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A New Approach of Stochastic Dominance for Ranking Transformations on the Discrete Random Variable

Jianwei Gao and Feng Zhao

Abstract

This paper develops some new stochastic dominance (SD) rules for ranking transformations on a random variable, which is the first time to study ranking approach for transformations on the discrete framework. By using the expected utility theory, the authors first present a sufficient condition for general transformations by first degree SD (FSD), and further develop it into the necessary and sufficient condition for the monotonic transformations. For the second degree SD (SSD) case, the authors divide the monotonic transformations into increasing and decreasing ones, and respectively derive the necessary and sufficient conditions for the two situations. For two different discrete random variables with the same possible states, they obtain the sufficient and necessary condition for FSD and SSD, respectively. The feature of the new SD rules is that each FSD condition is represented by the transformation functions and each SSD condition is characterized by the transformation functions and the probability distributions of the random variable. This is different from the existing SD approach where they are described by cumulative distribution functions. In this way, the authors construct a new theoretical paradigm for transformations on the discrete random variable. Finally, a numerical example is provided to show the effectiveness of the new SD rules.

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Keywords Stochastic dominance; transformation; utility theory; insurance

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1 Introduction

In real world, many human activities in insurance and financial fields induce risk transformations. For example, we assume that an investor owns a house where X denotes the value of the house (a random variable). The investor can insure the house with various levels of deductions. By choosing two different deduction policies, the investor creates different transformations $m(X)$ and $n(X)$. Then an interesting question occurs: which deduction policy (transformation) dominates the other? In other words, how to find an effective approach for ranking these transformations so as to choose the beneficial one?

Stochastic dominance (SD) is the most famous approach to compare pairs of prospects. Presented in the context of expected utility theory, SD approach has the advantage that it requires no restrictions on probability distributions. Well-known specifications of SD are first degree SD (FSD) and second degree SD (SSD), which attract by far the most attention in SD research. Due to the advantage mentioned above, SD approach has been proved to be a powerful tool for ranking random variables and employed in various areas of finance, decision analysis, economics and statistics (See e.g., Meyer, 1989; Levy, 1992, 2006; Chiu 2005; Li 2009; Blavaskyy, 2010, 2011; Tzeng et al.,2013; Loomes et al., 2014; Tsetlin et al., 2015). Unfortunately, SD approach for ranking random variables is inefficient to rank transformations on random variables because it relies on the cumulative distribution functions (CDFs) of random variables, which are hard to calculate. In other words, SD approach cannot be used directly to rank transformations.

To the best of the authors' knowledge, there are only three papers studying SD rules for transformations on the continuous random variables under the conditions of increasing, continuous, and piecewise differentiable transformations (see, Meyer,1989; Brooks and Levy, 1989; Levy,1992). However, there is little research which focuses on ranking transformations on the discrete framework. It should be pointed out that the outcomes of transformations for continuous random variables cannot be extended directly to the discrete system. In real life, we notice that the discrete random variables are ubiquitous and even the continuous random variables should be discretely handled in many cases, so it is significant to find new SD criteria for ranking transformations on the discrete random variable.

The paper aims to develop some new SD rules for ranking transformations on the discrete random variable, which is the first time to investigate the ranking approach for the discrete system. In order to construct such theoretical paradigm, we start from the FSD rule by applying the expected utility theory, and derive a sufficient condition (See Theorem 1). We further extend the sufficient condition into the sufficient and necessary condition by introducing the monotonicity of transformations, this FSD relation is determined only by the difference between the compared transformation functions (See Theorem 2). For the case of SSD, we first divide the monotonic transformation functions into increasing case and decreasing situation, respectively. For the increasing transformation functions, we respectively develop a sufficient condition and a sufficient and necessary condition for SSD by means of the transformation functions and the probability distributions of the random variable (See Theorem 3 and Theorem 4). For the decreasing transformation functions, we also respectively obtain a sufficient condition and a sufficient and necessary condition for SSD, while these dominant conditions are different from the increasing situations (See Theorem 5 and Theorem 6). In addition, for two different discrete random variables with the same possible states, we provide the sufficient and necessary conditions for FSD and SSD, respectively (See Theorem 7).

Compared with the existing SD rules, the advantages of the new SD rules we derived are as follows: (1) the new SD rules can rank transformations on a discrete random variable, while the existing SD rules do not work; (2) the new SD rules make us avoid the tedious computation of CDFs, whereas this can not be done in the existing SD rules. In this sense, the new theoretical paradigm we derived can be regarded as a useful complement to the existing SD theory. Finally, a numerical example is provided to show the effectiveness of the new SD rules.

The rest of this paper is organized as follows. Section 2 reviews the existing SD rules. Section 3 and Section 4 present the SD rules of transformations by FSD and SSD, respectively. Section 5 makes a comparison between the new SD rules and the existing SD rules. Section 6 gives a numerical example to show the efficiency of the new SD method and Section 7 draws the conclusions.

2 Preliminaries

This section introduces the definition of stochastic dominance, and the SD rules for transformation on the continuous random variable.

Let X and Y be two random variables with support in the finite interval $[a,b]$, and their CDFs will be denoted by $F(x)$ and $G(x)$, respectively. Define $F^{(n)}(x)=\int_a^x F^{(n-1)}(x)dx$ ($n=2,3,\cdots$)with $F^{(1)}(x)=F(x)$, and define $G^{(n)}(x)$ similarly. Moreover, we denote U_n as the class containing all the functions u with $(-1)^{k+1}u^{(k)}\geq 0\ (k=1,2,\cdots,n)$.

Definition 1. (Levy, 1992) (i) X dominates Y by FSD if $F(x)\leq G(x)$ for any real number x;

(ii) X dominates Y by SSD if $F^{(2)}(x)\leq G^{(2)}(x)$ for any real number x;

(iii) X dominates Y by n th degree SD ($n\geq 3$) if

$$F^{(k)}(b)\leq G^{(k)}(b) \quad \text{for} \quad k=1,2,\cdots,n, \tag{1}$$

and

$$F^{(n)}(x)\leq G^{(n)}(x) \quad \text{for all} \quad a\leq x\leq b. \tag{2}$$

The SD rules and the relevant class of preferences U_k are related in the following way:

X dominates Y by FSD if and only if $E[u(X)]\geq E[u(Y)]$ for any $u\in U_1$.

X dominates Y by SSD if and only if $E[u(X)] \ge E[u(Y)]$ for any $u \in U_2$.

X dominates Y by n th degree SD ($n \ge 3$) if and only if $E[u(X)] \ge E[u(Y)]$ for any $u \in U_n$ and $F^{(k)}(b) \le G^{(k)}(b)$ for $k = 1, 2, \cdots, n$.

Integral conditions (1) and (2) mean that SD approach relies on CDFs of the random variables, and it is inefficient to rank transformations on the random variable. To overcome this shortcoming, Meyer (1989) proposes the following results.

