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Nash equilibria in all-pay auctions with discrete strategy space

Zheng Li

Abstract
Using two-player all-pay auctions, the author fully characterizes the Nash equilibrium under a discrete bidding strategy space. In particular, he shows that under the random tie-breaking rule, the cardinality of the set of Nash equilibrium depends on the parity of the reward size and a continuum of Nash equilibria exists. Additionally, when a simple favor-one-sided tie-breaking rule is used, the equilibrium solution becomes independent of the reward size.

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1 Introduction

All-pay auctions with complete information have been widely used to model R&D, lobbying and tournaments (Baye, Kovenock and De Vries, 1993; Dasgupta, 1982; Hillman and Riley, 1989). Due to the tractability concern, most theoretical models assume a continuous strategy space (Baye, Kovenock and De Vries, 1996; Che and Gale, 1998). Moreover, even under a discrete strategy space, one could take a sequence of finer and finer grids, and the limit would be the equilibrium under a continuous strategy space (Dasgupta and Maskin, 1986).

However, the cardinality of the set of Nash equilibrium may be larger under a discrete strategy space than that under a continuous strategy space. Taking an all-pay auction for two homogeneous players as an example, i.e., the value of the object is the same for the two players, the equilibrium is unique and symmetric under the continuous strategy space (Baye et al., 1996). However, if the value of the object is an even number, there exists a continuum of symmetric equilibria in the discrete strategy space and the uniform equilibrium bidding distribution becomes a special case among all symmetric equilibria (Bouckaert, Degryse and De Vries, 1992). Additionally, whether an asymmetric Nash equilibrium exists or not has yet to be discussed. Therefore, it is important to complete the characterization of equilibria in all-pay auctions, and it may have implications for guiding experimental design and analysis.

Altogether, we fully characterize the Nash equilibrium in two-player all-pay auctions with a discrete strategy space for homogeneous bidders, i.e., $v_1 = v_2$. Furthermore, we extend our analysis to heterogeneous bidders, i.e., $v_1 > v_2$. We choose the two-player case because the corresponding Nash equilibrium under a continuous strategy space is unique, while with more than two players, unless we have $v_1 > v_2 > v_3$, a continuum of asymmetric equilibria already exists (Baye, Kovenock and De Vries, 1996). Therefore, with a two-player all-pay auction model, it is clear to compare the prediction difference between continuous and discrete strategy spaces.

We show that the characterization of equilibrium depends on the parity of the reward size, and revenue equivalence does not hold in certain conditions. In particular, a continuum of asymmetric Nash equilibria exists and individuals do not play uniformly. Moreover, we extend the model to the case in which a bidding cap is imposed, and present similar theoretical predictions. Additionally, we show that when a favor-one-sided tie-breaking rule is implemented, the Nash equilibrium does not depend on the parity of the reward size anymore.

2 Theoretical Model

In a two-player all-pay auction, each risk-neutral player $i \in \{x, y\}$ chooses b_i. The value of the object: Q_i, is a positive integer. Every bidder chooses a bid from $\{0, 1, \ldots, C-1, C\}$, and C is a bidding cap. Without loss of generality, the payoff function for player x under a random tie-breaking rule is given by:

$$\pi_x(b_x, b_y) = \begin{cases} -b_x, & b_x < b_y \\ \frac{Q_x}{2} - b_x, & b_x = b_y \\ Q_x - b_x, & b_x > b_y \end{cases}$$

2.1 Homogeneous Bidders without Caps

In this subsection, we characterize the Nash equilibrium for homogeneous bidders without caps, i.e., $Q_x = Q_y = Q$ and $C = \infty$.

The following proposition shows that when $Q = 2n$, there exists not only a continuum symmetric equilibria (Bouckaert et al., 1992), but also a continuum asymmetric equilibria. Furthermore, when $Q = 2n+1$, the unique symmetric Nash equilibrium is characterized in Bouckaert et al. (1992) and Schep (1995). We also include it in the following proposition for the completeness of the characterization.

Proposition 1. *1. When $Q = 2n$, all equilibria are characterized as follows:*

(a) *For player x,*

$$P_0^x = P_2^x = \cdots = P_{Q-2}^x = 2V_y/Q,$$
$$P_1^x = P_3^x = \cdots = P_{Q-1}^x = 2(1-V_y)/Q,$$

and $V_y \in [0,1]$.

(b) *For player y,*

$$P_0^y = P_2^y = \cdots = P_{Q-2}^y = 2V_x/Q,$$
$$P_1^y = P_3^y = \cdots = P_{Q-1}^y = 2(1-V_x)/Q,$$

and $V_x \in [0,1]$.

2. *When $Q = 2n+1$, the unique Nash equilibrium is symmetric (Bouckaert et al. 1992 & Schep 1995):*

$$P_0 = P_1 = \cdots = P_{Q-1} = \frac{1}{Q},$$

and $V_x = V_y = 1/2$.

Proof. See A. □

First, Proposition 1 indicates that when $Q = 2n$, the equilibria strategy is purely determined by V_x and V_y. Second, although players may not have equal probability on every integer from 0 to $Q-1$, each of them puts the same mass on all even numbers as well as the same probability on all odd numbers. Additionally, no player submits a bid which is equal to Q.

Second, it implies that the expected total bids no longer equal the value of the object, i.e., full dissipation. Instead, they are either lower or at most equal to the value of the object. Specifically, when $Q = 2n+1$, the expected total bid is $Q-1$. While it is $Q - V_x - V_y \in [Q-2, Q]$ for $Q = 2n$.

2.2 Heterogeneous Bidders without Caps

Now we consider the heterogeneous bidders' case, i.e., the value of the object or the bidding cost is different between the two players. Without loss of generality, we assume $Q_x - Q_y > 1$ and $C = \infty$. [1]

First, Baye et al. (1996) show that a unique equilibrium exists under the continuous strategy space when there are two players. Moreover, $V_x = Q_x - Q_y$ and $V_y = 0$. However, the equilibrium characterization is different in the discrete strategy space, as shown in the next proposition.[2]

Proposition 2. 1. *When $Q_y = 2n$, all equilibria are characterized as follows:*

(a) *For player x,*

$$P_0^x = P_2^x = \cdots = P_{Q_y}^x = 0,$$
$$P_1^x = P_3^x = \cdots = P_{Q_y-1}^x = 2/Q_y.$$

[1] When $Q_x - Q_y = 1$, the special solution in Proposition 2.2(a) is $P_0^x = P_2^x = \cdots = P_{Q_y-1}^x = 2V_y/Q_y$, $P_1^x = P_3^x = \cdots = P_{Q_y-2}^x = (2-2V_y)/Q_y$, $P_{Q_y}^x = (1-2V_y)/Q_y$, and $V_y \in [0, 1/2]$ in Proposition 2.2(c).

[2] Cohen and Sela (2007) point out one of the equilibria in the discrete strategy space, though the focus of the paper is not the complete characterization of the Nash equilibria.

(b) *For player y, the equilibrium strategy is determined by a linear system, and the special solution is:*

$$\begin{aligned}
&P_0^y = (V_x + 1)/Q_x,\\
&P_1^y = P_3^y = \cdots = P_{Q_y-1}^y = 0,\\
&P_2^y = P_4^y = \cdots = P_{Q_y-2}^y = 2/Q_x,\\
&P_{Q_y}^y = (Q_x - Q_y + 1 - V_x)/Q_x.
\end{aligned}$$

(c) *In all equilibria, $V_x \in [Q_x - Q_y - 1, Q_x - Q_y + 1]$ and $V_y = 0$.*

2. *When $Q_y = 2n + 1$ and $Q_x - Q_y > 1$, all equilibria are characterized as follows:*

(a) *For player x, the equilibrium strategy is determined by a linear system, and the special solution is:*

$$\begin{aligned}
&P_0^x = P_2^x = \cdots = P_{Q_y-1}^x = 0,\\
&P_1^x = P_3^x = \cdots = P_{Q_y-2}^x = 2/Q_y,\\
&P_{Q_y}^x = 1/Q_y.
\end{aligned}$$

(b) *For player y,*

$$\begin{aligned}
&P_0^y = (Q_x - Q_y + 1)/Q_x,\\
&P_1^y = P_3^y = \cdots = P_{Q_y}^y = 0,\\
&P_2^y = P_4^y = \cdots = P_{Q_y-1}^y = 2/Q_x.
\end{aligned}$$

(c) *In all equilibria, $V_x = Q_x - Q_y$ and $V_y = 0$.*

Proof. See B. □

Proposition 2 implies that the expected payoff for player y is always zero, which is the same as that under the continuous strategy space. Moreover, when $Q_y = 2n$, her expected bid can be any number in $[\frac{Q_y^2-2Q_y}{2Q_x}, \frac{Q_y^2+2Q_y}{2Q_x}]$. When $Q_y = 2n + 1$, it is always $\frac{Q_y^2-1}{2Q_x} < \frac{Q_y^2}{2Q_x}$.

