Size Effect, Neighbour Effect and Peripheral Effect in Cross-Border Tax Games

Xin Liu

Abstract

This paper analyses a game theoretic model of tax competition in a system where tax authorities are revenue optimisers and countries are differentiated by size. The model accommodates more than two countries. In equilibrium, larger countries set higher tax rates non-cooperatively. By applying the Hotelling linear model, this paper gives examples where the size effect, neighbourhood effect, and peripheral effect coexist and push up the tax rate in equilibrium.

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1. Introduction

A casual observation on Chinese internet shopping shows an interesting phenomenon. Type purchase intermediate/agent (in Chinese) in Taobao, the biggest Chinese and arguably international on-line retail platform, you may find over 70 million results.[2] The tariff and retail tax combined make the commodity price relatively higher than the international price. Hong Kong is another way to avoid the high tax. Mainland Chinese travel to Hong Kong not just for tourist pleasure, but also, if not more importantly, for shopping. The targets range from luxury accessories to basic food, even for milk. The price difference between mainland China and Hong Kong may be higher than the aggregate of plane tickets and overnight Hong Kong accommodation, which is considered to be the most expensive in the world. This kind of outward cross-border shopping is so big that the Chinese tax authority (custom and tax bureau) is thinking about lowering tariff and tax. Hong Kong is not the unique example. The generosity of Chinese customers becomes well known in Korea, Japan and most of neighbouring Asian countries. However, a simple fact is that the demand of China is bigger than any other Asian countries, and the difference will be even bigger in the foreseeable future. This is not a unique case in tax distribution.

In the European gasoline price index,[3] a country's petrol price seems related to its neighbors' geographic size, as well as its own. The Western European countries seem to have relatively higher petrol prices than the rest. France, which is the biggest country located at the middle of this group, has one of the highest prices, at 111 US Cents. The low-petrol-price countries agglomerate in the Eastern Europe, in which Romania has the lowest price at 53 US Cents, only half high of French petrol price. As the petrol price is largely related to the countries' taxation policies, we can have a glance at the European tax rates distribution in this petrol price index.[4] While the

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2 A random search showed 71469345 items on www.taobao.com, the top on-line shopping platform in China, at 11:37am., 2nd, April, 2014.
3 All data are given in Rietveld, van Woudenberg, 2005.
4 We do not include European countries, which are not member of EU, into this observation, e.g., Turkey and

countries own sizes' influence on the tax rates have been explored heavily, little attention has been devoted to the question of the neighbourhoods' influence on tax distribution.

Two questions are of particular interest. What is the country's size effect on fiscal policies? What is the neighbourhoods' effect on fiscal policies? In fact, combination of above questions is essence of geographic fiscal competition studies.

Kanbur and Keen (1993) initiated the study on commodity tax competition in a model of tax competition between sovereign governments. The authors argue that countries with higher population density tends to set higher tax rate. Ohsawa (1999) modelled a linear economy argues that geographically bigger country sets higher tax rate. Piertii (1999) and Wang (1999) introduced dynamic game into tax competition. Nielsen (2001) implanted transportation costs. Ohsawa and Koshizuka (2003) modelled the tax competition in a two-dimensional model. Bhaskar and To (2003) proposed a model of wage dispersion in oligopsonistic competition in the labour market, where they deduced the equilibrium wage distribution. The techniques of linear system analyses facilitate to seek unique PSNE in a model of n countries. Liu and Madden (2013) studied countries differentiated in both size and population density. Non-pure-strategy-equilibrium was found, and the results from mixed strategy equilibrium violated the previous observation that bigger country sets higher tax rate.

The purpose of this paper is to study tax competition amongst more than 2 tax authorities. We examine how the sizes of the countries (both in land sizes and in population) and their locations affect tax rates. The tax authorities of the countries are considered as players in tax game. We are interested in the answers to the two essential questions above: What is the country's size effect on fiscal policies? What is

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Russia. This is because their citizens cannot enjoy free cross-border traveling or shopping. We draw non-Schengen-Agreement Member states from this observation, e.g., the UK and the Republic of Ireland. Because visa is still needed for their citizens' European travel.
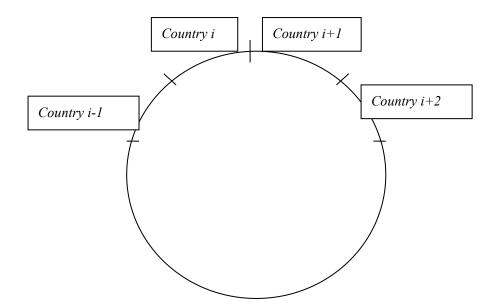
the neighbourhoods' effect on fiscal policies?

The rest of this paper is organized as follows. In section2, we formulate our general Salop tax competition model. Section 3 analyses the equilibrium. Section 4 models a Hotelling linear economy, and studies the coexistence of size effect, neighbour effect and peripheral effect. Section 5 concludes this paper.