Lemma 1. (Meyer, 1989) Given a continuous random variable X with the density $f(x)$ and support in the interval $[a,b]$. If $m(x)$ and $n(x)$ are non-decreasing, continuous and piecewise differential functions, then

(i) the transformed random variable $m(X)$ dominates $n(X)$ by FSD if and only if

$$\{m(x) - n(x)\} f(x) \ge 0 \quad \text{for all } x \text{ in } [a,b], \tag{3}$$

(ii) the transformed random variable $m(X)$ dominates $n(X)$ by SSD if and only if

$$\int_a^x \{m(t) - n(t)\} f(t) dt \ge 0 \quad \text{for all } x \text{ in } [a,b]. \tag{4}$$

Lemma 1 provides the FSD and SSD rules, which are only valid for non-decreasing, continuous and piecewise differential functions, and these SD rules cannot be directly applied to ranking transformations on the discrete random variable. However, in the real world, the discrete random variables are ubiquitous and even the continuous random variables should be discretely handled in many cases, so it is significant to find SD criteria for ranking transformations on the discrete random variable. Considering that FSD and SSD have more practical implication than higher degree SD rules, this paper will focus on FSD and SSD rules in the remaining part of the paper.

3 Dominance Conditions for FSD

Let X be a discrete random variable whose prospects are characterized by $\{p_1, x_1; \cdots, p_n, x_n\}$ with $x_1 < x_2 < \cdots < x_n$ and support in the finite interval $[a,b]$. Assume that two transformed random variables $m(X)$ and $n(X)$ are denoted as $\{p_1, m(x_1); \cdots, p_n, m(x_n)\}$ and $\{p_1, n(x_1); \cdots, p_n, n(x_n)\}$, or shortly as $\{p_1, m_1; \cdots, p_n, m_n\}$ and $\{p_1, n_1; \cdots, p_n, n_n\}$, respectively. Then, an interesting question arises: given by a discrete random variable X and two transformed random variables $m(X)$ and $n(X)$, under what conditions will one transformation dominate the other for a given order?

Since Lemma 1 is only suitable for transformations on the continuous random variable, and it is invalid for the discrete random variable case, we need to develop new SD rules for transformations on the discrete random variable.

We will first discuss the FSD conditions for ranking $m(X)$ and $n(X)$ by expected utility theory.

That is, when is $E[u(m(X)]$ larger than or equal to $E[u(n(X)]$ for all increasing utility functions?

Theorem 1. The transformed random variable $m(X)$ dominates $n(X)$ by FSD if

$$m_i \geq n_i \text{ for all } i = 1, 2, \cdots, n. \tag{5}$$

Proof. For any utility function $u(x) \in U_1$, we have

$$E[u(m(X)] - E[u(n(X)] = \sum_{i=1}^{n} [u(m_i) - u(n_i)] p_i = \sum_{i=1}^{n} u'(\xi_i)(m_i - n_i) p_i \geq 0 .$$

In order to better understand the meaning of Theorem 1, Fig. 1 shows its graphical illustration. Different from the existing SD rules, the areas $S_1, S_2, \cdots, S_n$ in Fig. 1 are derived by the difference between $m(x)$ and $n(x)$ at x_i ($i = 1, 2, \cdots, n$) multiplying the corresponding probability of x_i, rather than the CDFs. Here we let $d(x) = m(x) - n(x)$, then $d_i = m_i - n_i$ denotes the difference of $m(x)$ and $n(x)$ at x_i ($i = 1, 2, \cdots, n$). Furthermore, we take n points $A_1, A_2, \cdots, A_n$ at the horizontal axis, such that $OA_1 = p_1$, $A_1A_2 = p_2$, $\cdots$, $A_{n-1}A_n = p_n$. We stipulate that a geometric area takes a positive value when it lies above the horizontal axis and takes a negative value when it lies below the horizontal axis, and then we can use the notation S_i to represent the algebra value of a rectangular area whose base and height are denoted as p_i and d_i, respectively ($i = 1, 2, \cdots, n$). From Fig.1 we notice that $m_i \geq n_i$ is equivalent to $S_i \geq 0$ ($i = 1, 2, \cdots, n$). Therefore, we can conclude that $m(X)$ dominates $n(X)$ by FSD if all the n areas $S_1, S_2, \cdots, S_n$ are all non-negative.

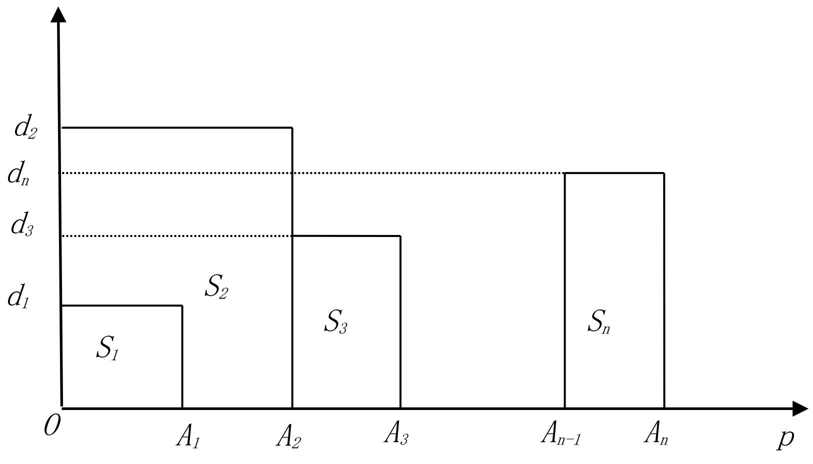

Figure 1: The graphical illustration about Theorem 1.

Theorem 1 presents a sufficient condition for determining FSD relations which only involves the transformation function. Apparently, it is much easier to compare the transformation functions than to compare the CDFs of transformed random variables.

However, we see that condition (5) is only a sufficient condition for $m(X)$ dominating $n(X)$ by FSD, rather than the necessary and sufficient condition. A natural question is whether condition (5) is also necessary? The following example shows that the answer is negative and the rest of this section is devoted to finding the necessary and sufficient condition.

Example 1. Suppose that a random variable X yields the outcomes -1, 0 and 1 with equal probabilities $\frac{1}{3}$. If $m(x)=1-x^2$, $n(x)=x^2-1$ and $p(x)=1-x$, then their probability distributions are shown as follows (See, Table 1).

Table 1: Probability distributions of $X, m(X), n(X)$ and $p(X)$.

X	-1	0	1
$m(X)$	0	1	0
$n(X)$	0	-1	0
$p(X)$	2	1	0
$\Pr(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(a) Table 1 reports that the transformed random variable $m(X)$ takes 0 with probability $\frac{2}{3}$ and 1 with probability $\frac{1}{3}$. Then, from Theorem 1, we see that $m(X)$ dominates X by FSD. However, it follows from Table 1 that $m(x_3)=0<1=x_3$. This fact shows that condition (5) is not necessary when the dominating transformation $m(x)$ is non-monotonic.