Next, the expected payoff for player x lies between $[Q_x - Q_y - 1, Q_x - Q_y + 1]$ for $Q_y = 2n$, while it is $Q_x - Q_y$ for $Q_y = 2n + 1$. Furthermore, when $Q_y = 2n$, her expected bid is $\frac{Q_y}{2}$, and it is $\frac{Q_y^2+1}{2Q_y} > \frac{Q_y}{2}$ with $Q_y = 2n + 1$.

Additionally, although both players' equilibrium bidding distribution may depend on a linear system, it is worthwhile to note that both of their expected bids are solely determined by the other player's expected payoff.

2.3 Homogeneous Bidders with Caps

Considering homogeneous bidders with a bidding cap. Che and Gale (1998) show that a unique symmetric Nash equilibrium exists under the continuous strategy space, and we prove that the set of Nash equilibrium is larger in the discrete case.

First, it is easy to show that when $Q > 2C$, the unique equilibrium is a pure strategy in which everyone bids C (Boudreau, 2011). Therefore, we focus on $Q \leq 2C$ in the following analysis.

Proposition 3. *1. When $Q = 2n \leq 2C$, all equilibria are characterized as follows:*

(a) *For player x,*

$$\begin{aligned}
&P_0^x = P_2^x = \cdots = P_{2C-Q}^x = 2V_y/Q,\\
&P_1^x = P_3^x = \cdots = P_{2C-Q-1}^x = 2(1-V_y)/Q,\\
&P_{2C-Q+1}^x = P_{2C-Q+2}^x = \cdots = P_{C-1}^x = 0,\\
&P_C^x = 2(Q-C-V_y)/Q,
\end{aligned}$$

where $V_y \in [0,1]$.

(b) *For player y,*

$$\begin{aligned}
&P_0^y = P_2^y = \cdots = P_{2C-Q}^y = 2V_x/Q,\\
&P_1^y = P_3^y = \cdots = P_{2C-Q-1}^y = 2(1-V_x)/Q,\\
&P_{2C-Q+1}^y = P_{2C-Q+2}^y = \cdots = P_{C-1}^y = 0,\\
&P_C^y = 2(Q-C-V_x)/Q,
\end{aligned}$$

where $V_x \in [0,1]$.

2. *When $Q = 2n+1 \le 2C$, the Nash equilibrium is unique and symmetric:*

$$\begin{aligned}
&P_0 = P_1 = \cdots = P_{2C-Q} = 1/Q,\\
&P_{2C-Q+1} = P_{2C-Q+2} = \cdots = P_{C-1} = 0,\\
&P_C = 2(Q-C-1/2)/Q,
\end{aligned}$$

and $V_x = V_y = 1/2$.

Proof. See C. □

Compared to the no-cap case in Section 2.1, Proposition 3 suggests that under a bidding cap C, neither player puts probability mass in $[2C-Q+1, Q-1]$. However, given Q, V_x and V_y, the expected bids are the same between no-cap and cap cases. Furthermore, when a bidding cap C is implemented, the difference of equilibrium characterization between continuous and discrete strategy spaces is similar to that without a bidding cap.

2.4 Heterogeneous Bidders with Caps

We consider heterogeneous bidders with a bidding cap in the following analysis. Without loss of generality, we assume $C < Q_y - 1$, $Q_x - Q_y > 1$ and $Q_y \le 2C$. [3]

Proposition 4. 1. *When $Q_y = 2n < 2C$, all equilibria are characterized as follows:*

(a) *For player x,*

$$\begin{aligned}
&P_0^x = P_2^x = \cdots = P_{2C-Q_y}^x = 0,\\
&P_1^x = P_3^x = \cdots = P_{2C-Q_y-1}^x = 2/Q_y,\\
&P_{2C-Q_y+1}^x = P_{2C-Q_y+2}^x = \cdots = P_{C-1}^x = 0,\\
&P_C^x = 2(Q_y-C)/Q_y.
\end{aligned}$$

[3] When $C = Q_y - 1$, it is same as heterogeneous bidders without caps. When $Q_x - Q_y = 1$, similar to footnote 2, the special solution in Proposition 4.2(a) is $P_0^x = 2/Q_y$, $P_1^x = 1/Q_y$, $P_2^x = P_4^x = \cdots = P_{2C-Q_y-1}^y = 2(1-V_y)/Q_y$, $P_3^x = P_5^x = \cdots = P_{2C-Q_y}^y = 2V_y/Q_y$, $P_{2C-Q_y+1}^x = P_{2C-Q_y+2}^x = \cdots = P_{C-1}^x = 0$, $P_C^x = 2(Q_y - C - V_y)/Q_y$, and $V_y \in [0, 1/2]$ in Proposition 4.2(c). When $Q_y > 2C$, the unique equilibrium is a pure strategy in which everyone bids C.

(b) *For player y, the equilibrium strategy is determined by a linear system, and the special solution is:*

$$
\begin{aligned}
&P_0^y = (V_x + 1)/Q_x, \\
&P_1^y = P_3^y = \cdots = P_{Q_y-1}^y = 0, \\
&P_2^y = P_4^y = \cdots = P_{2C-Q_y-2}^y = 2/Q_x, \\
&P_{2C-Q_y}^y = (Q_y - Q_x + 1 + V_x)/Q_x, \\
&P_{2C-Q_y+1}^y = P_{2C-Q_y+2}^y = \cdots = P_{C-1}^y = 0, \\
&P_C^y = (2Q_x - 2C - 2V_x)/Q_x.
\end{aligned}
$$

(c) *In all equilibria, $V_x \in [Q_x - Q_y - 1, Q_x - Q_y + 1]$ and $V_y = 0$.*

2. *When $Q_y = 2n + 1 < 2C$, all equilibria are characterized as follows:*

(a) *For player x, the equilibrium strategy is determined by a linear system, and the special solution is:*

$$
\begin{aligned}
&P_0^x = 0, \\
&P_1^x = 1/Q_y, \\
&P_2^x = P_4^x = \cdots = P_{2C-Q_y-1}^y = 2/Q_y, \\
&P_3^x = P_5^x = \cdots = P_{2C-Q_y}^y = 0, \\
&P_{2C-Q_y+1}^x = P_{2C-Q_y+2}^x = \cdots = P_{C-1}^x = 0, \\
&P_C^x = 2(Q_y - C)/Q_y.
\end{aligned}
$$

(b) *For player y,*

$$
\begin{aligned}
&P_0^y = (Q_x - Q_y + 1)/Q_x, \\
&P_1^y = P_3^y = \cdots = P_{2C-Q_y}^y = 0, \\
&P_2^y = P_4^y = \cdots = P_{2C-Q_y-1}^y = 2/Q_x, \\
&P_{2C-Q_y+1}^y = P_{2C-Q_y+2}^y = \cdots = P_{C-1}^y = 0, \\
&P_C^y = 2(Q_y - C)/Q_x.
\end{aligned}
$$

(c) *In all equilibria, $V_x = Q_x - Q_y$ and $V_y = 0$.*

Proof. See D. □

Similar to the payoff equivalence between no-cap and with-cap cases in a continuous strategy space (Che and Gale, 1998), Proposition 4 suggests that when a discrete strategy space is used, this property still holds. Combining Propositions 2 and 4, we also expect that in a discrete strategy space, the equilibrium bidding distribution for heterogeneous bidders is not uniform anymore, which is different from those in a continuous strategy space.

Altogether, when the bidding strategy space is discrete, for both with and without bidding cap cases, the characterization of the Nash equilibrium depends on the parity of the reward size. Moreover, a key assumption for all our analyses is the random tie-breaking rule in which one of the player is randomly selected as the winner whenever there is a tie. However, in many real-life examples, a favor-one-sided tie-breaking rule instead of a random tie-breaking rule is implemented. For example, two firms compete for a market in which consumers have stickiness. If one of the firm is the incumbent, whenever its advertisement expenditure is no less than the opponent firm, it can occupy the market.

