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Optimal Rules for Central Bank Interest Rates Subject to Zero Lower Bound

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Abstract
The celebrated Taylor rule provides a simple formula that aims to capture how the central bank interest rate is adjusted as a linear function of inflation and output gap. However, the rule does not take explicitly into account the zero lower bound on the interest rate. Prior studies on interest rate selection subject to the zero lower bound have not produced rigorous derivations of explicit rules. In this work, Taylor-like rules for central bank interest rates bounded below by zero are derived rigorously using a multi-parametric model predictive control (mpMPC) framework. Rules with or without inertia are included in the derivation. The proposed approach is illustrated through simulations on US economy data. A number of issues for future study are proposed.

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Keywords Taylor rule; zero lower bound; liquidity trap; model predictive control; multi-parametric programming

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1 Introduction and Motivation

The general form of the standard Taylor rule suggests that the short-term interest rate i_t applied by the central bank at time t can be set according to the formula

$$i_t = \phi_y(y_t - y^*) + \phi_\pi(\pi_t - \pi^*) + r^* + \pi^* \tag{1}$$

where y represents the output gap (deviation of real GDP from potential GDP as percent of potential GDP); π represents inflation rate; subscript t refers to the time the rule is applied, using information up to that time; superscript * represents the desired equilibrium value; $r \triangleq i - \pi$ is the real interest rate; and ϕ_y, ϕ_π are coefficients associated with the output gap and inflation rate respectively. In the original publication [1] it was assumed that $y^* = 0$, $\pi^* = 2\%$, $r^* = 2\%$, $\phi_\pi = 1.5$, $\phi_y = 0.5$, with quarterly data for output gap, and annual data for inflation rate. Variants of the above basic Taylor rule have been studied in literature, such as rules with an inertia term containing i_{t-1} and/or with projected future values of π and y in the right-hand side of eqn. (1) [2, and references therein]. The stated objective for inertia-based policies is interest rate smoothing, to avoid large variations in interest rates and to produce robust policy rules [3-5]. Additional variants of the Taylor rule containing more lagged terms of i have also appeared [6, 7].

While the initial inspiration for the Taylor rule was based on fitting actual historical data, Taylor rules and some of its variants can be derived by application of optimization theory on a quadratic objective function, using a small-scale model of the economy to capture the effect of interest rate on inflation and output gap [8-11]. Such derivations have mainly focused on the effect of the specific form of the quadratic objective function on the resulting rule. This approach, however, *has not been successful at producing a rigorous derivation of explicit Taylor rules when a zero lower bound (ZLB) on the interest rate is included in the optimization*. Nevertheless, a number of approaches for determining an optimal interest rate subject to ZLB have been proposed, which can be broadly classified into two categories:

The first category includes explicit rules that truncate to zero the interest rate i_t^{TR} calculated by an unconstrainted Taylor rule (i.e. $i_t = \max[0, i_t^{\mathrm{TR}}]$), to ensure that a non-negative interest rate i_t is produced [12-14]. The rationale behind approaches in this category relies on qualitative analysis of a ZLB-constrained quadratic optimization problem or on other qualitative analysis of optimal policy effects on inflation and output gap.

The second category does not produce explicit rules; rather, it employs numerical simulation, i.e. repeated numerical solution of a ZLB-constrained optimization problem, to determine the optimal values of interest rate for inflation and output gap values in a range of interest [8, 15-18]. Most studies in this category rely on a constrained dynamic programming formulation of the underlying optimization problem, whose explicit analytical solution is hard to get.

Interesting observations were made in these studies. For example, it was observed that resulting policies may be nonlinear, (rather than piecewise linear, according to truncated Taylor rules) and more aggressive for interest rates close to ZLB (a behavior characterized as pre-emptiveness). However, *a rigorous derivation of simple explicit Taylor rules subject to ZLB is, to our knowledge, not currently available*.

In this paper, we rigorously derive explicit rules for interest rate subject to ZLB. Our approach relies on a formalism known as multi-parametric (mp) programming, a technique applied by the engineering community to constrained model predictive control (MPC) [19] or constrained state estimation problems [20]. The following are the key elements of the proposed approach.