(b) From Table 1 we see that the transformed random variable $n(X)$ takes -1 with probability $\frac{1}{3}$ and 0 with probability $\frac{2}{3}$. Hence, by comparing CDFs of X and $n(X)$, from Theorem 1 we can deduce that X dominates $n(X)$ by FSD. On the other hand, Table 1 shows that $x_1=-1<0=n(x_1)$, which means that condition (5) is not necessary when the dominated transformation $n(x)$ is non-monotonic.

(c) We notice that from Table 1 the transformed random variable $p(X)$ takes values $0, 1$ and 2 with equal probability $\frac{1}{3}$. Then, the relative position of CDFs of X and $p(X)$ indicates that $p(X)$ dominates X by FSD, whereas we have $p(x_3)=0<1=x_3$, which indicates that condition (5) is still not necessary when one transformation is increasing and the other is decreasing.

Based on the above analysis, we conclude that condition (5) is not necessary if $m(x)$ and $n(x)$ are not comonotonic. The following theorem shows that condition (5) will be sufficient and necessary when the transformation functions are comonotonic.

Theorem 2. Suppose that the transformation functions $m(x)$ and $n(x)$ are comonotonic, we derive that the random variable $m(X)$ dominates $n(X)$ by FSD if and only if $m_i \geq n_i$ for all $i = 1, 2, \cdots, n$.

Proof. The sufficiency is obvious from Theorem 1. We then only need to prove the necessity, and this process is divided into the following three steps.

Step 1. Suppose that $m(x)$ and $n(x)$ are both increasing, we will use the reduction to absurdity to prove this conclusion.

If there exists a number j such that $1 \leq j \leq n$ and $m_j < n_j$, then we can define the utility function with the following form:

$$u(x) = \begin{cases} m_j, x < m_j \\ x, \quad m_j \leq x \leq n_j \\ n_j, x > n_j \end{cases}. \tag{6}$$

From the function (6), it is easy to see that $u(x) \in U_1$, $u(m_j) = m_j$, $u(n_j) = n_j$ and

(a) if $i < j$, then $u(m_i) = m_j$ and $u(m_i) \leq u(n_i)$ since m_j is the minimum of $u(x)$;

(b) if $i > j$, then from the monotonous property of $u(x)$ and $n(x)$, we get $u(n_i) = n_j$ and $u(m_i) \leq u(n_i)$ since n_j is the maximum of $u(x)$.

Therefore, from the analyses of (a) and (b), we can conclude that $u(m_i) \leq u(n_i)$ for all $1 \leq i \leq n$ and $i \neq j$. Combining it with the assumption $m_j < n_j$, we have that

$$E[u(m(X)] - E[u(n(X)] = \sum_{i=1}^{n} [u(m_i) - u(n_i)] p_i$$

$$= \sum_{i=1}^{j-1} [u(m_i) - u(n_i)] p_i + [u(m_j) - u(n_j)] p_j + \sum_{i=j+1}^{n} [u(m_i) - u(n_i)] p_i$$

$$\leq [u(m_j) - u(n_j)] p_j = (m_j - n_j) p_j < 0, \tag{7}$$

which is a contradiction with the assumption of $m(X)$ dominating $n(X)$ by FSD.

Step 2. We will prove that for any two random variables X and Y with the corresponding CDFs $F_X(x)$ and $F_Y(x)$, $-Y$ dominates $-X$ by FSD if and only if X dominates Y by FSD.

X dominates Y by FSD

$\Leftrightarrow \forall x \in R, F_X(x) \leq F_Y(x) \Leftrightarrow \forall x \in R, P(X \leq x) \leq P(Y \leq x)$

$\Leftrightarrow \forall x \in R, P(X < x) \leq P(Y < x) \Leftrightarrow \forall x \in R, P(-X > -x) \leq P(-Y > -x)$

$\Leftrightarrow \forall x \in R, 1 - P(-X > -x) \geq 1 - P(-Y > -x) \Leftrightarrow \forall x \in R, P(-Y \leq -x) \leq P(-X \leq -x)$

$\Leftrightarrow \forall x \in R, F_{-Y}(-x) \leq F_{-X}(-x) \Leftrightarrow \forall x \in R, F_{-Y}(x) \leq F_{-X}(x)$

$\Leftrightarrow -Y$ dominates $-X$ by FSD.

Step 3. When the transformations $m(x)$ and $n(x)$ are both decreasing, it is obvious that $-m(x)$ and $-n(x)$ are both increasing. Then from Step 1 and Step 2, we derive that

$m(X)$ dominates $n(X)$ by FSD

$\Leftrightarrow$ $-n(X)$ dominates $-m(X)$ by FSD

$\Leftrightarrow$ $-n(x_i) \geq -m(x_i)$ for all $i = 1, 2, \cdots, n$

$\Leftrightarrow$ $m(x_i) \geq n(x_i)$ for all $i = 1, 2, \cdots, n$.

Fig.1 can also illustrate the graphical presentation of Theorem 2, that is, $m(X)$ dominating $n(X)$ by FSD is equivalent to the situation that the n areas $S_1, S_2, \cdots, S_n$ are non-negative.

Remark 1. Theorem 2 provides a sufficient and necessary condition by introducing the monotonic condition of transformations, which only depends on the transformation functions, and in this way it presents a simple way for determining FSD relations between two comonotonic transformations. This situation is different from the case of the existing SD rules for FSD in which we need to take a tedious calculation to get the CDFs of the transformed random variables. Therefore, Theorem 2 plays an active part in dealing with realistic problems via the new SD rules we derived. The assumption of monotonous condition is appropriate because it seems to be a common feature of transformations in the fields of insurance and decision analysis (see Meyer 1989).

4 Dominance Conditions for SSD

In this section, we try to find some dominant rules for SSD. It is well known that SSD condition in the existing SD approach is more complicated than the case of FSD. In order to find SSD rules for ranking transformed discrete random variables, we will divide the monotonic transformation functions into increasing and decreasing ones, respectively.

Theorem 3. If $m(x)$ is increasing and

$$\sum_{i=1}^{k} (m_i - n_i) p_i \geq 0 \quad \text{for all} \quad k = 1, 2, \cdots, n, \tag{8}$$

then the transformed random variable $m(X)$ dominates $n(X)$ by SSD.

Proof. If $m(x_i) \geq n(x_i)$ holds for all $i = 1, 2, \cdots, n$, then by Theorem 1, we can derive that $m(X)$ dominates $n(X)$ by FSD. Hence, $m(X)$ dominates $n(X)$ by SSD via the hierarchical property of SD rules. Otherwise, let $\Phi = \{i \mid 1 \leq i \leq n, \mathit{and}\ m_i < n_i\} = \{j_1, j_2, \cdots, j_r\}$ denote the set of all the subscripts which violate the condition (1), where $1 \leq j_1 < j_2 < \cdots < j_r \leq n$.