In the next proposition, we show that when this favor-one-sided tie-breaking rule is used, the equilibrium strategy is independent of the parity of the reward size.[4]

Proposition 5. *Let $\tau = \min\{Q_x, Q_y, C\}$. When $Q_x \geq Q_y$ and the tie-breaking rule favors player x, the Nash equilibria are characterized as follows:*

1. *For player x, $P_0^x = P_1^x = \cdots = P_{\tau-1}^x = \frac{1}{Q_y}$ and $P_\tau^x = \frac{Q_y-\tau}{Q_y}$.*
2. *For player y, $P_0^y = \frac{V_x}{Q_x}$, $P_1^y = \cdots = P_{\tau-1}^y = \frac{1}{Q_x}$ and $P_\tau^y = \frac{Q_x-\tau+1-V_x}{Q_x}$.*
3. *In all equilibria, $V_x \in [Q_x - \tau, Q_x - \tau + 1]$ and $V_y = 0$.*

Proof. See E. □

Compared to the analyses for Propositions 2 and 4, Proposition 5 suggests that the equilibrium bidding distribution becomes uniform again with this favor-one-sided tie-breaking rule. Specifically, player x bids uniformly in $\{0, 1 \cdots \tau - 1\}$. Player y bids uniformly in $\{1, 2 \cdots \tau - 1\}$, and the probabilities for her to bid 0 and τ respectively are both determined by V_x. In addition, the expected bids for player x is $\frac{\tau}{2Q_y}(2Q_y - \tau - 1)$ and it is $\frac{\tau}{2Q_x}(2Q_x - \tau + 1 - 2V_x)$ for player y.

3 Conclusion

In this paper, we characterize the Nash equilibrium in a two-player complete information all-pay auction with a discrete strategy space. Compared to a continuous strategy space, the set of equilibrium bidding strategies under a discrete strategy space is much larger, and the characterization of the Nash equilibrium depends on the parity of the reward size. When $Q = 2n$, for both homogeneous and heterogeneous bidders, the equilibrium is not unique anymore and a continuum of Nash equilibrium exists. When $Q = 2n + 1$, as shown in Schep (1995), the equilibrium for homogeneous bidders is symmetric and unique, whereas multiple Nash equilibria still exist for heterogeneous bidders. Furthermore, we extend our analysis to the case of bidding with a cap and find that the introduction of the bidding cap only affects the bidding probability in the upper end. Additionally, when a favor-one-sided tie-breaking rule is used, the equilibrium characterization becomes independent of the reward size.

Altogether, our results complete the theoretical analysis of complete information all-pay auctions in the two-player case. In future work, several modeling assumptions could be relaxed, such as extending the analysis to $n > 2$ players as well as relaxing the assumption of risk neutrality. Second, our equilibrium characterization depends on the parity of the reward size, unless a favor-one-sided tie-breaking rule is used. Therefore, it would be interesting to test the effect of reward size and the effect of the tie-breaking rule using laboratory experiments.

[4] Otsubo (2015) studies a special case where $Q_x = Q_y = Q$ and $C = \infty$.

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Online Appendices

First, Bouckaert et al. (1992) have shown that all equilibria are characterized by the following matrix format:

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \frac{1}{2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & \frac{1}{2} \end{vmatrix} \begin{vmatrix} P_0^i \\ P_1^i \\ P_2^i \\ \vdots \\ P_\tau^i \end{vmatrix} \leq \begin{vmatrix} \frac{V_j}{Q_j} \\ \frac{V_j+1}{Q_j} \\ \frac{V_j+2}{Q_j} \\ \vdots \\ \frac{V_j+\tau}{Q_j} \end{vmatrix} \tag{1}$$

where $i, j \in \{x, y\}$ and $i \neq j$. Additionally, $\tau = \min\{Q_x, Q_y, C\}$. Based on the the complementary slackness condition, we have:

$$P_0^i + \cdots + \tfrac{1}{2}P_n^i < \tfrac{V_j+n}{Q_j} \Rightarrow P_n^j = 0,$$

$$P_n^i > 0 \Rightarrow P_0^j + \cdots + \tfrac{1}{2}P_n^j = \tfrac{V_i+n}{Q_i}.$$

Additionally, $P_n^i \geq 0$ and $P_0^i + P_1^i + \cdots + P_\tau^i = 1$.

In the following lemma, we show that $\forall n \in \{1, 2, \ldots, \tau - 1\}$, the upper bound of P_n^i is always $\frac{2}{Q_j}$.

Lemma 1. $P_n^i \leq \frac{2}{Q_j}$.

Proof. Without loss of generality, we prove $P_n^x \leq \frac{2}{Q_y}$ by contradiction. If $P_n^x > \frac{2}{Q_y}$, together with $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{Q_{n+1}}^x \leq \frac{V_y+n+1}{Q_y}$, we must have $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{Q_n}^x < \frac{V_y+n}{Q_y}$. Consequently, $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{Q_{n-1}}^x < \frac{V_y+n-1}{Q_y}$. Based on the slackness condition, $P_{n-1}^y = P_n^y = 0$. Therefore, bidding n for player x is strictly dominated by bidding $n-1$, and it contradicts the assumption: $P_n^x > \frac{2}{Q_y}$. □

A Proof of Proposition 1

The proof of Proposition 1 are contained in the following lemmas. The first lemma shows that in all equilibria, both players bids Q with zero probability.

Lemma 2. $P_Q^x = P_Q^y = 0$.

Proof. We prove this by contradiction. First, if $P_Q^x > 0$, bidding Q for player y is strictly dominated by bidding zero, i.e., $P_Q^y = 0$. Second, as the expected payoff for player x to bid Q is 0, we must have $P_0^y = 0$. Together, we have:

$$P_1^y + \cdots + P_{Q-1}^y = 1. \tag{1}$$

$$P_1^y + \cdots + \frac{1}{2}P_{Q-1}^y \leq \frac{Q-1}{Q}. \tag{2}$$

Therefore, we have $P_{Q-1}^y \geq \frac{2}{Q}$. By Lemma 1, $P_{Q-1}^y = \frac{2}{Q}$ must hold, and Equation 1 becomes:

$$P_1^y + \cdots + P_{Q-2}^y = \frac{Q-2}{Q}. \tag{3}$$

Next, we show that under the assumption: $P^x_Q > 0$, $P^y_{Q-2} = 0$ must hold. Again, we prove this by contradiction.

If $P^y_{Q-2} > 0$, given Equation 3, we have $P^y_0 + P^y_1 + \cdots + \frac{1}{2}P^y_{Q-2} < \frac{Q-2}{Q}$, and it implies $P^x_{Q-2} = 0$. Moreover, $P^y_{Q-2} > 0$ implies $P^x_0 + P^x_1 + \cdots + P^x_{Q-3} + \frac{1}{2}P^x_{Q-2} = \frac{V_y+Q-2}{Q}$. Together, we have the following equation:

$$P^x_0 + P^x_1 + \cdots + P^x_{Q-3} = \frac{V_y + Q - 2}{Q}. \tag{4}$$

As $P^y_{Q-1} = \frac{2}{Q}$, by the complementary slackness condition, we have:

$$P^x_0 + P^x_1 + \cdots + P^x_{Q-3} + \frac{1}{2}P^x_{Q-1} = \frac{V_y + Q - 1}{Q}. \tag{5}$$

Combining Equations 4 and 5, we have $P^x_{Q-1} = \frac{2}{Q}$. Therefore, we have $P^x_0 + P^x_1 + \cdots + P^x_{Q-3} + P^x_{Q-1} = \frac{V_y+Q}{Q}$, and it suggests $P^x_Q = 0$.

Similarly, we can show that $P^y_Q = P^y_{Q-2} = \cdots = 0$ and $P^y_{Q-1} = P^y_{Q-3} = \cdots = \frac{2}{Q}$.

When $Q = 2n$, using the complementary slackness condition, we can easily show that $P^x_0 + P^x_1 \geq \frac{2}{Q}$, $P^x_2 + P^x_3 \geq \frac{2}{Q}, \cdots, P^x_{Q-2} + P^x_{Q-1} \geq \frac{2}{Q}$. Altogether, they suggest that $P^x_Q = 0$, which contradicts $P^x_Q > 0$.

When $Q = 2n+1$, $P^y_0 = \frac{1}{Q}$, and it contradicts $P^y_0 = 0$.

□

By Lemma 2, the matrix format (1) becomes:

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \frac{1}{2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & \frac{1}{2} \end{vmatrix} \begin{vmatrix} P^i_0 \\ P^i_1 \\ P^i_2 \\ \vdots \\ P^i_{Q-1} \end{vmatrix} \leq \begin{vmatrix} \frac{V_j}{Q} \\ \frac{V_j+1}{Q} \\ \frac{V_j+2}{Q} \\ \vdots \\ \frac{V_j+Q-1}{Q} \end{vmatrix} \tag{6}$$

The next lemma shows that all "$\leq$" in (6) must be "$=$".

