$ or $\int_x^b \{m(t)-n(t)\}f(t)dt \leq 0$ for all x in $[a,b]$.

(2) Supposing that $m(x)$ is decreasing in $[a,b]$, $m(X)$ has more increasing risk than $n(X)$ if $\int_x^b \{m(t)-n(t)\}f(t)dt \geq 0$ or $\int_a^x \{m(t)-n(t)\}f(t)dt \leq 0$ for all x in $[a,b]$.

From this corollary, we can easily deduce that there exist a kind of risk transformations which lead to the SSD relation which is completely opposite to the conclusion of Theorem 5 in Levy (1992).

Example 2. Assume that the random variable X satisfies standard normal distribution. Define

$$m(x) = \begin{cases} -3x, & x \leq 0 \\ -x, & x > 0 \end{cases} \quad \text{and} \quad n(x) = \begin{cases} -2x, & x \leq 0 \\ 0, & x > 0 \end{cases}.$$

Obviously, $m(x)$ and $n(x)$ satisfy condition (2). According to Theorem 5 of Levy (1992), it should be concluded that $m(X)$ dominates $n(X)$ by SSD. But, the truth is on the opposite side. Actually, by Theorem 3, it is easy to prove the fact that $n(X)$ dominates $m(X)$ by SSD since $\int_x^{+\infty} \{n(t)-m(t)\}f(t)dt = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} > 0$ for all x in $(-\infty,+\infty)$.

5 Stochastic dominance criteria for general transformations on discrete random variables

By discussing Levy's dominance conditions for one transformation dominating another by FSD or SSD, we obtain several stochastic dominance criteria for transformations, which perfect and improve Levy and Meyer's results. It must be pointed out that all the conclusions, whether Levy and Meyer's results or the stochastic dominance criteria developed in the paper, are concentrating on transformations of continuous random variables. Actually, there exist similar stochastic dominance criteria for transformations on discrete random variables. Gao and Zhao (2017) have discussed the stochastic dominance relationship between two transformations on discrete random variables, and presents several sufficient conditions for ranking transformations on discrete random variables by FSD or SSD. Such conclusions can be summarized in the following theorem.

Theorem 4. Let X be a discrete random variable whose prospects are characterized by $\{p_1, x_1; \cdots, p_n, x_n\}$ with $x_1 < x_2 < \cdots < x_n$ and support in the finite interval $[a,b]$. For any two functions $m(x)$ and $n(x)$ defined on $[a,b]$, we get two transformed random variables $m(X)$ and $n(X)$, denoted as $\{p_1, m(x_1); \cdots, p_n, m(x_n)\}$ and $\{p_1, n(x_1); \cdots, p_n, n(x_n)\}$, or shortly as $\{p_1, m_1; \cdots, p_n, m_n\}$ and $\{p_1, n_1; \cdots, p_n, n_n\}$, respectively. Then we have:

(1) The transformed random variable $m(X)$ dominates $n(X)$ by FSD if $m_i \geq n_i$ for all $i = 1, 2, \cdots, n$.

(2) If $m(x)$ is increasing and $\sum_{i=1}^{k}(m_i - n_i)p_i \geq 0$ for all $k = 1, 2, \cdots, n$, then the transformed random variable $m(X)$ dominates $n(X)$ by SSD.

(3) If $m(x)$ is decreasing and $\sum_{i=k}^{n}(m_i - n_i)p_i \geq 0$ for all $k = 1, 2, \cdots, n$, then the transformed random variable $m(X)$ dominates $n(X)$ by SSD.

(4) Suppose that $m(x)$ is increasing and $E[m(X)] = E[n(X)]$. If $\sum_{i=1}^{k}(m_i - n_i)p_i \geq 0$ or $\sum_{i=k}^{n}(m_i - n_i)p_i \leq 0$ for all $k = 1, 2, \cdots, n$, then $m(X)$ has more increasing risk than $n(X)$.

(5) Suppose that $m(x)$ is decreasing and $E[m(X)] = E[n(X)]$. If $\sum_{i=k}^{n}(m_i - n_i)p_i \geq 0$ or $\sum_{i=1}^{k}(m_i - n_i)p_i \leq 0$ for all $k = 1, 2, \cdots, n$, then $m(X)$ has more increasing risk than $n(X)$.