2   Model

We consider a circular world, where n ($n \geq 2$) countries are located on the circumference of a circle. Those countries divide the circular world into n adjacent and non-overlapping segments. The segment of $[0, L_0]$ represent the land of country 0, $[L_0, L_1]$ represent the land of country 1, $[L_i, L_{i+1}]$ represent the land of country i, and so on and so forth. Each country has two neighbours, one to each side, i.e., country i is neighbouring country $i+1$ and country $i-1$, country 0 is neighbouring country $n-1$ and country 1. The population of country i is $A_i > 0$. The population is uniformly distributed over the whole world with density δ, which is normalized to unity without loss of generality. Notice that $A_i = \delta \cdot L_i = L_i$. Figure 1 describes this economy.
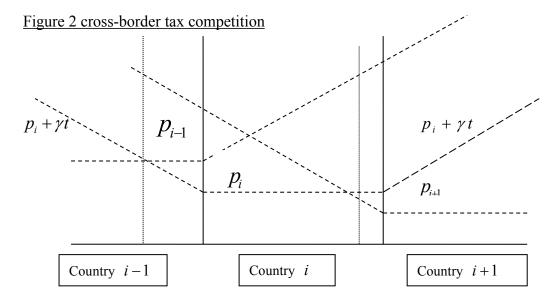
Figure 1.Salop model

Firms are located at each every point in all countries distributed in a continuum. All firms are producing one single type of consumable goods at a constant marginal cost C. We normalise the marginal cost to zero without loss of generality, i.e., $C = 0$. The production is able to meet all consumers wish to buy at their residence. As assumed in paper 2, all individual in the world wishes to buy one and only one unit of commodity inelastically. The consumers can either purchase domestic product without travelling, or shop at the border purchasing imported goods. In the latter case, a travelling cost of $\gamma \cdot x$ occurs to travel a distance x.

The governments of all countries impose commodity taxes on the purchase. Country i levies a tax of p_i for per unit of the goods from the consumer. Given the assumptions that the firms are located in a continuum and that the marginal cost normalised to zero, Betrand competition result in a pricing equilibrium of p_i inside the sovereign border of country i. Residents in country i have to travel to the border to make cross-border shopping from country $i+1$ or country $i-1$, where the total cost will be $p_{i+1} + \gamma \cdot x$ or $p_{i-1} + \gamma \cdot x$ if she has to travel a distance of x from home.

Figure 2 cross-border tax competition

If the tax rates are identical between neighbouring countries, where $p_i = p_{i+1}$, there will be no cross-border shopping between country i and $i+1$. If $p_i \geq p_{i+1}$, there will be outwards cross-border shopping from country i equal to the quantity such that

$$C_i(p_i - p_{i+1}) = \min\{(p_i - p_{i+1})|\gamma, A_i\} \qquad (2.1)$$

Governments are maximising the tax revenue by adjusting tax rates. Thus we can form a simultaneous move non-cooperative tax game as:

$$
\begin{aligned}
\pi_i(p_0,...,p_{n-1}) &= p_i D_i \\
&= \left(L_i + \tfrac{1}{\gamma}(p_{i-1} + p_{i+1} - 2p_i)\right) \cdot p_i
\end{aligned} \qquad (2.2)
$$

3. Salop Equilibrium

Notice that the payoff function given by (2.2) is a strictly concave function. Thus this globally strictly concavity enable the existence of a pure strategy Nash equilibrium.

The first order condition of π_i with respect to p_i gives the best response function:

$$p_i = \tfrac{1}{4}(\gamma L_i + p_{i-1} + p_{i+1}) \qquad (3.1)$$

Rearranging the best response equation, we get

$$-\tfrac{1}{4}p_{i-1} + p_i - \tfrac{1}{4}p_{i+1} = \tfrac{1}{4}\gamma L_i \qquad (3.2)$$

Based on (2.4), we formulate a linear system of best responses

$$
\begin{bmatrix} 1 & -\frac{1}{4} & 0 & \dots & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & \dots & 0 \\ 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & \dots \\ 0 & \dots & \dots & \dots & 0 \\ -\frac{1}{4} & 0 & \dots & -\frac{1}{4} & 1 \end{bmatrix} \times \begin{bmatrix} p_0 \\ \dots \\ p_i \\ \dots \\ p_{n-1} \end{bmatrix} = \frac{1}{4}\gamma \begin{bmatrix} L_0 \\ \dots \\ L_i \\ \dots \\ L_{n-1} \end{bmatrix} \qquad (3.3)
$$

Let

$$
B = \begin{bmatrix} 1 & -\frac{1}{4} & 0 & \dots & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & \dots & 0 \\ 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & \dots \\ 0 & \dots & \dots & \dots & 0 \\ -\frac{1}{4} & 0 & \dots & -\frac{1}{4} & 1 \end{bmatrix}
$$

And

$$
A = \frac{1}{4}\gamma \begin{bmatrix} L_0 \\ L_1 \\ \dots \\ L_{n-1} \end{bmatrix}
$$

The equilibrium tax rate is

$$
p = B^{-1}A \qquad (3.4)
$$

Notice that B is a symmetric circulant matrix, ensuring the existence of a unique Nash equilibrium.

Proposition 1. There always exists a unique Nash equilibrium, given by $P^* = B^{-1}A$.

Define a notation m that there are m neighbours on either side of i, which are i's neighbours, where $m \in [0, n/2]$ when n is even, and $m \in [0, (n-1)/2]$ if n is odd. Let #m(i) be the number of large countries in country i's neighbourhood. Define the expression $i\Delta j$ where for some m^*, if $m=m^*$, then #m(i)>#m(j), and if $m<m^*$, then #m(j)=#m(i). The intuition is that if $i\Delta j$, there are more large countries located the closest to country i than those closest to country j.

For example, if country j is a big country, and country j-1 and j+1, which are neighbouring j geographically, are both small country, then #1(j)=1, where $m \geq 1$. If

country j is a small country, #1(j)=0. If country j is a big country, j+1 is also a big country, and j-1 is small, then #2(j)=2.

We define S_h such that when n is odd, $S_h = 2\sum_{j=h}^{(n-1)/2} q_{0,j} + q_{0,\frac{n-1}{2}}$, or $S_h = 2\sum_{j=h}^{n/2} q_{0,j}$ when n is even. The intuition is that S_h is the summation of the weights ($q_{0,j}$) associated with all the countries' neighbours which are h places away or further.