According to the condition (8), we have that $\sum_{i=1}^{j_1} (m_i - n_i) p_i \geq 0$ and $\sum_{i=1}^{j_1 - 1} (m_i - n_i) p_i \geq (n_{j_1} - m_{j_1}) p_{j_1}$.

Then, there exist j_r numbers $p_{11}, p_{21}, \cdots, p_{j_r 1}$, such that

$$0 \le p_{i1} \le p_i (1 \le i < j_1),\ p_{i1} = 0 (j_1 \le i \le j_r), \tag{9}$$

and

$$\sum_{i=1}^{j_1 - 1} (m_i - n_i) p_{i1} = (n_{j_1} - m_{j_1}) p_{j_1} \quad . \tag{10}$$

Similarly, from $\sum_{i=1}^{j_2} (m_i - n_i) p_i \ge 0$, we get that

$$\sum_{i=1}^{j_1 - 1} (m_i - n_i) p_i + (m_{j_1} - n_{j_1}) p_{j_1} + \sum_{i=j_1+1}^{j_2 - 1} (m_i - n_i) p_i \ge (n_{j_2} - m_{j_2}) p_{j_2} . \tag{11}$$

Substituting Eqs. (9) and (10) into (11), we obtain that

$$\sum_{i=1}^{j_1 - 1} (m_i - n_i)(p_i - p_{i1}) + \sum_{i=j_1+1}^{j_2 - 1} (m_i - n_i)(p_i - p_{i1}) \ge (n_{j_2} - m_{j_2}) p_{j_2} .$$

Therefore, there exist j_r numbers $p_{12}, p_{22}, \cdots, p_{j_r 2}$ such that

$$0 \le p_{i2} \le p_i - p_{i1} (1 \le i < j_2),\quad p_{j_1 2} = 0,\quad p_{i2} = 0\ (j_2 \le i \le j_r), \tag{12}$$

and

$$\sum_{i=1}^{j_2 - 1} (m_i - n_i) p_{i2} = (n_{j_2} - m_{j_2}) p_{j_2} \quad . \tag{13}$$

Repeating this process for r times, we can conclude that there exist j_r numbers $p_{1r}, p_{2r}, \cdots, p_{j_r r}$ such that

$$0 \le p_{ir} \le p_i - p_{i(r-1)} (1 \le i < j_r),\ p_{j_{r-1} r} = p_{j_r r} = 0, \tag{14}$$

and

$$\sum_{i=1}^{j_r - 1} (m_i - n_i) p_{ir} = (n_{j_r} - m_{j_r}) p_{j_r} . \tag{15}$$

For any utility function $u(x) \in U_2$, by using the differential mean value theorem, we have that

$$\begin{aligned} & E[u(m(X)] - E[u(n(X)] \\ &= \sum_{i=1}^{n} [u(m_i) - u(n_i)] p_i = \sum_{i=1}^{n} u'(\xi_i)(m_i - n_i) p_i \\ &= \sum_{i=1}^{j_r} u'(\xi_i)(m_i - n_i) p_i + \sum_{i=j_r+1}^{n} u'(\xi_i)(m_i - n_i) p_i , \end{aligned} \tag{16}$$

where ξ_i is among m_i and n_i for all $1 \le i \le n$.

Obviously, $u'(\xi_i) \ge 0$ and $(m_i - n_i) p_i \ge 0$ for all $j_r + 1 \le i \le n$. Then, we have

$$E[u(m(X)]-E[u(n(X)]\geq\sum_{i=1}^{j_r}u'(\xi_i)(m_i-n_i)p_i$$

$$=\sum_{i=1}^{j_r-1}u'(\xi_i)(m_i-n_i)p_i+u'(\xi_{j_r})(m_{j_r}-n_{j_r})p_{j_r}$$

$$=\sum_{i=1}^{j_r-1}u'(\xi_i)(m_i-n_i)p_i-u'(\xi_{j_r})\sum_{i=1}^{j_r-1}(m_i-n_i)p_{ir}\,. \tag{17}$$

For any $1\leq i\leq j_r-1$, it follows from the increasing property of $m(x)$ and $n(x)$ that $m_i\leq m_{j_r-1}$ and $n_i\leq n_{j_r-1}$. Hence, we can derive that $\xi_i\leq\max\{m_{j_r-1},n_{j_r-1}\}=m_{j_r-1}\leq m_{j_r}$. Due to the decreasing property of $u'(x)$ and $m_{j_r}\leq\xi_{j_r}$, we can conclude that

$$E[u(m(X)]-E[u(n(X)]\geq[u'(m_{j_r})-u'(\xi_{j_r})]\sum_{i=1}^{j_r-1}(m_i-n_i)(p_i-p_{ir})\geq 0\,. \tag{18}$$

Fig. 2 makes a graphical explanation about Theorem 3. We investigate these n rectangular areas from left to right in Fig.2 and find that if the sum of the first k ($k=1,2,\cdots,n$) rectangular areas are all non-negative, then we can conclude that $m(X)$ dominates $n(X)$ by SSD. In other words, the condition of $\sum_{i=1}^{k}(m_i-n_i)p_i\geq 0$ is equivalent to the case of $\sum_{i=1}^{k}S_i\geq 0$ ($i=1,2,\cdots,n$, $k=1,2,\cdots,n$).

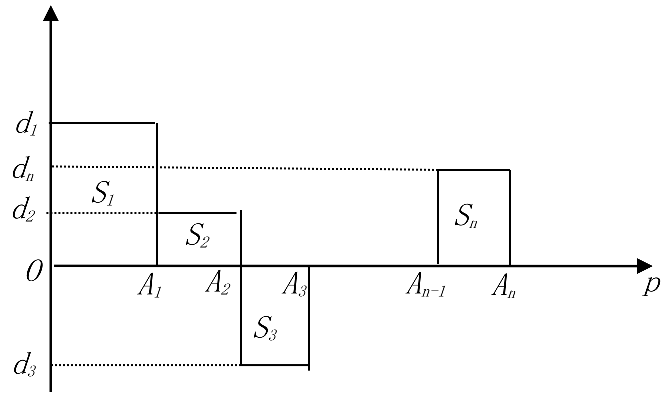

Figure 2: The graphical illustration about Theorem 3.

Remark 2. Theorem 3 presents a sufficient condition for one transformation dominating the other by SSD via the transformation functions and the probability distributions of the random variable, which makes us avoid the tedious computation of CDFs, whereas this can not be done in the existing SD theory.

Similar to Theorem 1, the condition (8) in Theorem 3 only indicates the sufficient condition rather than the necessary and sufficient condition. What we focus on is the sufficient and necessary condition, then, is the condition (8) necessary? The following example will answer this question.