Lemma 3. *There is no strict inequality in (6) .*

Proof. We prove Lemma 3 in the following steps.

1. We first show $P^i_0 + P^i_1 + \cdots + \frac{1}{2}P^i_{Q-1} = \frac{V_j+Q-1}{Q}$ by contradiction.

   If $P^i_0 + P^i_1 + \cdots + \frac{1}{2}P^i_{Q-1} < \frac{V_j+Q-1}{Q}$, $P^j_{Q-1} = 0$. Therefore, $V_i \geq 1$. Now we show $V_i = 1$ must hold.

   If $V_i > 1$, $P^i_{Q-1} = 0$ and $V_j > 1$. Based on (6) and the complementary slackness condition, it is easy to get $P^i_0 = P^i_1 = \cdots = P^i_{Q-2} = 0$. Therefore, we must have $V_i = 1$. Consequently, $P^j_0 + P^j_1 + \cdots + P^j_{Q-2} = 1$ and $P^j_0 + P^j_1 + \cdots + \frac{1}{2}P^j_{Q-2} \leq \frac{Q-1}{Q}$. Combining with Lemma 1, we have $P^j_{Q-2} = \frac{2}{Q}$.

   Next we show $P^j_{Q-3} = 0$ by contradiction.

   If $P^j_{Q-3} > 0$, $P^j_0 + P^j_1 + \cdots + \frac{1}{2}P^j_{Q-3} < \frac{Q-2}{Q}$, which implies $P^i_{Q-3} = 0$. Furthermore, the complementary slackness condition suggests that $P^i_0 + P^i_1 + \cdots + P^i_{Q-4} + \frac{1}{2}P^i_{Q-2} = \frac{V_j+Q-2}{Q}$ and

$P_0^i + P_1^i + \cdots + P_{Q-4}^i = \frac{V_j+Q-3}{Q}$. Altogether, $P_{Q-2}^i = \frac{2}{Q}$ must hold, and it implies $P_0^i + P_1^i + \cdots + P_{Q-2}^i = \frac{V_j+Q-1}{Q}$, which contradicts the prior assumption $P_0^i + P_1^i + \cdots + \frac{1}{2}P_{Q-1}^i < \frac{V_j+Q-1}{Q}$.

Similarly, it is easy to show that $P_{Q-1}^j = P_{Q-3}^j = \cdots = 0$ and $P_{Q-2}^j = P_{Q-4}^j = \cdots = \frac{2}{Q}$.

When $Q = 2n$, the complementary slackness condition suggests $P_0^i + P_1^i \le \frac{2}{Q}$, $P_2^i + P_3^i \le \frac{2}{Q}, \cdots$, $P_{Q-2}^i + P_{Q-1}^i \le \frac{2}{Q}$, $P_{Q-2}^i + P_{Q-1}^i < \frac{2}{Q}$, which contradicts $P_0^i + P_1^i + \cdots + P_{Q-1}^i = 1$.

When $Q = 2n+1$, $P_0^j = \frac{1}{Q}$. Based on the complementary slackness condition, $V_j = 0$, $P_0^i = P_2^i = \cdots = P_{Q-3}^i = 0$ and $P_1^i = P_3^i = \cdots = P_{Q-2}^i = \frac{2}{Q}$. Altogether, $P_Q^i = \frac{1}{Q}$, which is an contradiction to Lemma 2.

2. Next, we show $P_0^i + P_1^i + \cdots + \frac{1}{2}P_{Q-2}^i = \frac{V_j+Q-2}{Q}$ by contradiction.

   If $P_0^i + P_1^i + \cdots + \frac{1}{2}P_{Q-2}^i < \frac{V_j+Q-2}{Q}$, $P_{Q-2}^j = 0$. Combining with $P_0^i + P_1^i + \cdots + P_{Q-2}^i = \frac{2V_j+Q-2}{Q}$, we have $P_{Q-2}^i > \frac{2V_j}{Q} \ge 0$. Based on the complementary slackness condition, $P_0^j + P_1^j + \cdots + P_{Q-3}^j = \frac{V_i+Q-2}{Q}$. As $P_0^j + P_1^j + \cdots + P_{Q-3}^j = \frac{2V_i+Q-2}{Q}$, we must have $V_i = 0$.

   Similar to the proof in Step 1, we can show that $P_{Q-1}^j = P_{Q-3}^j = \cdots = \frac{2}{Q}$ and $P_{Q-2}^j = P_{Q-4}^j = \cdots = 0$.

   When $Q = 2n$, the complementary slackness condition suggests that $P_0^i + P_1^i \ge \frac{2}{Q}$, $P_2^i + P_3^i \ge \frac{2}{Q}$, $\cdots$, $P_{Q-2}^i + P_{Q-1}^i > \frac{2}{Q}$, which contradict $P_0^i + P_1^i + \cdots + P_{Q-1}^i = 1$.

   When $Q = 2n+1$, $P_0^j = \frac{1}{Q}$, and it contradicts $\frac{1}{2}P_0^j \le \frac{2V_i}{Q} = 0$.

3. Repeating the proof procedure in the prior steps, we can show that except $\frac{1}{2}P_0^i \le \frac{V_j}{Q}$, no strict inequality in (6) exists. Based on the complementary slackness condition and $P_0^j + P_1^j + \cdots + P_{Q-1}^j = 1$, it is also easy to show that $\frac{1}{2}P_0^i = \frac{V_j}{Q}$ must hold.

□

By Lemma 3, we solve the matrix format (6) and get the following equilibrium solutions:

$$\begin{aligned} P_0^i &= P_2^i = \cdots = \frac{2V_j}{Q}, \\ P_1^i &= P_3^i = \cdots = \frac{2-2V_j}{Q}. \end{aligned}$$

When $Q = 2n$, we have:

$$\begin{aligned} P_0^i &= P_2^i = \cdots = P_{Q-2}^i = \frac{2V_j}{Q}, \\ P_1^i &= P_3^i = \cdots = P_{Q-1}^i = \frac{2-2V_j}{Q}, \end{aligned}$$

where $V_j \in [0,1]$.

When $Q = 2n+1$, to guarantee $P_0^i + P_1^i + \cdots + P_Q^i = 1$, we must have $V_i = V_j = 1/2$. Therefore, we have:

$$P_0^i = P_1^i = \cdots = P_{Q-1}^i = \frac{1}{Q}.$$

B Proof of Proposition 2

Assuming $Q_x - Q_y > 1$, neither players would bid higher than $Q_y + 1$. Consequently, $V_x \geq Q_x - Q_y - 1 > 0$. The proof of Proposition 2 are contained in the following two lemmas, and we first show that player x never bids zero.

Lemma 4. $P_0^x = 0$.

Proof. We prove this by contradiction.

If $P_0^x > 0$, then $V_y > 0$. Therefore, $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{Q_y}^x < \frac{V_y + Q_y}{Q_y}$. Based on the complementary slackness condition, $P_{Q_y}^y = 0$.

Moreover, the complementary slackness condition also suggests $P_0^y = \frac{2V_x}{Q_x}$ and $P_1^y \leq \frac{2(1-V_x)}{Q_x}$. Consequently, $V_x \leq 1$. Together with $V_x \geq Q_x - Q_y - 1$, we get $Q_x \leq Q_y + 2$, and because of $Q_x - Q_y > 1$, $Q_x = Q_y + 2$ must hold.

When $Q_x = Q_y + 2$, $V_x = 1$. Therefore, $P_0^y + P_1^y + \cdots + \frac{1}{2}P_{Q_y}^y \leq \frac{1+(Q_x-2)}{Q_x}$. Together with $P_0^y + P_1^y + \cdots + P_{Q_y}^y = 1$, we get $P_{Q_y}^y \geq \frac{2}{Q_x}$, which is an contradiction to $P_{Q_y}^y = 0$. □

The next lemma shows that the probability of bidding zero for player y is greater than zero.

Lemma 5. $P_0^y > 0$.

Proof. We prove this by contradiction.

First, we show that if $P_0^y = 0$, $P_1^y > 0$ must hold.

If $P_1^y = 0$, together with $P_0^y = 0$, Lemma 4 and the complementary slackness condition suggests that $P_0^x = P_1^x = 0$. Moreover, by Lemma 1, $\frac{1}{2}P_2^x < \frac{V_x+2}{Q_y}$, and $P_2^y = 0$. Consequently, $P_2^x = 0$.