Proposition 2.[5] If $i\Delta j$, then $p_i > p_j$. If neither $i\Delta j$ nor $j\Delta i$, then $p_i = p_j$.

There are two major implications from proposition 2. Firstly, the country size is positively related to the country's tax rate. A big country will always set a higher tax rate than a small country.
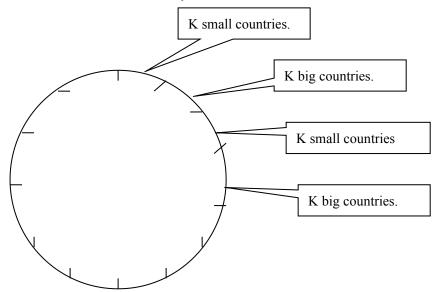
The second implication suggests that the number of big countries in the neighbourhood is positively related to the country's tax rate, and the number of small countries is negatively related to the tax rate. Less attention was devoted to the study of geographic environment on sovereignty country's taxation policy. Here we suggest that the location of the country plays an important role, just second to its own geographic characteristic, in the country's taxation decision. The intuition is that if a country is surrounded by big countries, it tends to set higher tax rate, while a country surrounded by small countries incline to set lower tax rate.

Examples may help us understand the property of proposition 1 and 2. We model the k-group world as following. Half of the countries are large countries at size L_L, and the rest of them are small. Suppose that there are k large countries located contiguously on the circle, followed by k small countries, followed by k large countries, followed by k small countries, and so on and so forth. Here k denotes the

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[5] See proof in appendix.

number of the countries of the same size located together as a group.

Figure 3 tax distribution in k-country model

Without loss of generality, we locate the first k large countries from the point 0 on the circumference, which are followed by k small countries. The second group of countries are followed by k large countries, which are followed by k small countries, and so on and so forth. Thus all n countries are located adjacently without overlapping area on the circumference, where each country has two neighbours.

Proposition 3.

In the k-group model, we argue that:

I, The size of the country is positively related to its tax rate.

II, The number of the adjacent large countries is positively related to the tax rate of the country.

Proposition 3 is the ramification of proposition 2 in the k-country model context. Three corollaries occur.

Corollary 1. Big countries set higher tax rates than small countries, i.e., if i is big country, and j is small country, then $p_i > p_j$.[6]
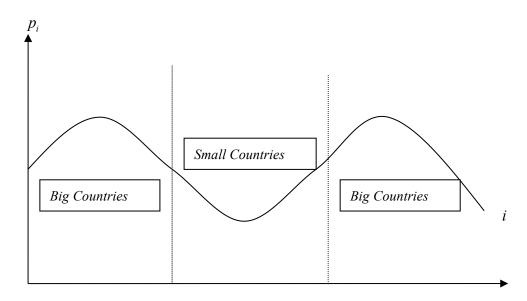
Corollary 2. Inside the group of large countries, the country (when n is odd) or the

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[6] This is the extreme case in the proof of proposition 2.

two countries (when n is even) at the geographical middle of the group set the highest tax rate. The tax rates decline from the middle to each end. The two countries bordering the small countries' group set the lowest tax rates in the group.[7]

<u>Corollary 3.</u> In the group of small countries, the two countries bordering the large countries set the highest tax rate. The lowest tax rate appears at the geographical middle of the group. The tax rates increase from the middle to each ends of the group.[8]

Figure 4. Tax distribution in k-country model

The K-country model give two direct implications. Firstly, the size is significant. The big countries can afford to set higher tax rate because they have bigger demand, while smaller countries have to lower their tax to attract foreign demand. That is partly why the international tax havens are always smaller economies, for instance, the Cayman islands.

Secondly, the big neighbours are positively related to local tax rate. From proposition 3, when there are big neighbours, a country tends to levy higher tax than other countries in its league. Surrounded by bigger countries means more cross-border demand from the outside world. On the contrary, the economies surrounded by small countries have to push tax lower to keep a decent fiscal income.

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[7] See the proof in appendix.
[8] Shown in the proof of proposition 2.

If we relax the unified country sizes in k-country models, we can have an interesting numerical example directly from proposition 2. China, Myanmar, Indonesia and Philippine are geographically on one curve neighbouring each other. We arbitrarily define the size of above four countries as L_0, L_1, L_2, L_3, where $L_0 = 10a, L_1 = a, L_2 = 2a, L_3 = a$. Here we assume the size of China is much bigger than the rest, and Indonesia is twice as big as that of Myanmar. The equilibrium tax rate will be $p_1 = \frac{4}{3}\gamma a, p_2 = \frac{7}{6}\gamma a$. The neighbouring effect dominates the size effect in this case. The huge demand from China may be so big that the smaller economy set higher tax rate than its big neighbour. However, if the size of China is not so big, say, $L_0 = 4a$. The equilibrium changes to $p_1 = \frac{20}{24}\gamma a, p_2 = \frac{21}{24}\gamma a$. Indonesia sets higher tax than Myanmar. The size effect dominates the neighbour effect. The bigger country set higher tax rate than the country with big neighbour.

The example also implies that the tax game is dynamic. It changes with the economic performance of the region. When a country grow fast, it may out-pace its neighbours and change the tax distribution of the whole region.

4. Hotelling model and peripheral effects

Peripheral effects are the distinguishing difference between Hotelling and Salop model. Hotelling model is a linear world, hence two countries on the ends occur inevitably. The peripheries have only one side open to other countries while the interior ones open in both sides. It brings periphery effects from those two special cases, where inward cross-border demand is limited.