Example 2. We assume that a random variable X yields the outcomes -1, 0 and 1 with equal probability $\frac{1}{3}$, and that $n(x)=x^2-1$ and $q(x)=-2x$. Their probability distributions are listed in Table 2.

Table 2. Probability distributions of X, $n(X)$ and $q(X)$.

X	-1	0	1
$n(X)$	0	-1	0
$q(X)$	2	0	-2
$\Pr(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

By analyzing the data in Table 2, we can make the following statements.

(a) We find that X dominates $n(X)$ by SSD via the hierarchical property of the SD rules. However, $[x_1-n(x_1)]p_1=-\frac{1}{3}<0$, which implies that condition (8) is not necessary if the dominating transformation is increasing and the dominated transformation is non-monotonic.

(b) We derive that X dominates $q(X)$ by SSD, while $[x_1-q(x_1)]p_1=-1<0$, meaning that the condition (8) is not necessary if the dominating transformation is increasing and the dominated transformation is decreasing.

(c) We have that $\sum_{i=1}^{k}[q(x_i)-x_i]p_i\geq 0 (k=1,2,3)$. However, step (b) tells us that X dominates $q(X)$ by SSD, rather than $q(X)$ dominates X. This implies that the condition (8) could not be sufficient if the dominating transformation is not increasing.

Based on the above statements (a)-(c), we conclude that if either of the two transformations is not increasing, the condition (8) will not be sufficient and necessary. This situation indicates that we should concentrate on the case of two increasing transformations.

Theorem 4. Suppose that $m(x)$ and $n(x)$ are both increasing. Then, $m(X)$ dominates $n(X)$ by SSD if and only if $\sum_{i=1}^{k}(m_i-n_i)p_i\geq 0$ for all $k=1,2,\cdots,n$.

Proof. The sufficiency of this theorem can be immediately obtained from Theorem 3. We then only need to prove the necessity.

Suppose that the condition of $\sum_{i=1}^{k}(m_i-n_i)p_i\geq 0$ (for all $k=1,2,\cdots,n$)is invalid, we then let S denote the set of all subscripts violating this condition, i.e., $S=\{k\mid\sum_{i=1}^{k}(m_i-n_i)p_i<0, 1\leq k\leq n\}$.

Let r be the minimum of S , we then have $m_r < n_r$. Define $u(x)=\begin{cases} x, \ x \le n_r \\ n_r, \ x > n_r \end{cases}$, we find that $u(x) \in U_2$.

According to the definition of $u(x)$ and the monotonicity of $m(x)$, $n(x)$ and $u(x)$, we conclude that

(a) if $1 \le i \le r$, we get $u(m_i)=m_i$, $u(n_i)=n_i$;

(b) if $r < i \le n$, we have $u(m_i) \le u(n_i)$ because $u(n_i)=n_r$ and n_r is the maximum of $u(x)$.

So,

$$E[u(m(X)]-E[u(n(X)] = \sum_{i=1}^{n}[u(m_i)-u(n_i)]p_i$$

$$= \sum_{i=1}^{r}[u(m_i)-u(n_i)]p_i + \sum_{i=r+1}^{n}[u(m_i)-u(n_i)]p_i$$

$$\le \sum_{i=1}^{r}(m_i-n_i)p_i < 0, \qquad (19)$$

which is a contradiction with $m(X)$ dominating $n(X)$ by SSD.

Theorem 4 can also be illustrated by Fig.2. Recall that in Fig. 2 the SSD conditions depend on the n rectangular areas, we investigate these rectangular areas from left to right and find that if the cumulative sum of the first $k(k=1,2,\cdots,n)$ rectangular areas is non-negative, then $m(X)$ dominates $n(X)$ by SSD; If the cumulative sum of the first $k(k=1,2,\cdots,n)$ rectangular areas is non-positive, then $n(X)$ dominates $m(X)$ by SSD; Otherwise, there is no SSD relation between $m(X)$ and $n(X)$.

Compared with Theorem 2, Theorem 4 reduces the requirement of the n rectangular areas. That is, any of the n rectangular areas except for the first one may take negative value, but the cumulative sum of the first $k(k=1,2,\cdots,n)$ rectangular areas (from left to right) must be non-negative. It also means that if $m(X)$ dominates $n(X)$ by FSD, then $m(X)$ dominates $n(X)$ by SSD, which is in accordance with the hierarchical property of SD rules.

Remark 3. Theorem 4 shows that we can determine the SSD relations between two transformed random variables by the transformation functions and the probability function of the original random variable, rather than CDFs of the transformed random variables. To be more precise, the signs of the tail conditional expectations $E[m(X)-n(X) \mid X \le x_k](k=1,2,\cdots,n)$ determine the SSD relations between the two transformed random variables. Compared with the existing SSD rule, CDFs of random variables are absent in Theorem 4, which make us avoid the tedious computation of CDFs. As a result, Theorem 4 provides us a simple way to determine the SSD relations only by means of the transformation functions and the probability function of the random variable.

Recall that when $m(x)$ and $n(x)$ are comonotonic, there exists a unified necessary and sufficient condition for FSD case (see Theorem 2). However, this statement is not valid for SSD. That is, we need to seek a necessary and sufficient condition for SSD if $m(x)$ and $n(x)$ are both decreasing. We first provide a sufficient condition as follows.

Theorem 5. If $m(x)$ is decreasing and

$$\sum_{i=k}^{n}(m_i - n_i)p_i \ge 0 \quad \text{for all} \quad k = 1, 2, \cdots, n, \tag{20}$$

then the transformed random variable $m(X)$ dominates $n(X)$ by SSD.

Proof. If $m(x_i) \ge n(x_i)$ holds for all $i = 1, 2, \cdots, n$, then by Theorem 1, we derive that $m(X)$ dominates $n(X)$ by FSD. We further conclude that $m(X)$ dominates $n(X)$ by SSD via the hierarchical property of SD rules. Otherwise, let $\Psi = \{i \mid 1 \le i \le n, \textit{ and } m_i < n_i\} = \{l_1, l_2, \cdots, l_r\}$ denote the set of all the subscripts which violate condition (20), where $1 \le l_r < \cdots < l_2 < l_1 \le n$.