Furthermore, by doing this repeatedly, we get $P_0^x = P_1^x = \cdots = P_{Q_y-1}^x = 0$ and $P_0^y = P_1^y = \cdots = P_{Q_y-1}^y = 0$. Then $P_{Q_y}^x = P_{Q_y}^y = 1$, which is an contradiction to $V_y \geq 0$.

When $P_1^y > 0$, the complementary slackness condition suggests that $P_1^x = \frac{2V_y+2}{Q_y} > 0$. Combing with $P_0^y = 0$, we have $P_1^y = \frac{2V_x+2}{Q_x}$. Based on Lemma 1, $V_x = 0$, and it contradicts $V_x > 0$. □

Lemmas 4 and 5 suggest $V_y = 0$. Now we present the proof of Proposition 2 for $Q_y = 2n$ and $Q_y = 2n + 1$ separately.

1. When $Q_y = 2n$,

   Proof. First, we show that $P_{Q_y}^x = 0$.

   If $P_{Q_y}^x > 0$, by the complementary slackness condition, $P_{Q_y}^y = 0$, which implies $P_{Q_y-1}^y > 0$ and $V_x = Q_x - Q_y$.

   Furthermore, similar to the proof in Lemma 2, we have $P_0^y = \frac{Q_x - Q_y}{Q_x}$, $P_1^y = P_3^y = \cdots = P_{Q_y-1}^y = \frac{2}{Q_x}$ and $P_2^y = P_4^y = \cdots = P_{Q_y}^y = 0$.

   Consequently, by the complementary slackness condition and Lemma 4, $P_0^x = P_2^x = \cdots = P_{Q_y}^x = 0$, which is an contradiction to $P_{Q_y}^x > 0$.

   Conditional on $V_y = 0$ and $P_{Q_y}^x = 0$, the matrix format (1) for player x becomes:

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & \frac{1}{2} \end{vmatrix} \begin{vmatrix} P_1^x \\ P_2^x \\ \vdots \\ P_{Q_y-1}^x \end{vmatrix} \leq \begin{vmatrix} \frac{1}{Q_y} \\ \frac{2}{Q_y} \\ \vdots \\ \frac{Q_y-1}{Q_y} \end{vmatrix} \tag{7}$$

Second, we show that there is no strict inequality in (7) by the following steps.

(a) If $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{Q_y-1}^x < \frac{Q_y-1}{Q_y}$, $P_{Q_y-1}^x > \frac{2}{Q_y}$, which is an contradiction to Lemma 1. Therefore, $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{Q_y-1}^x = \frac{Q_y-1}{Q_y}$ and $P_{Q_y-1}^x = \frac{2}{Q_y}$.

(b) If $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{Q_y-2}^x < \frac{Q_y-2}{Q_y}$, $P_{Q_y-2}^x > 0$ and $P_{Q_y-2}^y = 0$. Based on the complementary slackness condition and Lemma 1, $P_{Q_y-1}^y = P_{Q_y-3}^y = \frac{2}{Q_x}$.
Next, we show $P_{Q_y-4}^y = 0$ by contradiction. If $P_{Q_y-4}^y > 0$, the complementary slackness condition and Lemma 1 imply that $P_{Q_y-3}^x = \frac{2}{Q_y}$ and $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{Q_y-3}^x = \frac{Q_y-3}{Q_y}$, which contradict the assumption: $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{Q_y-2}^x < \frac{Q_y-2}{Q_y}$. Therefore, $P_{Q_y-4}^y = 0$. Based on the complementary slackness condition and Lemma 1, $P_{Q_y-5} = P_{Q_y-1}^y = P_{Q_y-3}^y = \frac{2}{Q_x}$. Repeatedly, we get $P_{Q_y-1}^y = P_{Q_y-3}^y = \cdots = P_1^y = \frac{2}{Q_x}$.
As $P_{Q_y-1}^y = P_{Q_y-3}^y = \cdots = P_1^y = \frac{2}{Q_x}$, the complementary slackness condition implies that $P_0^x + P_1^x \geq \frac{2}{Q_y}$, $P_2^x + P_3^x \geq \frac{2}{Q_y}$, $\cdots$, $P_{Q_y-2}^x + P_{Q_y-1}^x > \frac{2}{Q_y}$, which contradicts $P_0^x + P_1^x + \cdots + P_{Q-1}^x = 1$.

(c) Similar to the prior two steps, we show that no strict inequality exists in (7).

Now we solve (7) and it yields $P_0^x = P_2^x = \cdots = P_{Q_y}^x = 0$, $P_1^x = P_3^x = \cdots = P_{Q_y-1}^x = 2/Q_y$. Additionally, $P_{Q_y-1}^x = 2/Q_y > 0$ implies $V_x \leq Q_x - Q_y + 1$.

Furthermore, by the complementary slackness condition, the matrix format (1) for player y becomes:

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \frac{1}{2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \frac{1}{2} & 0 \\ 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \end{vmatrix} \begin{vmatrix} P_0^y \\ P_1^y \\ P_2^y \\ \vdots \\ P_{Q_y-1}^y \\ P_{Q_y}^y \end{vmatrix} \begin{matrix} \leq \\ = \\ \leq \\ \\ = \\ \leq \end{matrix} \begin{vmatrix} \frac{V_x}{Q_x} \\ \frac{V_x+1}{Q_x} \\ \frac{V_x+2}{Q_x} \\ \vdots \\ \frac{V_x+Q_y-1}{Q_x} \\ \frac{V_x+Q_y}{Q_x} \end{vmatrix} \tag{8}$$

Since $Q_x - Q_y > 1$, it is easy to show that there exists at least one strict inequality. Therefore, the equilibrium strategy for player y is determined by the linear system (8).

Furthermore, $P_0^y = (V_x+1)/Q_x$, $P_{Q_y}^y = (Q_x - Q_y + 1 - V_x)/Q_x$, $P_2^y = \cdots = P_{Q_y-2}^y = 2/Q_x$, $P_1^y = P_3^y = \cdots = P_{Q_y-1}^y = 0$ is a special solution of the linear system (8). The number of free parameters is $Q_y/2$, and the range of the parameters is determined by the inequality in (8). In addition, the homogeneous solution yields $P_1 + 2P_2 + \cdots + Q_y P_{Q_y} = 0$.

□

2. When $Q_y = 2n+1$,

Proof. First, we show that $P_{Q_y}^y = 0$.

If $P_{Q_y}^y > 0$, similar to the proof in Lemma 2, we get $P_1^x = P_3^x = \cdots = P_{Q_y}^x = 0$ and $P_2^x = P_4^x = \cdots = P_{Q_y-1}^x = \frac{2}{Q_y}$. However, it suggests $P_0^x = \frac{1}{Q_y}$, which contradicts Lemma 4.

As $P^y_{Q_y} = 0$, the matrix format (1) for player y becomes:

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & \frac{1}{2} \end{vmatrix} \begin{vmatrix} P^y_0 \\ P^y_1 \\ \vdots \\ P^y_{Q_y-1} \end{vmatrix} \le \begin{vmatrix} \frac{V_x}{Q_x} \\ \frac{V_x+1}{Q_x} \\ \vdots \\ \frac{V_x+Q_y-1}{Q_x} \end{vmatrix} \quad (9)$$

Second, we show that one and only one strict inequality exists in (9) by the following steps.

(a) If $P^y_0 + P^y_1 + \cdots + \frac{1}{2}P^y_{Q_y-1} < \frac{V_x+Q_y-1}{Q_x}$, $P^x_{Q_y-1} = 0$. Similar to the proof in Lemma 2, $P^x_1 = P^x_3 = \cdots = P^x_{Q_y-2} = \frac{2}{Q_y}$ and $P^x_{Q_y} = \frac{1}{Q_y}$. Consequently, $V_x = Q_x - Q_y$.

By inserting $V_x = Q_x - Q_y$ into $P^y_0 + P^y_1 + \cdots + \frac{1}{2}P^y_{Q_y-1} < \frac{V_x+Q_y-1}{Q_x}$, $P^y_0 + P^y_1 + \cdots + \frac{1}{2}P^y_{Q_y-1} < \frac{Q_x-1}{Q_x}$, and it suggests $P^y_{Q_y-1} > \frac{2}{Q_x}$, which is an contradiction to Lemma 1.

Therefore, $P^y_0 + P^y_1 + \cdots + \frac{1}{2}P^y_{Q_y-1} = \frac{V_x+Q_y-1}{Q_x}$, which implies $P^y_{Q_y-1} = \frac{2(Q_x-Q_y+1-V_x)}{Q_x}$. Furthermore, Lemma 1 implies that $V_x \ge Q_x - Q_y$.