The Hotelling model is similar to Salop but in the peripheries. Consider a linear world, where countries are located on a line. Countries divide the circular world into n adjacent and non-overlapping segments. The segment of $[0, L_0]$ represent the land of country 0, $[L_0, L_1]$ represent the land of country 1, $[L_i, L_{i+1}]$ represent the land of country i, and so on and so forth. The countries at the two ends of the line are called peripheral countries, while the rest are defined as interior countries. Each interior country has two neighbours, one to each side, i.e., country i is neighbouring country $i+1$ and country $i-1$. The peripheral countries are country 0 and country n-1. Country 0 is neighbouring only country 1, while country n-1 is neighbouring country n-2 only.

The cross-border shopping demand is

$$C_i(p_i - p_{i+1}) = \min\{(p_i - p_{i+1}) | \gamma, A_i\} \qquad (4.1)$$

The tax revenue in the interior countries are

$$\pi_i(p_0, ..., p_{n-1}) = \left(L_i + \tfrac{1}{\gamma}(p_{i-1} + p_{i+1} - 2p_i)\right) \cdot p_i \qquad (4.2)$$

$$\text{where} \quad i = 1, 3 ... n - 2$$

Tax revenue for peripheral countries is

$$\pi_0(p_0, ..., p_{n-1}) = \left(L_0 + \tfrac{1}{\gamma}(p_1 - p_0)\right) \cdot p_0$$

$$\pi_{n-1}(p_0, ..., p_{n-1}) = \left(L_{n-1} + \tfrac{1}{\gamma}(p_{n-2} - p_{n-1})\right) \cdot p_{n-1}$$

$$(4.3)$$

(4.2) and (4.3) are strictly concave functions in tax rates. Thus the FOC indicates best response.

$$p_i = \tfrac{1}{4}(\gamma L_i + p_{i-1} + p_{i+1}) \qquad (4.4)$$

$$\text{where } i = 1, 2, ... n - 2$$

From (4.3), the interior countries' best response functions are

$$p_0 = \frac{1}{2}(\gamma L_0 + p_1)$$
$$\qquad (4.5)$$
$$p_{n-1} = \frac{1}{2}(\gamma L_{n-1} + p_{n-2})$$

From (4.4) and (4.3), a linear system of the best response function can be given:

$$\overline{A} = \overline{B} \cdot \overline{P} \qquad (4.6)$$

where

$$\overline{A} = \tfrac{1}{4}\gamma \begin{bmatrix} L_0 \\ L_1 \\ ... \\ L_{n-1} \end{bmatrix} \text{ and } \overline{B} = \begin{bmatrix} \tfrac{1}{2} & -\tfrac{1}{4} & 0 & ... & 0 \\ -\tfrac{1}{4} & 1 & -\tfrac{1}{4} & ... & 0 \\ 0 & -\tfrac{1}{4} & 1 & -\tfrac{1}{4} & ... \\ 0 & ... & ... & ... & 0 \\ 0 & 0 & ... & -\tfrac{1}{4} & \tfrac{1}{2} \end{bmatrix} = \begin{bmatrix} b_0 & c_0 & & & 0 \\ a_1 & b_1 & c_1 & & \\ & ... & ... & ... & \\ & & & ... & c_{n-2} \\ 0 & & ... & a_{n-1} & b_{n-1} \end{bmatrix}$$

The tax rates index is denoted by $\overline{P}$. Equilibrium tax rates are given by $P^* = \overline{B}^{-1} \cdot \overline{A}$.

$\overline{B}$ is different from B given in section 3 only on the first and last row. $\overline{B}$ is an invertible tridiangnal matrix, thus P* presents a unique Nash equilibrium to this linear system. Hence proposition 4.

<u>Proposition 4.</u>[9] There always exists a unique tax equilibrium in the Hotelling tax competition. Equilibrium tax rates are given by following difference equation:

$$\begin{cases} p_{n-1} = \dfrac{L_{n-1} + \dfrac{1}{4}y_{n-2}}{1 - \dfrac{1}{4}\beta_{n-2}} \\ p_i = y_i - \beta_i \cdot p_{i+1} \end{cases}$$

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[9] Please find the proof in appendix.

where $\begin{cases} y_1 = 2L_1 \\ y_i = \dfrac{L_i + \dfrac{y_{i-1}}{4}}{1 - \dfrac{1}{4}\beta_{i-1}} \end{cases}$, and $\beta_i = -\dfrac{(2+\sqrt{3})^{i-1} + (2-\sqrt{3})^{i-1}}{(2+\sqrt{3})^i + (2-\sqrt{3})^i}$.

Proposition 4 gives analytical solution to any Hotelling tax game. It suggests that with the three effects' coexistence, there is no simple tax distribution in the tax game with more than 3 countries. However, it does suggests that size, neighbour and peripheral effect have positive relation with tax rate. Numerical example may help us understand proposition 4.

We take possible tax game in Asia as an example. Thailand, Hong Kong and Taiwan are located on a curve neighbouring each other, with land size L_0, L_1, L_2 respectively. We arbitrarily assume that they are of the same size, where $L_0 = L_1 = L_2 = a$. Apply proposition 4, we have $p_0 = p_2 = \frac{5}{6}\gamma a$, while $p_1 = \frac{2}{3}\gamma a$. The peripheral effect significantly pushes up the tax rate for the countries on the geographic ends.

When the effects are effective together, the picture may be blurred. Instead of Hong Kong, if we have China in the middle as interior country, the tax equilibrium changes dramatically. Now the region consists Thailand, China and Taiwan. Considering the huge size of China, we let $L_1 = 4a$. The tax equilibrium is now $p_0 = p_2 = \frac{4}{3}\gamma a, p_1 = \frac{5}{3}\gamma a$. the size effect dominates the peripheral effect. If we increase the number of participant countries, we can also test the effect of neighbours.