- When a ZLB is present, explicit rules can be developed that produce a value for the interest rate through application of one from a finite number of explicit formulas. These formulas entail a finite number of Taylor-like rules. To know which of these formulas will be applied at any time, one has to simply pick an entry from a look-up table, based on checking which inequality is satisfied out of a finite number of a priori developed mutually exclusive linear inequalities on the inflation and output gap.
- Various forms of Taylor-like rules result rigorously from the particular form of the quadratic objective used in MPC. For example, Taylor rules with inertia terms arise from inclusion of a quadratic penalty on the rate of change of the interest rate (rather than on the interest rate itself).
- Application of any interest rate policy, Taylor-like or not, essentially creates a closed-loop feedback controlled economy. Therefore, any policy should, at the very least, result in a stable closed loop. Additionally, it should be fairly robust, namely it should produce sensible results in the presence of discrepancies between assumed economy models and the actual economy.

In the rest of the paper we first provide some background on MPC and mpMPC, and elaborate on the small-scale economy model used. Within this setting, we derive a number of Taylor-like rules, based on a number of MPC quadratic objectives, and examine their dependence on relative weights of various terms in the MPC objective. The effect of these rules on the resulting closed-loop behavior is examined. Comparison with the standard Taylor rule and actual interest rates implemented by the Central bank is provided. Finally, future extensions are proposed.

2 Preliminaries: Model Predictive Control (MPC) and Taylor rules

MPC is a class of model-based feedback control algorithms for systems with constraints [21, 22]. MPC finds the value of the manipulated input (interest rate in our case) of a controlled process at each point in time by setting up and solving a constrained optimization problem at that time. The optimization involves an objective function (usually quadratic) over a future horizon. The objective contains terms involving future predictions of the controlled variables (output gap and inflation in our case) as well as penalty terms on manipulated inputs within the horizon. Future output predictions are established in terms of a model and existing measurements.

As will be made clear below, MPC (also known as "open-loop optimal feedback") differs from stochastic dynamic programming (also known as "closed-loop optimal feedback") in that MPC does not explicitly account for information that is now expected to be available in the future, thus avoiding the computational complexity of the nested optimization (curse of dimensionality from Bellman's principle of optimality) which burdens stochastic dynamic programming.

Next, we first provide a description of the model we use, and subsequently explain its use in formulating the MPC optimization.

2.1 Economy model structure

A semi-empirical linear model around a baseline can describe the evolution of the economy as

$$y_{t+1} = \rho y_t - \xi \left(i_t - \pi_t \right) + e^y_{t+1}, \tag{2}$$

$$\pi_{t+1} = \pi_t + \alpha y_t + e^\pi_{t+1}. \tag{3}$$

[11] where y, π, and i are as above; α and ξ are positive constants; $\rho \in [0,1)$; e^y_{t+1} and e^π_{t+1} are zero-mean white noise signals; and the sampling period (time interval from t to $t+1$) is one year. The above model is similar in spirit to more complicated models used by many central banks. The model's main purpose is to capture the overall dynamic causal relationship between the manipulated input i and the two controlled outputs, y, π. Note that eqns. (2) and (3) capture the aggregate effect of the interest rate on the economy, namely effects due to phenomena such as rational expectations are assumed to have been incorporated in the model structure. Other kinds of models can also be converted to the aggregate form of eqns. (2) and (3) [17].

At steady state (equilibrium point), we have $i_t = i^*$, $y_t = 0$ and $\pi_t = \pi^*$, with $r^* = i^* - \pi^*$. Hence, in the terms of deviation variables from the equilibrium point, eqns. (2) and (3), can be written as

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}u_t + \boldsymbol{\varepsilon}_{t+1}, \tag{4}$$

where

$$\mathbf{x} \triangleq \begin{bmatrix} \Delta y \\ \Delta \pi \end{bmatrix} \triangleq \begin{bmatrix} y - y^* \\ \pi - \pi^* \end{bmatrix}, \; u \triangleq \Delta i \triangleq i - i^*, \; \boldsymbol{\varepsilon} \triangleq \begin{bmatrix} e^y \\ e^\pi \end{bmatrix}, \tag{5}$$

$$\mathbf{A} \triangleq \begin{bmatrix} \rho & \xi \\ \alpha & 1 \end{bmatrix}, \tag{6}$$

$$\mathbf{B} \triangleq \begin{bmatrix} -\xi \\ 0 \end{bmatrix}. \tag{7}$$

Using the above model, the optimal k-step-ahead prediction for the state $\mathbf{x}$ with initial condition $\mathbf{x}_t$ is

$$\hat{\mathbf{x}}_{t+k|t} = \sum_{\ell=0}^{k-1} \mathbf{A}^\ell \mathbf{B} u_{t+k-\ell-1|t} + \mathbf{A}^k \mathbf{x}_t \,, \tag{8}$$

[23] where $\hat{\mathbf{x}}_{t+k|t}$ stands for the expected value of $\mathbf{x}$ at time $t+k$ using all information available at time t. The above prediction will be used in the formulation of the MPC objective below.