(b) If $P^y_0 + P^y_1 + \cdots + \frac{1}{2}P^y_{Q_y-2} < \frac{V_x+Q_y-2}{Q_x}$, $P^y_{Q_y-2} > \frac{2(Q_y-Q_x+V_x)}{Q_x} \ge 0$ and $P^x_{Q_y-2} = 0$. By Lemma 1, $V_x < Q_x - Q_y + 1$, which implies $P^y_{Q_y-1} > 0$. Moreover, using the complementary slack condition, we recursively get $P^x_2 = \cdots = P^x_{Q_y-1} = \frac{2}{Q_y}$, and it implies $P^x_0 = \frac{1}{Q_y}$, which is an contradiction to Lemma 4.

(c) Similar to Steps 1 and 2, we can show that except $\frac{1}{2}P^y_0 \le \frac{V_x}{Q_x}$, all "≤" in (9) must be "=". Then solving (9) yields $P^y_0 = \frac{Q_x-Q_y+1}{Q_x}$, $P^y_1 = P^y_3 = \cdots = P^y_{Q_y-2} = \frac{2(Q_y-Q_x+V_x)}{Q_x}$ and $P^y_2 = P^y_4 = \cdots = P^y_{Q_y-1} = \frac{2(Q_x-Q_y+1-V_x)}{Q_x}$.

(d) In Step 1, we have shown that $V_x \ge Q_x - Q_y$. When $V_x > Q_x - Q_y$, $P^y_1 = P^y_3 = \cdots = P^y_{Q_y-2} = \frac{2(Q_y-Q_x+V_x)}{Q_x} > 0$. Based on the complementary slack condition and Lemma 4, we can easily show that $P^x_0 = P^x_2 = \cdots = P^x_{Q_y-1} = P^x_{Q_y} = 0$ and $P^x_1 = P^x_3 = \cdots = P^x_{Q_y-2} = \frac{2}{Q_y}$, which contradicts $P^x_0 + P^x_1 + \cdots + P^x_{Q_y} = 1$.

As $V_x = Q_x - Q_y$ and $Q_x > Q_y + 1$, $\frac{1}{2}P^y_0 = \frac{Q_x-Q_y+1}{2Q_x} < \frac{V_x}{Q_x}$.

Now we solve (9) and it yields $P^y_1 = P^y_3 = \cdots = P^y_{Q_y-2} = 0$ and $P^y_2 = P^y_4 = \cdots = P^y_{Q_y-1} = \frac{2}{Q_x}$. Furthermore, given the complementary slackness condition, the matrix format (1) for player x becomes:

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \frac{1}{2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \frac{1}{2} & 0 \\ 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \end{vmatrix} \begin{vmatrix} P^x_1 \\ P^x_2 \\ P^x_3 \\ \vdots \\ P^x_{Q_y-1} \\ P^x_{Q_y} \end{vmatrix} \begin{vmatrix} \le \\ = \\ \le \\ \\ = \\ \le \end{vmatrix} \begin{vmatrix} \frac{1}{Q_y} \\ \frac{2}{Q_y} \\ \frac{3}{Q_y} \\ \vdots \\ \frac{Q_y-1}{Q_y} \\ \frac{Q_y}{Q_y} \end{vmatrix} \quad (10)$$

As $Q_y = 2n+1$, there exists at least one strict inequality. Therefore, the equilibrium strategy for player x is determined by the linear system (10). Furthermore, $P_0^x = P_2^x = \cdots = P_{Q_y-1}^x = 0$, $P_1^x = P_3^x = \cdots = P_{Q_y-2}^x = 2/Q_y$ and $P_{Q_y}^x = 1/Q_y$ is a special solution of the linear system (10). The number of free parameters is $(Q_y-1)/2$, and the range of the parameters is determined by the inequality in (10). Additionally, the homogeneous solution yields $P_1 + 2P_2 + \cdots + Q_y P_{Q_y} = 0$.

□

C Proof of Proposition 3

As the equilibrium solution for $Q = C+1$ is the same as that for no cap case, we focus on $Q > C+1$ in the following proof. The proof of Proposition 3 are contained in the following lemmas.

We first show that bidding C is not dominated by other strategies.

Lemma 6. $P_0^i + P_1^i + \cdots + \frac{1}{2}P_C^i = \frac{V_j+C}{Q}$.

Proof. We discuss the following three different cases:

1. If $P_0^x + P_1^x + \cdots + \frac{1}{2}P_C^x < \frac{V_y+C}{Q}$ and $P_0^y + P_1^y + \cdots + \frac{1}{2}P_C^y < \frac{V_x+C}{Q}$, $P_C^x = P_C^y = 0$. Therefore, $P_{C-1}^i \geq \frac{2(Q-C+1-V_j)}{Q}$, and Lemma 1 implies $V_i \geq Q - C > 1$.

   When $V_x > 1$, $\frac{1}{2}P_0^y < \frac{V_x}{Q}$ must hold. Otherwise, $P_1^y \leq \frac{2-2V_x}{Q} < 0$. Conditional on $\frac{1}{2}P_0^y < \frac{V_x}{Q}$, $P_0^y = 0$. Combining $P_0^y = 0$ with Lemma 1, we get $\frac{1}{2}P_1^y < \frac{1+V_x}{Q}$, which implies $P_1^x = 0$. Repeatedly, we have $P_0^x = P_1^x = \cdots = P_C^x = 0$.

2. If $P_0^x + P_1^x + \cdots + \frac{1}{2}P_C^x = \frac{V_y+C}{Q}$ and $P_0^y + P_1^y + \cdots + \frac{1}{2}P_C^y < \frac{V_x+C}{Q}$, $P_C^x = 0$. Combining $P_C^x = 0$ with $P_0^x + P_1^x + \cdots + P_{C-1}^x = 1$ and $P_0^x + P_1^x + \cdots + P_{C-1}^x = \frac{V_y+C}{Q}$, we have $V_y = Q - C > 1$, and it implies $P_0^y = 0$.

   Similar to the proof in Lemma 2, $P_{C-1}^x = P_{C-3}^x = \cdots = \frac{2}{Q}$, $P_C^x = P_{C-2}^x = \cdots = 0$ and $P_0^x > 0$. Together with $P_0^y = 0$, $V_x = 0$ must hold.

   When $C = 2n$, $P_{C-1}^y = \frac{2}{Q}$ and $P_0^y + P_1^y + \cdots + \frac{1}{2}P_{C-1}^y = \frac{C-1}{Q}$, which contradicts $P_0^y + P_1^y + \cdots + \frac{1}{2}P_C^y < \frac{C}{Q}$.

   When $C = 2n+1$, using the complementary slackness condition, we have $P_0^y + P_1^y \leq \frac{2}{Q}$, $P_2^y + P_3^y \leq \frac{2}{Q}$, $\cdots$, $P_{C-1}^y + P_C^y < \frac{2}{Q}$, which contradicts $P_0^x + P_1^x + \cdots + P_C^x = 1$.

3. Similar to the previous case, we show that $P_0^x + P_1^x + \cdots + \frac{1}{2}P_C^x < \frac{V_y+C}{Q}$ and $P_0^y + P_1^y + \cdots + \frac{1}{2}P_C^y = \frac{V_x+C}{Q}$ can not hold.

□

In the following lemma, we show the boundary of V_x and V_y under the assumption of $Q \leq 2(C+1)$.

Lemma 7. *When* $Q \leq 2(C+1)$, $V_x, V_y \in [0,1]$.

Proof. Similarly, we discuss the following three different cases:

1. If $V_x > 1$ and $V_y > 1$, $P_0^x = P_1^x = \cdots = P_{C-1}^x = 0$ and $P_0^y = P_1^y = \cdots = P_{C-1}^y = 0$. However, $V_x = V_y = \frac{1}{2}Q - C \leq 1$, which contradicts $V_x > 1$ and $V_y > 1$.

2. If $V_x > 1$ and $V_y \in [0,1]$, $P_0^x = 0$.

   First, we show that $V_y = 0$ by contradiction.

   Assuming $V_y > 0$, given $P_0^x = 0$, we have $\frac{1}{2}P_0^x < \frac{V_y}{Q}$. Furthermore, the complementary slackness condition suggests $P_0^y = 0$.

Next, Lemma 1 suggests that $\frac{1}{2}P_1^x < \frac{V_y+1}{Q}$, which implies $P_1^y = 0$. Consequently, $P_0^y + \frac{1}{2}P_1^y < \frac{V_x}{Q}$, which suggests $P_1^x = 0$.