When the peripheral effect exists in the tax game, both size effect and the neighbourhood effect still emerge and significantly influence the equilibrium. When they put each impact on the tax distribution, the game becomes more dynamic and interesting.

5. Conclusion

The central message of this paper is to describe the impact of cross-border shopping on regional tax rates distribution. The sizes of the countries, the neighbours of the countries and the countries' geographic locations determine the tax distribution. We show that the size effect, neighbour effect and peripheral effect are positively related to tax rates.

We provide a simple general model of a circular world with future potential in this paper. This multi-country model enables geographical size differentiation. Analyses of the game explain the tax mechanism in the international trade context with tax authorities governing national sovereignties in different sizes. By analysing the equilibrium, we realize the impact from cross-border shopping and the geographic features on the regional tax distribution.

The two main findings from the first part of the paper are: country sizes are positively related to the countries' own tax rates; the sizes of the countries' tax rates are positively related to their neighbouring countries sizes as well as their own. These findings have been tested by a set of numeric examples. The first one is coherent with existing literature, confirming the circular model's solidity in explaining cross-border shopping. It provides possibility of applying country size differentiation in computing and analysing equilibrium in a model with more than 3 countries. The second finding is new to existing knowledge in the international taxation study: tax rate is influenced by the geographic sizes of the country's neighbours. The tax rates of the neighbouring countries are positively related to the subject country's tax rate. Both findings can be reflected in real-world examples, e.g., the European petrol price index.

The second part of the paper shows a Hotelling model of regional tax competition. It facilitates us to study the coexistence and correlation of size effect, neighbourhood

effect and peripheral effect in one model. We argue that the peripheral effect can dictate tax distribution when country size difference is not big enough.

Refrences:

1. *Acton, 1990, Numerical Methods That Work, 2nd printing. Washington, DC: Math. Assoc. Amer.*
2. *Baskar and To, 2003, Oligopsony and the distribution of wages, European Economic Review, 47 (2),371-399.*
3. *Bucovessky, 1991, Asymmetric Tax Competition, Journal of Urban Economics 30, 167-181.*
4. *Cremer, Gahvari, 2000, Tax evasion, fiscal competition and economic integration, European economic review, 44, 1633-1657.*
5. *Esteller-Moré and Solé-Ollé, Vertical income and tax externalities and fiscal interdependence: evidence from the US, 2000, Regional science and urban economics, 31, 247-272.*
6. *Kanbur, Keen, 1993, Tax competition and tax coordination when countries differ in size, American Economic Review 83, 877-892.*
7. *Liu and Madden, 2013, Small country may have higher tax rate, Studies in Microeconomic Studies, November, 2013*
8. *Mieszkowski and Zodrow, 1986, Pigou, Tiebout, property taxation and the underprovision of local public goods, Journal of urban economics, 19, 1986, 356-370.*
9. *M ieszkowski and Zodrow, Taxation and the Tiebout model: the differential effects of head taxes, taxes on land rents, and property taxes, Journal of economic literature, 27-3, September 1986, 1098-1146.*
10. *Nielson,2001, A simple model of commodity taxation and cross-border shopping, Scandinavian journal of economics, 103, 599-623.*
11. *Ohsawa, 1999, Cross-Border Shopping and Commodity tax competition among governments, Regional Science and Urban Economics 29, 33-51*
12. *Ohsawa, 2003, A spatial tax harmonization model, European economic review, 47 (2003), 443-459.*
13. *Trandel, 1994, Interstate commodity tax differentiatials and the distribution of residents, Journal of public economics, 53, 435-457.*
14. *Vickerman, 1992, The single European Market, Harvester Wheatsheaf, Herefordshire.*
15. *Wang You-qiang, 1999, Commodity taxes under fiscal competition: Stackelberg equilibrium and optimality, the American economic review, September 1999, 89-4, p974*
16. *Wellisch and Richter, 1994, the provision of local public goods and factors in the presence of firm and household mobility, Journal of public economics, 60, 1996, 73-93.*
17. *Wilson, 1997, property taxation, congestion, and local public goods, Journal of public economics, 64, 1997, 207-217.*
18. *Wilson, 1991, Tax Competition with Interregional Differences in Factor Endowments, Regional Science and Urban Economics 21, 423-451.*
19. *Zodrow and Mieszkowski, 1986, Pigou, Tiebout, property taxation and the*

Appendix:

1. Proof of proposition 2.

To analyse the equilibrium, we let

$$Q = B^{-1} = \begin{bmatrix} 1 & -\frac{1}{4} & 0 & \cdots & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & \cdots & 0 \\ 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ -\frac{1}{4} & 0 & \cdots & -\frac{1}{4} & 1 \end{bmatrix}^{-1}$$

Q is the inverse of B, which is also a symmetric circulant matrix, which is defined by its first row. From (3.7), the first row of Q is $q_0 = (q_{0,0}, q_{0,1}, q_{0,2}, \ldots q_{0,n-1})$. To ensure $QB=I$, q_0 must solve the following system:

$$q_{0,0} - \tfrac{1}{4} q_{0,1} - \tfrac{1}{4} q_{0,n-1} = 1 \qquad (A.1.1)$$

$$-\tfrac{1}{4} q_{0,j-1} + q_{0,j} - \tfrac{1}{4} q_{0,j+1} = 1 \qquad (A.1.2.)$$

$$-\tfrac{1}{4} q_{0,0} - \tfrac{1}{4} q_{0,n-2} + q_{0,n-1} = 1 \qquad (A.1.3.)$$

(A.1.2.) is a second order linear difference equation with characteristic roots $\lambda = 2 - \sqrt{3}$ *and* $\mu = 2 + \sqrt{3}$. Notice that $0 < \lambda < 1 < \mu$. The general solution to (A.1.2.) is, for some constant a and b:

$$q_{0,j} = a\lambda^j + b\mu^j \qquad (A.1.4)$$

Substitute (A.1.4.) into (A.1.1.) and (A.1.3.), we solve for a and b.

$$a = \frac{2}{(1 - \lambda^n)\sqrt{3}}$$

$$b = \frac{2}{(\mu^n - 1)\sqrt{3}} \qquad (A.1.5.)$$

Notice that $q_{0,j} > 0, \forall j \in N$.