According to condition (20), we have $\sum_{i=l_1}^{n}(m_i - n_i)p_i \ge 0$ and $\sum_{i=l_1+1}^{n}(m_i - n_i)p_i \ge (n_{l_1} - m_{l_1})p_{l_1}$. Then there exist $n - l_r$ numbers $p_{11}, \cdots, p_{(n-l_r)1}$, such that $0 \le p_{i1} \le p_i \; (l_1 < i \le n)$, $p_{i1} = 0 \, (l_r < i \le l_1)$ and

$$\sum_{i=l_1+1}^{n}(m_i - n_i)p_{i1} = (n_{l_1} - m_{l_1})p_{l_1}. \tag{21}$$

Similarly, we can derive $\sum_{i=l_2}^{n}(m_i - n_i)p_i \ge 0$, and then

$$\sum_{i=l_2+1}^{l_1-1}(m_i - n_i)p_i + (m_{l_1} - n_{l_1})p_{l_1} + \sum_{i=l_1+1}^{n}(m_i - n_i)p_i \ge (n_{l_2} - m_{l_2})p_{l_2},$$

or

$$\sum_{i=l_2+1}^{l_1-1}(m_i - n_i)(p_i - p_{i1}) + \sum_{i=l_1+1}^{n}(m_i - n_i)(p_i - p_{i1}) \ge (n_{l_2} - m_{l_2})p_{l_2}.$$

Hence, there exist $n - l_r$ numbers $p_{12}, p_{22}, \cdots, p_{(n-l_r)2}$, such that $0 \le p_{i2} \le p_i - p_{i1}(l_2 < i \le n)$, $p_{l_1 2} = 0$, $p_{i2} = 0 \, (l_r < i \le l_2)$ and

$$\sum_{i=l_2+1}^{n}(m_i - n_i)p_{i2} = (n_{j_2} - m_{j_2})p_{j_2}. \tag{22}$$

After repeating this process for r times, we can draw the conclusion that there exist $n - l_r$ numbers $p_{1r}, p_{2r}, \cdots, p_{(n-l_r)r}$ such that $0 \le p_{ir} \le p_i - p_{i(r-1)}(l_r < i \le n)$, $p_{l_{r-1}r} = 0$ and

$$\sum_{i=l_r+1}^{n}(m_i - n_i)p_{ir} = (n_{l_r} - m_{l_r})p_{l_r}. \tag{23}$$

For any utility function $u(x) \in U_2$, by using the differential mean value theorem, we have that

$$E[u(m(X)] - E[u(n(X)] = \sum_{i=1}^{n} [u(m_i) - u(n_i)] p_i$$

$$= \sum_{i=1}^{n} u'(\xi_i)(m_i - n_i) p_i \text{ (} \xi_i \text{ is among } m_i \text{ and } n_i \text{)}$$

$$= \sum_{i=1}^{l_r - 1} u'(\xi_i)(m_i - n_i) p_i + \sum_{i=l_r}^{n} u'(\xi_i)(m_i - n_i) p_i \,. \tag{24}$$

It is obvious that $u'(\xi_i) \geq 0$ and $(m_i - n_i) p_i \geq 0$ for all $1 \leq i \leq l_r - 1$. Then

$$E[u(m(X)] - E[u(n(X)] \geq \sum_{i=l_r}^{n} u'(\xi_i)(m_i - n_i) p_i$$

$$= \sum_{i=l_r+1}^{n} u'(\xi_i)(m_i - n_i) p_i + u'(\xi_{l_r})(m_{l_r} - n_{l_r}) p_{l_r}$$

$$= \sum_{i=l_r+1}^{n} u'(\xi_i)(m_i - n_i) p_i - \sum_{i=l_r+1}^{n} u'(\xi_{l_r})(m_i - n_i) p_{ir} \tag{25}$$

Since $u'(x)$ is decreasing, $\xi_i \leq \max\{m_{l_r+1}, n_{l_r+1}\} = m_{l_r+1} \leq m_{l_r}$ for any $l_r + 1 \leq i \leq n$ and $m_{l_r} \leq \xi_{l_r}$, we get that

$$E[u(m(X)] - E[u(n(X)] \geq [u'(m_{l_r}) - u'(\xi_{l_r})] \sum_{i=l_r+1}^{n} (m_i - n_i)(p_i - p_{ir}) \geq 0 \,. \tag{26}$$

In order to better understand the meaning of Theorem 5, Fig.3 shows the graphical illustration about Theorem 5. We investigate these n rectangular areas from right to left (just the opposite directions as what we do with Fig. 2.), and obtain that if the cumulative sum of the first $k(k = 1, 2, \cdots, n)$ rectangular areas is non-negative, then $m(X)$ dominates $n(X)$ by SSD. In other words, the condition of $\sum_{i=k}^{n} (m_i - n_i) p_i \geq 0$ can be expressed by $\sum_{i=k}^{n} \mathrm{S}_i \geq 0$ ($k = 1, 2, \cdots, n$).

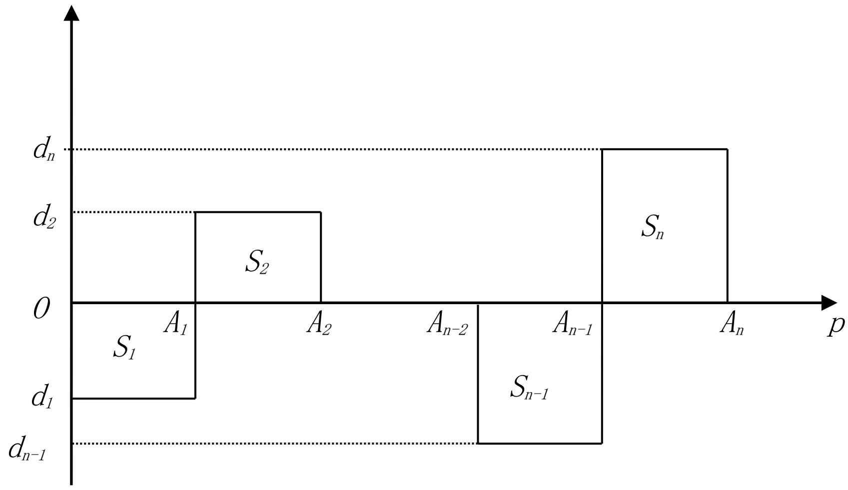

Figure 3: The graphical illustration about Theorem 5.

Remark 4. Compared with Theorem 3, a new condition (20) in Theorem 5 is introduced to substitute condition (8) in Theorem 3. This new formulation is suitable for discussing the SSD relations of the decreasing transformations.

Theorem 6. Suppose that $m(x)$ and $n(x)$ are both decreasing. Then, $m(X)$ dominates $n(X)$ by SSD if and only if $\sum_{i=k}^{n}(m_i - n_i)p_i \geq 0$ for all $k = 1, 2, \cdots, n$.

Proof. The sufficiency of this theorem can be immediately obtained from Theorem 5. We then only need to prove the necessity.

Suppose that the condition of $\sum_{i=k}^{n}(m_i - n_i)p_i \geq 0$ (for $k = 1, 2, \cdots, n$)is invalid, we then let T denote the set of all subscripts violating this condition, i.e., $T = \{k \mid \sum_{i=k}^{n}(m_i - n_i)p_i < 0, 1 \leq k \leq n\}$. In addition, we denote l as the maximum of T. Let $u(x) = \begin{cases} x, & x < n_l \\ n_l, & x \geq n_l \end{cases}$, we then have $u(x) \in U_2$. According to the definition of $u(x)$ and the monotonicity of $m(x), n(x)$ and $u(x)$, we conclude that

(a) if $1 \leq i < l$, we get $u(n_i) = n_l$ and $u(m_i) \leq u(n_i)$ since n_l is the maximum of $u(x)$;

(b) if $l \leq i \leq n$, we have $u(m_i) = m_i$, $u(n_i) = n_i$.