The proofs of the first three items in Theorem 4 are in Gao and Zhao (2017), and the proofs of the last two items follow from them and are omitted. Theorem 4 presents several dominance conditions for ranking transformations on discrete random variables by FSD or SSD, and it overcomes the drawbacks of Meyer and Levy's results that cannot deal with transformations on discrete random variables.

6 Applications in the option strategy

It is well-known that put and call option contracts can modify the value of common stock. These contracts provides the buyer of the option with the right to either buy (call) or sell (put) shares of common stock at a fixed price referred to as the striking price. On the other hand, the seller of such an option contract incurs the obligation to either sell or buy the common stock at the agreed upon striking price if the contract purchaser decides to exercise the option. To model one such option transaction using the transformation notation, let X represent the random value of 100 shares of a given common stock and assume that its support is the interval $[a,b]$. An investor who owns the common stock can sell a call contract (100 shares) with striking price x_m for a price of p_m. This investment of selling a call option while owning the common stock can be represented by the following transformation $m(x)=\begin{cases} x+p_m, & x<x_m \\ x_m+p_m, & x\geq x_m \end{cases}$. The original random value x becomes $m(x)$ when the stock is held and the call option is sold. That is, this sale of the call option while holding the common stock alters the value of the total investment by adding the option price to the stock value in the event that the option is not exercised, and fixes the investment's value at the option price plus the string price if the option is exercised. A similar option strategy with striking price x_n and option price p_n defines transformation $n(x)$. It is certainly that $m(x)$ and $n(x)$ are both increasing in $[a,b]$. Then, how to choose the better option strategy?

To answer this question, we first form the difference of $m(x)$ and $n(x)$. Assuming that $x_m < x_n$, then we have

$$m(x)-n(x)=\begin{cases} p_m-p_n, & a\leq x\leq x_m \\ (x_m+p_m)-(x+p_n), & x_m\leq x\leq x_n \\ (x_m+p_m)-(x_n+p_n), & x_n<x\leq b \end{cases}.$$

Of course, experience in choosing option strategies with varying sizes for the striking price indicates that it is unlikely for the option price charged to be smaller with lower striking price. Furthermore, it is typical for the reduction in the option price to be a fraction of the increase in the striking price. Thus it is further assumed that $p_m > p_n$, and that $x_n - x_m > p_m - p_n$. Under this restriction, the difference $m(x)-n(x)$ is first positive and constant, then declines with slope minus 1, and finally is constant and negative.

From the definition of FSD, it is easily to declare that there is no FSD relation between $m(X)$ and $n(X)$. However, if $\int_a^b \{m(x)-n(x)\}f(x)dx \geq 0$ ($f(x)$ denote the probability distribution function of X, then by Theorem 2 we deduce that $m(X)$ dominates $n(X)$ by SSD. That is, if the mean value of $m(X)$ is at least large as the mean value of $n(X)$, then $m(X)$ is a better choice for all risk-averse investors.

While this example deals with the selling of a call option, the purchase of a put option contract can also be modeled using a similar transformation. One can also model the simultaneous purchase or sale of put or call contracts with different striking prices, although the transformations involved become cumbersome.

7 Conclusion

We first present a counterexample to show that Levy's result with respect to SSD does not hold. Then, we give the revised exact dominance condition for one transformation dominating another by SSD. Next, we propose several stochastic dominance criteria for the most general transformations, which can be viewed as a further improvement of Theorem 5 in Levy (1992). Moreover, we further generalize these stochastic dominance criteria for transformations on continuous random variables to the discrete case. Finally, we employ the SD approach to analyze the transformations resulting from holding a stock with the corresponding call option.

Whether on theory or in applications, much can still be done concerning transformations and stochastic dominance. It would be useful to extend such stochastic dominance criteria to transformations on more than one random variables and to consider higher-degree SD rules for transformations. In addition, we will further apply these results to the analysis of transformations resulting from economic and financial issues.