Two properties of Q need to be addressed. First of all, Q is a symmetric circulant matrix, it has a property that $q_{0,j} = q_{0,n-j}$. Secondly, in matrix Q, we can show that in the first row of it, $q_{0,0} > q_{0,1} > q_{0,2} > ... > q_{0,\frac{n}{2}}$, $\quad q_{0,\frac{n}{2}} < q_{0,\frac{n}{2}+1} < q_{0,\frac{n}{2}+2} < ... < q_{0,n-1}$ if n is even. It also follows the same logic when n is odd. The intuition is that the elements of q_0 initially decreases from left to the right, and then increase from middle to the right end. Following two lemmas establish these two properties.

Lemma 1. when Q is a symmetric circulant matrix as we defined above, $q_{0,j} = q_{0,n-j}$.

Proof:

$$q_{0,j} - q_{0,n-j} = a\lambda^j + b\mu^j - (a\lambda^{n-j} + b\mu^{n-j})$$

$$= \frac{2(2-\sqrt{3})^j}{[1-(2-\sqrt{3})^n]\sqrt{3}} + \frac{2(2+\sqrt{3})^j}{[(2+\sqrt{3})^n - 1]\sqrt{3}} - \left\{ \frac{2(2-\sqrt{3})^{n-j}}{[1-(2-\sqrt{3})^n]\sqrt{3}} + \frac{2(2+\sqrt{3})^{n-j}}{[(2+\sqrt{3})^n - 1]\sqrt{3}} \right\}$$

$$= 2\frac{(2-\sqrt{3})^j[(2+\sqrt{3})^n - 1] + (2+\sqrt{3})^j[1-(2-\sqrt{3})^n] - (2-\sqrt{3})^{n-j}[(2+\sqrt{3})^n - 1] - (2+\sqrt{3})^{n-j}[1-(2-\sqrt{3})^n]}{[1-(2-\sqrt{3})^n][(2+\sqrt{3})^n - 1]\sqrt{3}}$$

$$= 2\frac{(2+\sqrt{3})^{n-j} - (2-\sqrt{3})^j + (2+\sqrt{3})^j - (2-\sqrt{3})^{n-j} - (2+\sqrt{3})^j + (2-\sqrt{3})^{n-j} - (2+\sqrt{3})^{n-j} + (2-\sqrt{3})^j}{[1-(2-\sqrt{3})^n][(2+\sqrt{3})^n - 1]\sqrt{3}}$$

$$= 0$$

$$\Leftrightarrow q_{0,j} = q_{0,n-j}$$

QED.

Lemma 2. When Q is a symmetric circulant matrix as we described above, $q_{0,j} > q_{0,j+1}$, where $j \in [0, n/2-1]$ when even, and $j \in [0, (n-1)/2-1]$ when odd, $\forall n \geq 3$.

Proof:

Let's consider the case when n is even. First we need to prove that $q_{0,0} > q_{0,1}$.

$$q_{0,j} - q_{0,j+1} = a\lambda^j + b\mu^j - (a\lambda^{j+1} + b\mu^{j+1})$$

$$\Leftrightarrow q_{0,0} - q_{0,1} = a + b - (a\lambda + b\mu)$$

$$= \frac{2}{3} \cdot \frac{-1 + (2+\sqrt{3})^n - (2-\sqrt{3}) + (2+\sqrt{3})^{n-1} - 1 - (2+\sqrt{3}) + (2-\sqrt{3})^n + (2-\sqrt{3})^{n-1}}{[-1+(2-\sqrt{3})^n][1-(2+\sqrt{3})^n]}$$

$$= \frac{2}{3} \cdot \frac{(2+\sqrt{3})^n + (2+\sqrt{3})^{n-1} + (2-\sqrt{3})^n + (2-\sqrt{3})^{n-1} - 6}{[-1+(2-\sqrt{3})^n][1-(2+\sqrt{3})^n]}$$

Notice that $n \geq 3$ as we assumed. Herein

$(2+\sqrt{3})^n + (2+\sqrt{3})^{n-1} + (2-\sqrt{3})^n + (2-\sqrt{3})^{n-1} - 6 \geq 0$. The numerator is positive.

$-1 + (2-\sqrt{3})^n < 0$ and $1 - (2+\sqrt{3})^n < 0$, thus $[-1+(2-\sqrt{3})^n][1-(2+\sqrt{3})^n] > 0$.

The denominator is positive. Thus we show $q_{0,0} > q_{0,1}$.