It should be noted that the idea here is not to fully explain the complex dynamics of the economy with such a simple linear model. Rather, the intended use of the above model is to help understand how optimal monetary policies are affected by various objective functions and by a ZLB on the interest rate when constrained MPC is used to derive such policies. The dimension of the state vector $\mathbf{x}$ is also limited to two, so that the solution of the constrained MPC optimization problem can be easily understood graphically in 2-D and 3-D plots using the mpMPC approach.

2.2 Economy model calibration

The economy model expressed by eqns. (2) and (3) is calibrated using US revised economy data over the time period 1976-2007. The annual revised output gap data is taken from the Congressional Budget Office [24]. Inflation is calculated as annual percentage change in the GDP deflator Q4/Q4 basis (Bureau of economic Analysis). The real interest rate, r, is calculated as the annual average of the interest rate deflated by the annual inflation rate. Interest rates are taken from the database of the Federal Reserve System. Figure 1 plots these data for the time period 1976-2010. Based on these data, Table 1 presents estimated values of parameters for the economy model, obtained using the prediction error method. Based on the parameter estimates in Table 1, the matrix $\mathbf{A}$, eqn. (6) turns out to be

$$\mathbf{A} = \begin{bmatrix} 0.63 & 0.19 \\ 0.12 & 1 \end{bmatrix}. \tag{9}$$

The eigenvalues of $\mathbf{A}$ are 0.58 and 1.05, suggesting that the economy model for the US economy is mildly unstable. Consequently, whatever control policy ones chooses to control the US economy, such a policy must be, at the very least, a stabilizing policy. We develop such a policy below via MPC.

2.3 Formulation of MPC optimization

The central bank's generalized loss function projected to infinity at time t is generally of the form

$$\sum_{k=0}^{\infty} \beta^k L(\hat{\mathbf{x}}_{t+k|t}, u_{t+k|t}). \tag{10}$$

After minimizing the above objective at time t, the first element $u_{t|t}^{\text{opt}}$ of the optimal sequence $\{u_{t|t}^{\text{opt}}, u_{t+1|t}^{\text{opt}}, ...\}$ is implemented, and the system (i.e. the economy) runs until the next decision making point, $t+1$. At time $t+1$ the optimization problem in eqn. (10) is reformulated, solved, the first element $u_{t+1|t+1}^{\text{opt}}$ of the optimal sequence $\{u_{t+1|t+1}^{\text{opt}}, u_{t+2|t+1}^{\text{opt}}, ...\}$ is implemented, the system runs until the next time, and the process continues to infinity. The difference $(t+1)-t$ is selected here to be one quarter. It should be stressed that, in general, $u_{t+1|t+1}^{\text{opt}} \neq u_{t+1|t}^{\text{opt}}$ because of modeling uncertainty and external disturbances.

It has been shown [25] that for quadratic $L(\hat{\mathbf{x}}_{t+k|t}, u_{t+k|t})$, stability of constrained MPC can be ensured if the objective in eqn. (10), which involves an infinite number of terms, is replaced by an equivalent objective that involves summation of a finite number of terms plus a terminal cost and/or terminal constraints. A particular realization of this idea can take the form

$$\min_{\mathbf{u}} \left\{ \sum_{k=0}^{N-1} \beta^k \left(\hat{\mathbf{x}}_{t+k|t}^T \mathbf{Q} \hat{\mathbf{x}}_{t+k|t} + R^2 u_{t+k|t}^2 + S^2 \delta u_{t+k|t}^2 \right) + \hat{\mathbf{x}}_{t+N|t}^T \beta^N \bar{\mathbf{Q}} \hat{\mathbf{x}}_{t+N|t} + \beta^N S^2 \delta u_{t+N|t}^2 \right\}, \tag{11}$$

subject to the model constraints

$$\hat{\mathbf{x}}_{t+k|t} = \sum_{\ell=0}^{k-1} \mathbf{A}^{\ell} \mathbf{B} u_{t+k-\ell-1|t} + \mathbf{A}^k \mathbf{x}_t, \ k = 1, ..., N, \tag{12}$$