Repeatedly, we get $P_C^x = P_C^y = 1$, which contradicts $V_x > V_y$.

Next, since $V_y = 0$, $P_C^x = \frac{2Q-2C}{Q}$ and $P_0^x + P_1^x + \cdots + P_{C-1}^x = \frac{2C-Q}{Q}$. Therefore, $P_0^x + P_1^x + \cdots + \frac{1}{2}P_{2C-Q+1}^x < \frac{2C-Q+1}{Q}$, and it implies $P_{2C-Q+1}^y = 0$.

Similarly, $P_{2C-Q+1}^y = P_{2C-Q+2}^y = \cdots = P_{C-1}^y = 0$. Combining it with $P_C^y = \frac{2Q-2C-2V_x}{Q}$, we have $P_0^y + P_1^y + \cdots + P_{2C-Q}^y = \frac{2V_x+2C-Q}{Q}$. However, $P_0^y + P_1^y + \cdots + P_{2C-Q}^y \leq \frac{V_x+2C-Q+1}{Q}$. Altogether, they suggest $V_x \leq 1$, which is an contradiction to $V_x > 1$.

3. Similar to the prior case, we show that $V_x \in [0, 1]$ and $V_y > 1$ can not hold.

□

In the following lemma, we show that neither players bids $2C - Q + 1, 2C - Q + 2, \cdots, C - 1$ with positive probability.

Lemma 8. $P_{2C-Q+1}^i = P_{2C-Q+2}^i = \cdots = P_{C-1}^i = 0$.

Proof. We separate the proof for the following two cases.

1. When $Q \leq 2(C+1)$

   First, Lemma 6 suggests that $P_C^i = \frac{2Q-2C-2V_j}{Q}$, and consequently $P_0^i + P_1^i + \cdots + P_{C-1}^i = \frac{2V_j+2C-Q}{Q}$. Combining it with Lemma 7, $P_0^i + P_1^i + \cdots + \frac{1}{2}P_{2C-Q+2}^i < \frac{V_j+2C-Q+2}{Q}$ must hold, and it suggests $P_{2C-Q+2}^j = 0$.

   Similarly, we have $P_{2C-Q+2}^i = \cdots = P_{C-1}^i = 0$.

   Next, we show $P_{2C-Q+1}^i = 0$ by contradiction.

   If $P_{2C-Q+1}^i > 0$, $P_0^j + P_1^j + \cdots + P_{2C-Q+1}^j = \frac{2V_i+2C-Q}{Q}$ and $P_0^j + P_1^j + \cdots + \frac{1}{2}P_{2C-Q+1}^j = \frac{V_i+2C-Q+1}{Q}$. They suggest $P_{2C-Q+1}^j = \frac{2(V_i-1)}{Q}$. Consequently, $V_i = 1$ and $P_{2C-Q+1}^j = 0$ must hold.

   Similar to the proof in Lemma 2, we have $P_C^j = \frac{2Q-2C-2}{Q}$, $P_{2C-Q}^j = P_{2C-Q-2}^j = \cdots = \frac{2}{Q}$ and $P_{2C-Q-1}^j = P_{2C-Q-3}^j = \cdots = 0$.

   When $Q = 2n$, $P_0^j = \frac{2}{Q}$. Consequently, $P_0^i = \frac{2V_j}{Q}$, which implies $P_1^i + P_2^i + \cdots + P_{2C-Q+1}^i = \frac{2C-Q}{Q}$. However, given the complementary slack condition, we have $P_1^i + P_2^i \geq \frac{2}{Q}$, $P_3^i + P_4^i \geq \frac{2}{Q}$, $\cdots$, $P_{2C-Q-1}^i + P_{2C-Q}^i \geq \frac{2}{Q}$. Consequently, $P_1^i + P_2^i + \cdots + P_{2C-Q}^i \geq \frac{2C-Q}{Q}$, which contradicts $P_{2C-Q+1}^i > 0$.

   When $Q = 2n + 1$, $P_0^j = \frac{1}{Q} < \frac{2V_i}{Q}$, and it suggests that $P_0^i = 0$ and $V_j = 0$. Furthermore, $P_1^i = P_3^i = \cdots = P_{2C-Q}^i = \frac{2}{Q}$ and $P_C^i = \frac{2Q-2C}{Q}$, which contradicts $P_0^i + P_1^i + \cdots + P_C^i = 1$.

2. When $Q > 2(C+1)$

   In this case, the expected payoff for bidding C is at least $\frac{1}{2}Q - C > 1$. Therefore $V_x > 1$ and $V_y > 1$. Similar to the proof in Lemma 6, we have $P_0^x = P_1^x = \cdots = P_{C-1}^x = 0$ and $P_0^y = P_1^y = \cdots = P_{C-1}^y = 0$.

□

Lemma 8 implies that when $2C - Q < 0$, a unique Nash equilibrium exists, and both players bid C.

When $2C - Q \geq 0$, the matrix format (1) becomes:

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \frac{1}{2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & \frac{1}{2} \end{vmatrix} \begin{vmatrix} P_0^i \\ P_1^i \\ P_2^i \\ \vdots \\ P_{2C-Q}^i \end{vmatrix} \leq \begin{vmatrix} \frac{V_j}{Q_j} \\ \frac{V_j+1}{Q_j} \\ \frac{V_j+2}{Q_j} \\ \vdots \\ \frac{V_j+2C-Q}{Q_j} \end{vmatrix} \tag{11}$$

The next lemma shows that all "$\leq$" in (11) must be "$=$".

Lemma 9. *No strict inequality exists in (11) .*

Proof. Lemma 3 suggests that when there is no cap, $P_{2C-Q+1}^i + P_{2C-Q+2}^i + \cdots + P_{Q-1}^i = \frac{Q-1-2C+Q-1}{2} \cdot \frac{2}{Q} + \frac{2-2V_j}{Q} = \frac{2Q-2C-2V_j}{Q}$. Therefore, by Lemma 3, when $P_0^i + P_1^i + \cdots + P_{2C-Q}^i = 1 - \frac{2Q-2C-2V_j}{Q}$, there is no strict inequality in

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \frac{1}{2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & \frac{1}{2} \end{vmatrix} \begin{vmatrix} P_0^i \\ P_1^i \\ P_2^i \\ \vdots \\ P_{2C-Q}^i \end{vmatrix} \leq \begin{vmatrix} \frac{V_j}{Q_j} \\ \frac{V_j+1}{Q_j} \\ \frac{V_j+2}{Q_j} \\ \vdots \\ \frac{V_j+2C-Q}{Q_j} \end{vmatrix} \tag{12}$$

Because $P_C^i = \frac{2Q-2C-2V_j}{Q}$, this is the same as (11).

□

By Lemma 9, we solve the matrix format (11) as follows:

$$\begin{aligned} P_0^i &= P_2^i = \cdots = \frac{2V_j}{Q}, \\ P_1^i &= P_3^i = \cdots = \frac{2-2V_j}{Q}. \end{aligned}$$

When $Q = 2n$, we have:

$$\begin{aligned} P_0^i &= P_2^i = \cdots = P_{2C-Q}^i = \frac{2V_j}{Q}, \\ P_1^i &= P_3^i = \cdots = P_{2C-Q-1}^i = \frac{2-2V_j}{Q}. \end{aligned}$$

and $P_C^i = \frac{2Q-2C-2V_j}{Q}$, where $V_j \in [0, 1]$.

When $Q = 2n + 1$, $V_i = V_j = 1/2$ must hold to guarantee $P_0^i + P_1^i + \cdots + P_Q^i = 1$. Therefore, we have:

$$P_0^i = P_1^i = \cdots = P_{2C-Q}^i = \frac{1}{Q}.$$

and $P_C^i = \frac{2Q-2C-1}{Q}$.

D Proof of Proposition 4

As the proof is similar to Propositions 2 and 3, we only present the sketch of the proof and omit the details. Without loss of generality, we assume $C < Q_y - 1$, $Q_x - Q_y > 1$ and $Q_y \leq 2C$.

Proof. We prove it in the following steps.