Now we are going to show $q_{0,j} > q_{0,j+1}$, $\forall j \in [0, n/2]$.

$$q_{0,j} - q_{0,j+1} = a\lambda^j + b\mu^j - (a\lambda^{j+1} + b\mu^{j+1})$$

$$= \frac{2\lambda^j}{[1-\lambda^n]\sqrt{3}} + \frac{2\mu^j}{[\mu^n - 1]\sqrt{3}} - \left\{ \frac{2\lambda^{j+1}}{[1-\lambda^n]\sqrt{3}} + \frac{2\mu^{j+1}}{[\mu^n - 1]\sqrt{3}} \right\}$$

$$= \frac{2}{3} \cdot \frac{\lambda^j + \lambda^{j+1} - \lambda^j \mu^n - \lambda^{j+1}\mu^n + \mu^j + \mu^{j+1} - \lambda^n \mu^j - \lambda^n \mu^j}{(-1+\lambda^n)(-1+\mu^n)}$$

Notice that $\lambda \cdot \mu = 1$. Then

$$q_{0,j} - q_{0,j+1} = \frac{2}{3} \cdot \frac{\lambda^j + \lambda^{j+1} - \mu^{n-j} - \mu^{n-(j+1)} + \mu^j + \mu^{j+1} - \lambda^{n-j} - \lambda^{n-(j+1)}}{(-1+\lambda^n)(-1+\mu^n)}$$

Notice that $\lambda^j + \lambda^{j+1} - \mu^{n-j} - \mu^{n-(j+1)} + \mu^j + \mu^{j+1} - \lambda^{n-j} - \lambda^{n-(j+1)} < 0$, the numerator is

negative, as $\lambda < 1 < \mu$. And $(-1+\lambda^n)(-1+\mu^n) < 0$, the denominator is negative.

$$\Leftrightarrow q_{0,j} > q_{0,j+1}.$$

When n is odd, in the same way we can prove $q_{0,j} > q_{0,j+1}$.

QED.

Lemma 3: $S_{h+1} < q_{0,h}, \forall h \in \{0,1,2,...,n/2-1\}$.

Proof:

When n is odd, $q_i > q_{i+1}, \forall i \in [0, n/2-1]$. From (7), we can get

$$q_{0,j} = 4q_{0,j+1} - q_{0,j+2} \Leftrightarrow q_{0,j} > 3q_{0,j+1} \qquad (3.15)$$

When j=n/2-1, $q_{0,j+2} = q_{0,j}$. That implies $q_{0,j} > 2q_{0,j+1}$. Using this result, when n is odd we have:

$$S_h = 2\sum_{j=h}^{(n-1)/2-1} q_{0,j} + q_{0,(n-1)/2} < 2\sum_{j=h}^{(n-1)/2-1} q_{0,h} \cdot (\tfrac{1}{3})^{h-j} + q_{0,h}(\tfrac{1}{3})^{(n-1)/2-h-1}$$

$$< 2q_{0,h}\sum_{j=0}^{(n-1)/2-h-1} (\tfrac{1}{3})^{h-j} + q_{0,h}(\tfrac{1}{3})^{(n-1)/2-h-1}$$

$$< 2q_{0,h} \cdot \tfrac{3}{2} + q_{0,h}(\tfrac{1}{3})^{n/2-h-1} \approx 2q_{0,h} \cdot \tfrac{3}{2} = 3q_{0,h} < q_{0,h-1}$$

$$\Leftrightarrow S_h < q_{0,h-1}$$

Notice again that from (3.9), $q_{0,j} > 3q_{0,j+1} \Leftrightarrow q_{0,j+1} < \dfrac{1}{3}q_{0,j}$.

When *n* is even, similarly get

$$S_h = 2\sum_{j=h}^{n/2-1} q_{0,j} < 2q_{0,h}\sum_{j=0}^{n/2-h-1} (\tfrac{1}{3})^{h-j} < 2q_{0,h} \cdot \tfrac{3}{2} = 3q_{0,h} < q_{0,h-1}$$

QED.

Proposition 2. If $i\Delta j$, then $p_i > p_j$. If neither $i\Delta j$ nor $j\Delta i$, then $p_i = p_j$.

Proof:

First consider two countries, i and j, where i is large and j is small. We set this example to extreme, where all i's neighbours are all small countries, while all j's neighbour's are large countries. We define these two countries located exactly the same. The tax rates difference between these two countries cannot be higher than this example, where #m(i)=1 and #m(j)=0. The tax rate difference is:

$$p_i - p_j = (q_{0,i} - S_{i+1})(L_L - L_S) > 0$$

Notice that here q0,i=q0,j, as both countries are located identically.

From Lemma 3, $q_{0,i} - S_{i+1} > 0$. And $L_L - L_S > 0$ is defined. So $\#m_i(1)=1$ and $\#m_j(1)=0$, $p_i > p_j$. As we assumed, j's neighbours are all large countries, and this is the extreme case of all possibilities, where the tax rates difference is the lowest when j is small and i is big. This property implies that large countries always set higher tax rates than small countries.

Secondly, we can assume two big countries i and j, where country i has one big neighbour, while j has none. Again, we make this example extreme, where all countries that are 2 countries away from i are all small, and those 2 countries from j are all large. Again, we let these two countries located at the same place. Now the tax rates difference is:

$$p_i - p_j$$
$$= [(q_{0,i} + q_{0,i+1})L_L + q_{0,i-1}L_S + S_{i+2}L_S] - [q_{0,j}L_L + (q_{0,j+1} + q_{0,j-1})L_S + S_{j+2}L_L]$$
$$= (q_{0,i+1} - S_{i+2})(L_L - L_S) > 0$$

$$\Leftrightarrow p_i > p_j.$$

Consider the general case $\#m*(i) = \#m*(j)+1$, and all country j's neighbours who are $m*+1$ away are all big countries and country i's neighbours who are $m*+1$ away are all small countries. This is an extreme case, where the difference between p_i and p_j is no less than this extreme case. Notice here $p_i = q_i L$, $p_j = q_j L$.

$$p_i - p_j = q_i L - q_j L$$
$$\geq q_{0,m*}(L_L - L_S) + (S_{m*+1}L_S - S_{m*+1}L_L)$$
$$= (q_{0,m} - S_{m+1})(L_L - L_S) > 0$$
$$\Leftrightarrow p_i - p_j > 0$$

As we argued before, no other instance will result in greater difference than this extreme case. This general case shows us the first part of this proposition.