So,

$$E[u(m(X))] - E[u(n(X))] = \sum_{i=1}^{n}[u(m_i) - u(n_i)]p_i$$

$$= \sum_{i=1}^{l-1}[u(m_i) - u(n_i)]p_i + \sum_{i=l}^{n}[u(m_i) - u(n_i)]p_i$$

$$\leq \sum_{i=l}^{n}(m_i - n_i)p_i < 0, \tag{27}$$

which is a contradiction with the assumption that $m(X)$ dominates $n(X)$ by SSD.

The graphical illustration about Theorem 6 can also be expressed by Fig.3. We investigate these n rectangular areas in Fig. 3 from right to left. If the cumulative sum of the first $k (k = 1, 2, \cdots, n)$ rectangular areas is non-negative (resp. non-positive), then $m(X)$ dominates $n(X)$ (resp. $n(X)$ dominates $m(X)$) by SSD. Otherwise, there is no SSD relation between $m(X)$ and $n(X)$.

In addition, from Fig.1, Fig.2 and Fig.3, we find that Theorem 6 and Theorem 4 both reduce the condition requirement of Theorem 2 in which all the n rectangular areas are non-negative, that is, Theorem 6 and Theorem 4 argue that these n rectangular areas can take negative values, but the precondition is that the cumulative sum of the first $k (k = 1, 2, \cdots, n)$ rectangular areas should be non-negative. The difference between Theorem 6 and Theorem 4 is that Theorem 6 sums up the rectangular areas from right to left, while Theorem 4 does it from left to right. Undoubtedly, Theorem 6 and Theorem 4 are homogeneous in essence except that the former deals with increasing transformations while the latter copes with decreasing ones.

5 Comparison of the new SD Rules and the existing SD Rules

Notice that the existing SD rules only rank transformations of the continuously distributed random variables with the piecewise differentiable transformations, while the new SD rules we developed can rank transformations of the discrete random variable. Therefore, the new SD rules we developed can be regarded as an extension to the existing SD rules. In particular, the new SD rules are used to rank the transformed random variables, while the existing SD rules are applied to compare the risk of any two random variables, and the existing SD rules is ineffective when dealing with transformations on the same random variable. In this sense, the new SD rules developed in Section 3 and Section 4 remedy the weakness of the existing SD approach.

With regard to the expression form, the major difference is that the existing SD approach is presented in the framework of CDFs while the new SD rules are expressed by transformation functions and the probability function of the original random variable, so it avoids the tedious computation of CDFs and their integral.

To better understand these two types SD rules, Table 3 shows their main differences from three aspects: application scope, expression approach and the specific criteria.

Table 3. Comparison of the new SD rules and the existing SD rules.

	The new SD rules	The existing SD rules
Application scope	two transformed random variables $m(X)$ and $n(X)$	two ordinary random variables X and Y
Expression approach	transformation functions $m(x)$, $n(x)$ and probability function $p(x)$ of X	difference of CDFs of X and Y
FSD rule	$m(x_i) \ge n(x_i)$ for all $i = 1, 2, \cdots, n$ ($m(x)$ and $n(x)$ are comonotonic)	$F(x) \le G(x)$ for all real numbers x
SSD rule	(1) $\sum_{i=1}^{k}(m_i - n_i)p_i \ge 0$ for all $k = 1, 2, \cdots, n$ ($m(x)$ and $n(x)$ are both increasing) (2) $\sum_{i=k}^{n}(m_i - n_i)p_i \ge 0$ for all $k = 1, 2, \cdots, n$ ($m(x)$ and $n(x)$ are both decreasing)	$F^{(2)}(x) \le G^{(2)}(x)$ for all real numbers x

Remark 5. It should be pointed out that as a partial order relation, the existing SD can not rank all the random variables. However, as a screening device, the existing SD rules can divide the whole decision making set into an efficient set and an inefficient one, and then the decision maker can make decision under the efficient set (See, e.g., Li 2009; Blavaskyy, 2010, 2011; Tzeng et al.,2013; Loomes et al., 2014; Tsetlin et al., 2015). Such statements are also suitable to our new SD rules.

We further study the intrinsic links between the two types SD rules in the following. Note that the two transformations $m(X)$ and $n(X)$ on a random variable can also be regarded as two special random variables, then there should exist the corresponding random variables X and Y generating similar SD rules as $m(X)$ and $n(X)$ behave. Given two comonotonic transformation functions $m(X)$ and $n(X)$, if we apply these functions to a discrete random variable, we get the same pair of random variable $X=\{p_1,x_1;\cdots,p_n,x_n\}$ with $x_1<x_2<\cdots<x_n$ and $Y=\{p_1,y_1;\cdots,p_n,y_n\}$ with $y_1<y_2<\cdots<y_n$. On the contrary, if the above two discrete random variables have the same possible states, then they can be regarded as two increasing transformation functions applied to a discrete random variable. Following Theorem 2 and Theorem 4, we have the following conclusion:

Theorem 7. If the two discrete random variables $X=\{p_1,x_1;\cdots,p_n,x_n\}$ with $x_1<x_2<\cdots<x_n$ and $Y=\{p_1,y_1;\cdots,p_n,y_n\}$ with $y_1<y_2<\cdots<y_n$ have the same possible future states, then we have:

(1) X dominates Y by FSD if and only if $x_i\geq y_i$ for all $i=1,2,\cdots,n$;

(2) X dominates Y by SSD if and only if $\sum_{i=1}^{k}(x_i-y_i)p_i\geq 0$ for all $k=1,2,\cdots,n$.

Theorem 7 shows that the new SD rules can rank not only the transformations on a discrete random variable, but also different random variables with the same possible future states.

6 Numerical Example

This section provides a numerical example to illustrate how to determine the SD relations for transformations by using the new SD rules and further determine the efficient set of decision making set.

With the accelerating trend of population aging, the pension fund gap of China is becoming increasingly wide. For example, the World Bank stated that the size of China's 2001 to 2075 pension fund gap is 9.15 trillion Yuan (Wang et al. 2014). To effectively control the pension fund gap, one of the important approaches is to increase the investment return of the pension fund. However, in real world, the government pension sector of China can not directly invest the pension fund, but authorize several institutional investors to invest. Then, there is a principal-agent relationship between the government pension sector of China and each institutional investor. For a given return rate on investment, different institutional investors may provide different revenue-sharing proposals except for the common commission for the agency. Notice that as a screening device, the new SD rules can divide the whole decision set into an efficient and inefficient sets. Then, how to determine the efficient set of these revenue-sharing proposals is an interesting question.