$$\hat{\mathbf{x}}_{t|t} = \mathbf{x}_t, \tag{13}$$

the unstable mode stabilization constraints

$$\tilde{\mathbf{v}}_{\text{u}}^T \left[\mathbf{A}^{N-1}\mathbf{B}, \mathbf{A}^{N-2}\mathbf{B}, ..., \mathbf{B} \right] \mathbf{u} = -\tilde{\mathbf{v}}_{\text{u}}^T \mathbf{A}^N \mathbf{x}_t, \tag{14}$$

the input move restriction constraints

$$u_{t+k|t} = u_{t+m-1|t}\,,\ k = m,\ldots,N-1, \tag{15}$$

and the inequality constraints

$$u_{t+k|t} \geq -i^*\,,\ k = 0,\ldots,N-1, \tag{16}$$

where

$$\mathbf{u} \triangleq \begin{bmatrix} \Delta i_t \\ \Delta i_{t+1|t} \\ . \\ . \\ \Delta i_{t+N-1|t} \end{bmatrix}, \tag{17}$$

$$\delta u_{t+k|t} \triangleq u_{t+k|t} - u_{t+k-1|t}\,,\ k = 0,\ldots,N\,, \tag{18}$$

$$\mathbf{Q} \triangleq \begin{bmatrix} 1-\lambda & 0 \\ 0 & \lambda \end{bmatrix} \succ 0\,,\ \ 0 < \lambda < 1 \tag{19}$$

$$\bar{\mathbf{Q}} \triangleq \frac{\mathbf{v}_{\mathrm{s}}^T \mathbf{Q} \mathbf{v}_{\mathrm{s}}}{1-\beta J_{\mathrm{s}}^2} \tilde{\mathbf{v}}_s \tilde{\mathbf{v}}_{\mathrm{s}}^T \succ 0\,, \tag{20}$$

(see Appendix A) with the vectors $\mathbf{v}_{\mathrm{s}}$ and $\tilde{\mathbf{v}}_{\mathrm{s}}$ coming from the diagonalization of the matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1} = \left[\begin{array}{c:c} \mathbf{v}_{\mathrm{u}} & \mathbf{v}_{\mathrm{s}} \end{array}\right] \begin{bmatrix} J_{\mathrm{u}} & 0 \\ 0 & J_{\mathrm{s}} \end{bmatrix} \left[\begin{array}{c} \tilde{\mathbf{v}}_{\mathrm{u}}^T \\ \hdashline \tilde{\mathbf{v}}_{\mathrm{s}}^T \end{array}\right], \tag{21}$$

where J_{u} and J_{s} refer to the unstable and stable eigenvalues of the matrix $\mathbf{A}$ with corresponding eigenvectors, $\mathbf{v}_{\mathrm{u}}$ and $\mathbf{v}_{\mathrm{s}}$, respectively.

The main rationale behind the above formulation is that closed-loop stability can be guaranteed by including the terminal penalty term $\hat{\mathbf{x}}_{t+N|t}^T \bar{\mathbf{Q}} \hat{\mathbf{x}}_{t+N|t}$ in the objective, eqn.

(11), and by explicitly forcing a terminal constraint, eqn. (14), to stabilize the unstable mode corresponding to the eigenvalue J_{u}. The values of the weights R and S determine the aggressiveness of the resulting control action, with small values of R and S encouraging more aggressive action and faster closed-loop response, at the cost of decreased closed-loop robustness [5, 10]. In particular, higher values of S are preferred when persistent external disturbances force the input i away from its nominal equilibrium value i^*. Finally, the values of $1-\lambda$ and λ in eqn. (19) determine the relative attention paid by the policy to output gap and inflation, respectively.

3 Taylor rules from MPC

In this section we show how Taylor rules can be derived from unconstrained MPC. Specifically, in section 3.1 we derive rules that follow the Taylor structure (eqn. (1)) while in section 3.3 we show how Taylor rules with inertia can be naturally derived from MPC with an additional quadratic penalty on the rate of change of interest rate. For both cases we examine the effects of MPC weights (λ, R, or S in eqn. (11)).