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Appendix A

Proof of Theorem 1. For an arbitrary utility function $u(x) \in U_2$, where U_2 denotes the set of utility function u satisfying the first derivative $u' \geq 0$ and the second derivative $u'' \leq 0$. By the second-order Taylor expansion, for all $x \in [a,b]$, we have $u(n(x)) = u(m(x)) + u'(m(x))[n(x) - m(x)] + \frac{1}{2}u''(\xi_x)[n(x) - m(x)]^2$, i.e.

$$u(m(x)) - u(n(x)) = u'(m(x))[m(x) - n(x)] - \tfrac{1}{2}u''(\xi_x)[m(x) - n(x)]^2, \tag{A1}$$

where ξ_x is among $m(x)$ and $n(x)$. Then

$$E(u(m(X)) - E(u(n(X)) = \int_a^b [u(m(x)) - u(n(x))] f(x)dx$$

$$= \int_a^b u'(m(x))[m(x) - n(x) f(x)dx + \int_a^b [-\tfrac{1}{2}u''(\xi_x)][m(x) - n(x)]^2 f(x)dx$$

$$= \int_a^b u'(m(x))d[\int_a^x [m(t) - n(t)] f(t)dt] + \int_a^b [-\tfrac{1}{2}u''(\xi_x)][m(x) - n(x)]^2 f(x)dx$$

$$= u'(m(b))\int_a^b [m(t) - n(t)] f(t)dt + \int_a^b [-u''(m(x))]m'(x)\int_a^x [m(t) - n(t)] f(t)dtdx$$

$$+\int_a^b [-\tfrac{1}{2}u''(\xi_x)][m(x) - n(x)]^2 f(x)dx \,. \tag{A2}$$

Then, by the definition of $u(x)$ and the supposition that $\int_a^x \{m(t)-n(t)\}f(t)dt \ge 0$ for all x in $[a,b]$, we derive the conclusion that $E(u(m(X))-E(u(n(X))\ge 0$. □

Appendix B

Proof of Theorem 2. (1) For an arbitrary utility function $u(x)\in U_1$, where U_1 denotes the set of utility function u satisfying the first derivative $u'\ge 0$. By the differential mean value theorem, we have

$$u(m(x))-u(n(x))=u'(\xi_x)[m(x)-n(x)] \text{ for all } x\in[a,b], \tag{B1}$$

where ξ_x is among $m(x)$ and $n(x)$. Then we have

$$\begin{aligned} &E(u(m(X))-E(u(n(X)) \\ &=\int_a^b [u(m(t))-u(n(t))]f(t)dt=\int_a^b u'(\xi_t)[m(t)-n(t)]f(t)dt \\ &=\int_S u'(\xi_t)[m(t)-n(t)]f(t)dt+\int_{[a,b]-S} u'(\xi_t)[m(t)-n(t)]f(t)dt . \end{aligned} \tag{B2}$$

Since S is a set of measure zero, we have

$$\int_S u'(\xi_t)[m(t)-n(t)]f(t)dt=0 . \tag{B3}$$

For the second term on the right of (B2), by the definitions of U_1 and S, we know that $u'(\xi_t)$ and $[m(t)-n(t)]f(t)$ are non-negative for all $t\in[a,b]$, which implies that

$$\int_{[a,b]-S} u'(\xi_t)[m(t)-n(t)]f(t)dt\ge 0 . \tag{B4}$$

Substitute (B3), (B4) into (B2), we have $E(u(m(X))-E(u(n(X))\ge 0$, i.e. $m(X)$ dominates $n(X)$ by FSD.

(2) If $\{m(x)-n(x)\}f(x)\ge 0$ holds almost everywhere in $[a,b]$, then from Theorem 2(1) we conclude that $m(X)$ dominates $n(X)$ by FSD, and then it still holds for SSD via the hierarchical property of SD rules. Therefore, we only need to consider the case that $\int_S [m(x)-n(x)]f(x)dx\ne 0$. In this case, S must consist of one or more intervals where hold that $[m(x)-n(x)]f(x)<0$, here we neglect single-point sets of S for they are sets of measure zero. Without loss of generality, we suppose that there are k intervals $(c_{i,}d_i)$ $(i=1,\cdots,k)$ with $c_1<d_1<c_2<d_2<\cdots<c_k<d_k$. According to the given condition that $\int_a^x \{m(t)-n(t)\}f(t)dt\ge 0$ for all x in $[a,b]$, for the first interval $[c_1,d_1]$ we have

$$\int_a^{d_1}[m(t)-n(t)]f(t)dt=\int_a^{c_1}[m(t)-n(t)]f(t)dt+\int_{c_1}^{d_1}[m(t)-n(t)]f(t)dt\ge 0 .$$

This means that there must exist a previous subset A_1 such that

$$\int_{A_1}[m(t)-n(t)]f(t)dt=\int_{c_1}^{d_1}|\, m(t)-n(t)\,|f(t)dt . \tag{B5}$$

Similarly, for (c_2,d_2), we have $\int_a^{d_2}[m(t)-n(t)]f(t)dt=\int_a^{c_2}[m(t)-n(t)]f(t)dt+\int_{c_2}^{d_2}[m(t)-n(t)]f(t)dt\geq 0$.