1. Similar to the proof in Lemma 6, $P_0^i + P_1^i + \cdots + \frac{1}{2}P_C^i = \frac{V_j+C}{Q_j}$.
2. Similar to the proof in Lemma 4, $P_0^x = 0$.
3. Similar to the proof in Lemma 5, $P_0^y > 0$ and $V_y = 0$.
4. Similar to the proof in Lemma 8, $P_{2C-Q_y+1}^i = P_{2C-Q_y+1}^i = \cdots = P_{C-1}^i = 0$.
5. When $Q_y = 2n$, we can solve the equilibrium strategy for player x:

$$\begin{aligned}
&P_0^x = P_2^x = \cdots = P_{2C-Q_y}^x = 0,\\
&P_1^x = P_3^x = \cdots = P_{2C-Q_y-1}^x = 2/Q_y,\\
&P_{2C-Q_y+1}^x = P_{2C-Q_y+2}^x = \cdots = P_{C-1}^x = 0,\\
&P_C^x = 2(Q_y - C)/Q_y.
\end{aligned}$$

   The equilibrium strategy for player y is determined by a linear system with special solution: $P_0^y = \frac{1+V_x}{Q_x}$, $P_1^y = P_3^y = \cdots = P_{2C-Q_y-1}^y = 0$, $P_2^y = P_4^y = \cdots = P_{2C-Q_y-2}^y = \frac{2}{Q_x}$, $P_{2C-Q_y}^y = \frac{Q_y-Q_x+1+V_x}{Q_x}$, $P_C^y = \frac{2Q_x-2C-2V_x}{Q_x}$, $V_x \in [Q_x - Q_y - 1, Q_x - Q_y + 1]$. Additionally, the number of free parameters is: $\frac{2C-Q_y}{2}$.

6. When $Q_y = 2n + 1$, we can solve the equilibrium strategy for player y:

$$\begin{aligned}
&P_0^y = (Q_x - Q_y + 1)/Q_x,\\
&P_1^y = P_3^y = \cdots = P_{2C-Q_y}^y = 0,\\
&P_2^y = P_4^y = \cdots = P_{2C-Q_y-1}^y = 2/Q_x,\\
&P_{2C-Q_y+1}^y = P_{2C-Q_y+2}^y = \cdots = P_{C-1}^y = 0,\\
&P_C^y = 2(Q_y - C)/Q_x.
\end{aligned}$$

   The equilibrium strategy for player x is determined by a linear system with special solution: $P_0^x = 0$, $P_2^x = \cdots = P_{2C-Q_y-1}^x = 2/Q_y$, $P_3^x = P_5^x = \cdots = P_{2C-Q_y}^x = 0$, $P_C^x = 2(Q_y - C)/Q_y$.

   Additionally, the number of free parameters is: $\frac{2C-Q_y-1}{2}$.

□

E Proof of Proposition 5

First, we define $\tau = \min\{Q_x, Q_y, C\}$. Since the tie-breaking rule favors player x, all equilibria are characterized by the following matrix format:

$$\begin{vmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & 0 \end{vmatrix} \begin{vmatrix} P_0^x \\ P_1^x \\ P_2^x \\ \vdots \\ P_\tau^x \end{vmatrix} \leq \begin{vmatrix} \frac{V_y}{Q_y} \\ \frac{V_y+1}{Q_y} \\ \frac{V_y+2}{Q_y} \\ \vdots \\ \frac{V_y+\tau}{Q_y} \end{vmatrix} \tag{13}$$

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & 1 \end{vmatrix} \begin{vmatrix} P_0^y \\ P_1^y \\ P_2^y \\ \vdots \\ P_\tau^y \end{vmatrix} \leq \begin{vmatrix} \frac{V_x}{Q_x} \\ \frac{V_x+1}{Q_x} \\ \frac{V_x+2}{Q_x} \\ \vdots \\ \frac{V_x+\tau}{Q_x} \end{vmatrix} \tag{14}$$

The proof of Proposition 5 are contained in the following lemmas.

First, we show that $\forall n \in \{1, 2, \ldots, \tau - 1\}$, the upper bound of P_n^x is always $\frac{1}{Q_y}$.

Lemma 10. $P_n^x \leq \frac{1}{Q_y}$.

Proof. We prove it by contradiction. If $P_n^x > \frac{1}{Q_y}$, together with $P_0^x + P_1^x + \cdots + P_n^x \leq \frac{V_y+n+1}{Q_y}$, we get $P_0^x + P_1^x + \cdots + P_{n-1}^x < \frac{V_y+n}{Q_y}$, and it implies $P_n^y = 0$. Consequently, bidding n for player x is strictly dominated by bidding $n-1$, and consequently $P_n^x = 0$, which contradicts $P_n^x > \frac{1}{Q_y}$. ☐

The next lemma shows that all "$\leq$" in (13) must be "$=$".

Lemma 11. *No strict inequality exists in (13) .*

Proof. We prove it in the following steps.

1. First, we show $P_0^x = \frac{V_y+1}{Q_y}$ by contradiction.

   Assuming $P_0^x < \frac{V_y+1}{Q_y}$, Lemma 10 suggests that $P_0^x + P_1^x < \frac{V_y+2}{Q_y}$. By using Lemma 10 repeatedly, all "$\leq$" in (13) must be "$<$". Given the complementary slackness condition, $P_1^y = P_2^y = \cdots = P_\tau^y = 0$. Furthermore, $P_1^x = P_2^x = \cdots = P_\tau^x = 0$. Altogether, $P_0^x = P_0^y = 1$. However, as $Q_y > 1$, $P_0^x = P_0^y = 1$ is not an equilibrium.

2. Next, we show $P_0^x + P_1^x + \cdots + P_n^x = \frac{V_y+n+1}{Q_y}$ by induction.

   First, we assume that $\forall m \in \{0, 1, \ldots, n-1\}$, $P_0^x + P_1^x + \cdots + P_n^x < \frac{V_y+n+1}{Q_y}$ and $P_0^x + P_1^x + \cdots + P_m^x = \frac{V_y+m+1}{Q_y}$.

By using the complementary slackness condition and Lemma 10, we solve (13) and get:

$$
\begin{aligned}
P_0^x &= \frac{1+V_y}{Q_y}, \\
P_1^x &= P_2^x = \cdots = P_{n-1}^x = \frac{1}{Q_y}, \\
P_n^x &= \frac{Q_y - n - V_y}{Q_y}.
\end{aligned}
$$

When $V_y > 0$, $P_0^y = 0$ must hold. Combining $P_0^y = 0$ with $P_0^x > 0$, we have $V_x = 0$.
In contrast, $P_{n+1}^y = P_{n+2}^y = \cdots = P_\tau^y = 0$ suggests that the expected payoff for player x for bidding n is at least $Q_x - n > 0$, and it contradicts $V_x = 0$.
Altogether, $V_y = 0$ and $P_n^x = \frac{Q_y - n}{Q_y}$. By Lemma 10, $n = Q_y - 1$.
However, when $n = Q_y - 1$, $P_0^x + P_1^x + \cdots + P_n^x = \frac{n+1}{Q}$, which contradicts the assumption that $P_0^x + P_1^x + \cdots + P_n^x < \frac{V_y + n + 1}{Q_y}$.

□

By Lemma 11, we solve (13) and get:

$$
\begin{aligned}
P_0^x &= \frac{1+V_y}{Q_y}, \\
P_1^x &= P_2^x = \cdots = P_{\tau-1}^x = \frac{1}{Q_y}, \\
P_\tau^x &= \frac{Q_y - \tau - V_y}{Q_y}.
\end{aligned}
$$

Next, we show $V_y = 0$ by contradiction.

If $V_y > 0$, $P_0^y = 0$. Combining $P_0^y = 0$ with $P_0^x > 0$, we get $V_x = 0$. Furthermore, since the expected payoff for player x to bid τ is at least $Q_x - \tau$, we have $Q_x = \tau$. However, as $Q_y \leq Q_x$, $Q_x = \tau$ suggests that $P_\tau^x = \frac{Q_y - \tau - V_y}{Q_y} < 0$, which is an contradiction.

Altogether, the equilibrium strategy for player x is:

$$
\begin{aligned}
P_0^x &= P_1^x = P_2^x = \cdots = P_{\tau-1}^x = \frac{1}{Q_y}, \\
P_\tau^x &= \frac{Q_y - \tau}{Q_y}.
\end{aligned}
$$

Using the complementary slackness condition, we solve (14) and the equilibrium strategy for player y is:

$$
\begin{aligned}
P_0^y &= \frac{V_x}{Q_x}, \\
P_1^y &= P_2^y = \cdots = P_{\tau-1}^y = \frac{1}{Q_x}, \\
P_\tau^y &= \frac{Q_x - \tau + 1 - V_x}{Q_x}.
\end{aligned}
$$

Additionally, $V_x \in [Q_x - \tau, Q_x - \tau + 1]$.