If neither $i\Delta j$ nor $j\Delta i$, the two countries have the same size and exactly the same neighbourhoods. Those two countries, with identical geographic characteristics and identical international taxation environment, set the same tax rate.

Proof of corollary 2.

Corollary 2. Inside the group of large countries, the country (when n is odd) or the two countries (when n is even) at the geographical middle of the group set the highest tax rate. The tax rates decline from the middle to each end. The two countries bordering the small countries' group set the lowest tax rates in the group.

Proof:

In the big countries' group, the country in the middle has the most big neighbours. If country λ is in the geographic middle of this group, then $\#m(\lambda) > \#m(j)$, where j is any other country in the same group. According to proposition 2, country λ sets the highest tax rate in this group. If k is even, the two countries in the middle will share the same highest tax rate.

Generally, $i\Delta j$ when $j < i < (h+1)\cdot k - j$, where i, j are in the same big countries group h. Those countries located closer to the middle have more big countries in their neighbourhoods, i.e., $\#m^*(i) > \#m^*(j)$, and thus $i\Delta j$, country i sets higher tax rates.

QED.

Proof of proposition 4.

Proposition 4. There always exists a unique tax equilibrium in the Hotelling tax competition. Equilibrium tax rates are given by following difference equation:

$$
\begin{cases}
p_{n-1} = \dfrac{L_{n-1} + \dfrac{1}{4} y_{n-2}}{1 - \dfrac{1}{4} \beta_{n-2}} \\
p_i = y_i - \beta_i \cdot p_{i+1}
\end{cases}
$$

where
$$\begin{cases} y_1 = 2L_1 \\ y_i = \dfrac{L_i + \dfrac{y_{i-1}}{4}}{1 - \dfrac{1}{4}\beta_{i-1}} \end{cases}, \text{ and } \beta_i = -\dfrac{(2+\sqrt{3})^{i-1} + (2-\sqrt{3})^{i-1}}{(2+\sqrt{3})^{i} + (2-\sqrt{3})^{i}}.$$

Proof:

Let $\overline{B} = W \cdot U$, where W and U are given by

$$W = \begin{bmatrix} \alpha_0 & & & & 0 \\ \delta_1 & \alpha_1 & & & \\ & \delta_2 & \dots & & \\ & & \dots & \dots & \\ 0 & & & \delta_{n-1} & \alpha_{n-1} \end{bmatrix}, \; U = \begin{bmatrix} 1 & \beta_0 & & & \\ & 1 & \beta_1 & & \\ & & 1 & & \\ & & & \dots & \beta_{n-1} \\ & & & & 1 \end{bmatrix}$$

where $\alpha_0 = \dfrac{1}{2}$, $\beta_0 = -\dfrac{1}{2}$, $\gamma_i = -\dfrac{1}{4}, \forall i$.

Apply following linear relations:[10]

$$\begin{cases} \alpha_i = b_i - a_i\beta_{i-1} \\ \beta_i = \dfrac{c_i}{b_i - a_i\beta_{i-1}} \end{cases} \qquad \text{(A.3.1.)}$$

$$\Rightarrow \beta_i = \dfrac{-1}{4 + \beta_{i-1}} \quad \text{(A.3.2.)}.$$

Let $\beta_i = -\dfrac{h_i}{k_i}$, where $h_i, k_i > 0$. Apply it into (A.3.2)

$$\Leftrightarrow \dfrac{h_i}{k_i} = \dfrac{k_{i-1}}{4k_{i-1} - h_{i-1}}.$$

Let

$$\begin{cases} h_i = k_{i-1} \\ k_i = 4k_{i-1} - h_{i-1} \end{cases}$$

The difference system gives

$$k_i = 4k_{i-1} - k_{i-2}$$

Eigenvalues from (A.3.3.) are $2 \pm \sqrt{3}$.

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[10] Detailed proof see Acton 1990, pp331-334.

Thus

$$k_i - (2+\sqrt{3})k_{i-1} = (2-\sqrt{3})[k_{i-1} - (2+\sqrt{3})k_{i-2}]$$

$$k_i - (2-\sqrt{3})k_{i-1} = (2+\sqrt{3})[k_{i-1} - (2-\sqrt{3})k_{i-2}]$$

Solve for k:

$$k_i = \frac{(2+\sqrt{3})^i + (2-\sqrt{3})^i}{2}$$

$$h_i = \frac{(2+\sqrt{3})^{i-1} + (2-\sqrt{3})^{i-1}}{2}$$

$$\Leftrightarrow \beta_i = -\frac{(2+\sqrt{3})^{i-1} + (2-\sqrt{3})^{i-1}}{(2+\sqrt{3})^i + (2-\sqrt{3})^i}$$

Notice that $WUP = A$, which implies $WP = y, Wy = A$.

Thus the equilibrium is given by the difference system

$$\begin{cases} y_1 = 2L_1 \\ y_i = \dfrac{L_i + \dfrac{y_{i-1}}{4}}{1 - \dfrac{1}{4}\beta_{i-1}} \end{cases} \text{ and } \begin{cases} p_{n-1} = \dfrac{L_{n-1} + \dfrac{1}{4}y_{n-2}}{1 - \dfrac{1}{4}\beta_{n-2}} \\ p_i = y_i - \beta_i \cdot p_{i+1} \end{cases}$$

QED.

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