From Equation (B5) and $c_1<d_1<c_2<d_2$, we can derive that

$$\int_a^{c_2}[m(t)-n(t)]f(t)dt=\int_B[m(t)-n(t)]f(t)dt\,, \tag{B6}$$

where $B=[a,c_2]/\{A_1\cup(c_1,d_1)\}$ denotes the set of all the elements of $[a,c_2]$ except for $A_1\cup(c_1,d_1)$. So there exists a subset $A_2\subseteq B$ lying on the left-hand side of (c_2,d_2) and satisfying

$$\int_{A_2}[m(t)-n(t)]f(t)dt=\int_{c_2}^{d_2}|\,m(t)-n(t)\,|f(t)dt\,,\text{ and } A_1\cap A_2=\Phi\,. \tag{B7}$$

By the mathematical induction, we can draw the conclusion that for any interval (c_i,d_i), there exists a subset A_i of $[a,b]$ satisfying the following properties:

(i) $\forall x\in A_i$, we have $x\leq c_i$;

(ii) if $x\in A_i$, then $[m(x)-n(x)]f(x)>0$;

(iii) $\int_{A_i}[m(t)-n(t)]f(t)dt=\int_{c_i}^{d_i}|\,m(t)-n(t)\,|f(t)dt$.

(iv) all the subsets $A_i(i=1,\cdots,k)$ are disjoint with each other.

Notice that for $u(x)\in U_2$, we have $u''\leq 0$, meaning that $u'(x)$ is decreasing. Then, according to property (i) – (iv) and the monotonous condition of $u'(x)$ and $m(x)$, we can make the following statements.

(a) Note that $[m(x)-n(x)]f(x)<0$ for any interval $(c_i,d_i)(i=1,\cdots,k)$. By differential mean value theorem, we have $u(m(x))-u(n(x))=u'(\xi_x)[m(x)-n(x)]$ for all x in (c_i,d_i) and $u(x)\in U_2$, where ξ_x is among $m(x)$ and $n(x)$, i.e., $m(x)\leq\xi_x\leq n(x)$. Hence,

$$\begin{aligned}&\int_{c_i}^{d_i}[u(m(t))-u(n(t))]f(t)dt\\&=\int_{c_i}^{d_i}u'(\xi_t)[m(t)-n(t)]f(t)dt=\int_{c_i}^{d_i}u'(\xi_t)[m(t)-n(t)]f(t)dt\\&\geq\int_{c_i}^{d_i}u'(m(t))[m(t)-n(t)]f(t)dt\geq\int_{c_i}^{d_i}u'(m(c_i)[m(t)-n(t)]f(t)dt\\&=-u'(m(c_i)\int_{c_i}^{d_i}[n(t)-m(t)]f(t)dt\,.\end{aligned} \tag{B8}$$

(b) Similarly, for the subset A_i of $[a,b]$, from property (ii) and the differential mean value theorem we have $n(x)\leq\xi_x\leq m(x)$, and

$$\begin{aligned}&\int_{A_i}[u(m(t))-u(n(t))]f(t)dt\\&=\int_{A_i}u'(\xi_t)[m(t)-n(t)]f(t)dt\geq\int_{A_i}u'(m(t))[m(t)-n(t)]f(t)dt\end{aligned}$$

$$\geq u'(m(c_i))\int_{A_i}[m(t)-n(t)]f(t)dt = u'(m(c_i))\int_{c_i}^{d_i}[n(t)-m(t)]f(t)dt . \tag{B9}$$

Combining with expressions (B8) and (B9), we obtain that

$$\int_{c_i}^{d_i}[u(m(t))-u(n(t))]f(t)dt+\int_{A_i}[u(m(t))-u(n(t))]f(t)dt \geq 0 .$$

Then we have

$$E(u(m(X))-E(u(n(X))=\int_a^b[u(m(t))-u(n(t))]f(t)dt$$

$$\geq \sum_{i=1}^{k}\{\int_{c_i}^{d_i}[u(m(t))-u(n(t))]f(t)dt+\int_{A_i}[u(m(t))-u(n(t))]f(t)dt\}$$

$$\geq 0 . \ \square \tag{B10}$$

Appendix C

Proof of Theorem 3. Suppose that $\int_x^b\{m(t)-n(t)\}f(t)dt \geq 0$ for all x in $[a,b]$ and $m(x)$ is decreasing, we now prove $m(X)$ dominates $n(X)$ by SSD.