Owing to the uncertainty of the stock market, there is an enormous risk for investing money in the stock market. Let X denote the rate of return on investing money in stock market and its probability distribution is shown as in Table 4. Assume that there are four institutional investors who can provide revenue-sharing proposals denoted by X, $m(X)$, $n(X)$ and $r(X)$, respectively. Table 5 shows the probability distributions of the four different revenue-sharing proposals.

Table 4. The probability distribution of the rate of return on investment X .

X	-50%	-10%	5%	20%	50%
$\Pr(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

We assume that revenue-sharing proposals $m(X)$, $n(X)$ and $r(X)$ are as follows:

$$m(x)=\begin{cases}\frac{2}{5}x, x\le 0\\ \frac{4}{5}x, x>0\end{cases}, \quad n(x)=\begin{cases}\frac{4}{5}x, x\le 0\\ x, \quad x>0\end{cases}, \quad \text{and} \quad r(x)=\frac{3}{5}x+5\% .$$

Here, $m(X)$ denotes that the institutional investor A will allocate four fifths of investment return rate to the government pension sector of China if the return rate is positive, and he will allocate two fifths of the the return rate to the government pension sector if the return rate is negative. While $n(X)$ represents the institutional investor B will distribute the total return rate to the government pension sector if the return rate is positive (meaning he only takes the commission in this situation), otherwise, he will distribute four fifths to the government pension sector if the return rate is negative. The meaning of $r(X)$ is that institutional investor C will allocate three fifths of the return rate plus five percent (regarded as risk-free interest rate) to the government pension sector. In addition, the implication of the revenue-sharing proposal X is that the institutional investor D will assign the total return rate to the government pension sector and he only takes the commission.

Table 5. Probability distributions of revenue-sharing proposals X, $m(X)$, $n(X)$, $r(X)$.

X	-50%	-10%	5%	20%	50%
$m(X)$	-20%	-4%	4%	16%	40%
$n(X)$	-40%	-8%	5%	20%	50%
$r(X)$	-25%	-1%	8%	17%	35%
$\Pr(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

To find the efficient set and the inefficient set, we need to perform pairwise comparisons. Note that the four revenue-sharing proposal functions are all increasing, so we can determine the FSD and SSD relations between them by Theorem 2 and Theorem 4, respectively. The following analyses (a1) to (a3) are to determine the FSD relations and (b1) to (b2) are to determine the SSD relations.

(a1) The revenue-sharing proposal X does not dominate any other revenue-sharing proposals by FSD because $x_1 = -50\%$ is far less than the values of m_1, n_1 and r_1; The proposal $m(X)$ does not dominate X by FSD because $m_4 < x_4$; The proposal $n(X)$ dominates X by FSD because $n_i \geq x_i$ holds for $i = 1, 2, \cdots, 5$. Therefore, X should be ruled out from the efficient set of the revenue-sharing proposals, and only $m(X), n(X)$ and $r(X)$ are kept in the efficient set.

(a2) There is no FSD relation between $m(X)$ and $n(X)$ because $m_1 > n_1$ while $m_5 < n_5$. Similarly, there is no FSD relation between $m(X)$ and $r(X)$ for $m_1 > r_1$ while $m_3 < r_3$. So, no revenue-sharing proposals in this step will be deleted from the efficient set.

(a3) There is no FSD relation between $n(X)$ and $r(X)$ because $n_1 < r_1$ but $n_5 > r_5$. Therefore, in this step, there will be no revenue-sharing proposals removed from the efficient set.

(b1) In this step, we will determine the SSD relations between $m(X)$ and $n(X)$.

Since $(m_1 - n_1)p_1 = 0.025$, $(m_2 - n_2)p_2 = 0.01$, $(m_3 - n_3)p_3 = -0.0025$, $(m_4 - n_4)p_4 = -0.01$ and $(m_5 - n_5)p_5 = -0.0125$,it is easy to verify that $\sum_{i=1}^{k}(m_i - n_i)p_i \geq 0$ for all $k = 1, 2, \cdots, 5$. Therefore, according to Theorem 4, we can conclude that $m(X)$ dominates $n(X)$ by SSD. That is, $n(X)$ should be ruled out from the efficient set, and only $m(X)$ and $r(X)$ are kept in the efficient set.

(b2) In this step, we will determine the SSD relations between $m(X)$ and $r(X)$.

On the one hand, from Table 5 we obtain that $(m_1 - r_1)p_1 = 0.00625$, $(m_2 - r_2)p_2 = -0.0075$ and $(m_1 - r_1)p_1 + (m_2 - r_2)p_2 = -0.00125 < 0$, so $m(X)$ does not dominate $r(X)$ by SSD. On the other hand, $r_1 < m_1$ implies that $r(X)$ does not dominate $m(X)$ by SSD.

Therefore, according to Theorem 2 and Theorem 4, we can determine the FSD relations and SSD relations among the four revenue-sharing proposals. The above computational procedure shows that $n(X)$ dominates X by FSD, and $m(X)$ dominates $n(X)$ by SSD, so X and $n(X)$ are ruled out from the efficient set. That is, $m(X)$ and $r(X)$ are kept in the efficient set, and X and $n(X)$ are included in the inefficient set.

7 Conclusion

Very often insurance activities induce transformations of an initial risk, which results in a new problem of how to rank transformations on the same random variable. This paper developed the new FSD and SSD rules for ranking transformations on a discrete random variable, which is the first time to consider the ranking approach for transformations on the discrete system. We start from the FSD rule by applying the expected utility theory, and derive the sufficient condition by FSD, and further extend the sufficient condition into the sufficient and necessary condition by introducing the monotonicity of transformations. For the case of SSD, we first divide the transformations into increasing and decreasing ones, and then respectively derive the necessary and sufficient conditions for the increasing and decreasing situations. For two different discrete random variables with the same possible states, we present the sufficient and necessary conditions for FSD and SSD, respectively.

The feature of our new SD rules lies in that each FSD condition is represented by the transformation functions and each SSD condition is characterized by the transformation functions and the probability distributions of the original random variable. This feature is different from the existing SD rules where FSD and SSD are described by CDFs. We notice that the SD rules proposed by Meyer (1989) are only suitable for the continuous random variables under the conditions with the piecewise differentiable transformations. Therefore, the existing SD rules can not be applied directly to the discrete system, in addition, it is very difficult for the existing SD rules to get the CDFs of the transformed random variables. However, the new SD rules we derived just overcame the above limitations. In this sense, the new theoretical paradigm we derived can be regarded as a useful complement for the existing SD approach.

The new SD rules we derived only study transformations on the same random variable. In the future study, it is significant to extend the ranking transformations on different random variables, and it would be interesting to consider higher-degree SD rules for transformations.

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