If $[m(x)-n(x)]f(x) \geq 0$ is true almost everywhere in $[a,b]$, then by Theorem 1 we conclude that $m(X)$ dominates $n(X)$ by FSD and correspondingly, $m(X)$ dominates $n(X)$ by SSD with the hierarchical property of SD relations.

Otherwise, we divide the set of points satisfying $[m(x)-n(x)]f(x)<0$ into k intervals $(c_{i,}d_i)(i=1,\cdots,k)$ with $c_1<d_1<c_2<d_2<\cdots<c_k<d_k$. Here, the discrete points which satisfy $[m(x)-n(x)]f(x)<0$ can be omitted because these points make no contribution to the expected utility of $m(X)$ and $n(X)$.

Since $\int_x^b[m(t)-n(t)]f(t)dt \geq 0$ for all x in $[a,b]$, we can draw the conclusion that for any interval $(c_{i,}d_i)(i=1,\cdots,k)$, there exists a subset A_i in $[a,b]$ satisfying:

(i) $\forall x \in A_i$, we have $x \geq d_i$;

(ii) if $x \in A_i$, then $[m(x)-n(x)]f(x)>0$;

(iii) $\int_{A_i}[m(t)-n(t)]f(t)dt=\int_{c_i}^{d_i}|m(t)-n(t)|f(t)dt$;

(iv) all the subsets $A_i(i=1,\cdots,k)$ are disjoint with each other.

Suppose that the utility function $u(x)$ is increasing and concave, then by the differential mean value theorem, we derive $u(m(x))-u(n(x))=u'(\xi_x)[m(x)-n(x)]$ for all $x \in [a,b]$, where ξ_x is among $m(x)$ and $n(x)$. Since $u'(x)$ and $m(x)$ are both decreasing in $[a,b]$, we have

(1) For $x \in (c_i,d_i)$, we have $m(x) \leq \xi_x \leq n(x)$ and

$$\int_{c_i}^{d_i}[u(m(t))-u(n(t))]f(t)dt=\int_{c_i}^{d_i}u'(\xi_t)[m(t)-n(t)]f(t)dt$$

$$\geq \int_{c_i}^{d_i} u'(m(t))[m(t)-n(t)]f(t)dt \geq \int_{c_i}^{d_i} u'(m(d_i)[m(t)-n(t)]f(t)dt$$

$$= -u'(m(d_i)\int_{c_i}^{d_i}[n(t)-m(t)]f(t)dt\,. \tag{C1}$$

(2) For $x \in A_i$, we have $n(x) \leq \xi_x \leq m(x)$ and

$$\int_{A_i}[u(m(t))-u(n(t))]f(t)dt = \int_{A_i} u'(\xi_t)[m(t)-n(t)]f(t)dt$$

$$\geq \int_{A_i} u'(m(t))[m(t)-n(t)]f(t)dt \geq u'(m(d_i))\int_{A_i}[m(t)-n(t)]f(t)dt$$

$$= u'(m(d_i))\int_{c_i}^{d_i}[n(t)-m(t)]f(t)dt\,. \tag{C2}$$

Thus,

$$\int_{c_i}^{d_i}[u(m(t))-u(n(t))]f(t)dt + \int_{A_i}[u(m(t))-u(n(t))]f(t)dt \geq 0\,. \tag{C3}$$

According to expressions (C1), (C2) and (C3), we obtain

$$E(u(m(X))-E(u(n(X)) = \int_a^b [u(m(t))-u(n(t))]f(t)dt$$

$$\geq \sum_{i=1}^{k}\{\int_{c_i}^{d_i}[u(m(t))-u(n(t))]f(t)dt + \int_{A_i}[u(m(t))-u(n(t))]f(t)dt\}$$

$$\geq 0\,.\square \tag{C4}$$

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