3.1 Taylor rules from MPC without zero lower bound

In the absence of ZLB, eqn. (16), and without penalty on the change of interest rate ($S=0$), the MPC optimization with objective function in eqn. (11) subject to equality constraints in eqns. (12)-(15) results in the unconstrained quadratic minimization

$$\min_{\mathbf{u}_m}\left[\frac{1}{2}\mathbf{u}_m^T\mathbf{H}\mathbf{u}_m+\mathbf{x}_t^T\mathbf{F}\mathbf{u}_m+\frac{1}{2}\mathbf{x}_t^T\mathbf{Y}\mathbf{x}_t\right], \tag{22}$$

where $\mathbf{H} \in \Re^{(m-1)\times(m-1)}$, $\mathbf{F} \in \Re^{2\times(m-1)}$, $\mathbf{Y} \in \Re^{2\times 2}$ are function of $\mathbf{A}$, $\mathbf{B}$, β, N, m, and the weights R and λ; and the decision variable is

$$\mathbf{u}_m \hat{=} \begin{bmatrix} \Delta i_t \\ \Delta i_{t+1|t} \\ . \\ . \\ \Delta i_{t+m-2|t} \end{bmatrix}. \tag{23}$$

(see Appendix A). The minimum in eqn. (22) is attained at $\mathbf{u}_m^{\text{opt}} = -\mathbf{H}^{-1}\mathbf{F}^T\mathbf{x}_t$, resulting in the optimal interest rate

$$i_t = -\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}}_{m-1}\mathbf{H}^{-1}\mathbf{F}^T\mathbf{x}_t + r^* + \pi^* = \phi_y(y_t - y^*) + \phi_\pi(\pi - \pi^*) + r^* + \pi^*. \tag{24}$$

at time t, which is clearly a Taylor-like rule, as in eqn. (1). It is also clear that ϕ_y, ϕ_π are functions of the economic model matrices $\mathbf{A}$, $\mathbf{B}$, and of the weights R, λ, given N, m and β.

3.1.1 Choice of prediction horizon length, N

For an unstable system such as the one described by eqns. (2) and (3), the horizon length, N, should be made long enough to ensure that the MPC optimization problem is feasible and ensure closed-loop stability. Systematic methods can be used for selecting N [26-28].

In all subsequent developments we will consider $N = 80$.

3.1.2 Choice of control horizon length, m

As eqn. (15) indicates, only a small number of inputs are included as decision variables in the MPC optimization. In addition to convenience (i.e. a small number of decision variables) there are deeper reasons for this choice.

First, increasing the value of m (with $1 \leq m \leq N$) quickly reaches a point of diminishing returns, namely no appreciable change in the closed-loop dynamics. Table 2 substantiates this claim by example, showing that the closed-loop poles remain almost unchanged after increasing the value of m beyond 4. The associated Table 3 shows the resulting coefficient for the Taylor-like solution provided by MPC.

Second, it has been rigorously shown that keeping m small improves the robustness of the closed loop, namely it helps maintain closed-loop stability in the presence of discrepancies between the model used by MPC and the actual system under control [29-31].

In all subsequent developments we will consider $m = 4$.

3.1.3 Choice of discount factor, β

Following the literature [15, 16] we use a value of the discount factor $\beta = 0.99$, except in situations where we explicitly specify a different value. We will comment below on how different values of β affect the resulting Taylor rules and closed-loop stability and performance.

3.1.4 Effects of MPC objective function weights on resulting Taylor rules

For the choice of $N = 80$, $m = 4$, and $\beta = 0.99$, discussed in the preceding sections, we now proceed to examine the effect of R and λ on the resulting Taylor rules, via eqn.

(24). Following the calculations in Appendix A, the matrices $\mathbf{H}$ and $\mathbf{F}$ in eqn. (22) are calculated as functions of R and λ, and coefficients of the output gap and inflation in the Taylor rule or eqn. (1) are expressed analytically in terms of R and λ, as

$$\phi_y = \frac{q_{y,3}R^6 + q_{y,2}(\lambda)R^4 + q_{y,1}(\lambda)R^2 + q_{y,0}(\lambda)}{p_3R^6 + p_2(\lambda)R^4 + p_1(\lambda)R^2 + p_0(\lambda)}, \tag{25}$$

$$\phi_\pi = \frac{q_{\pi,3}R^6 + q_{\pi,2}(\lambda)R^4 + q_{\pi,1}(\lambda)R^2 + q_{\pi,0}(\lambda)}{p_3R^6 + p_2(\lambda)R^4 + p_1(\lambda)R^2 + p_0(\lambda)}, \tag{26}$$

respectively, where the values of the corresponding parameters are shown in Table 4. In general, the numerator and denominator for ϕ_y and ϕ_π are polynomial functions of degree $m-1$ in both R^2 and λ.

Figure 2 employs the preceding eqns. (25) and (26) to calculate the policy coefficients ϕ_y, ϕ_π for a range of values of R and λ. The point corresponding to the original Taylor rule ($\phi_y = 0.5$, $\phi_\pi = 1.5$) is not present in Figure 2. However, various values of R and λ result in ϕ_y in the range of 1 to 3 (Figure 3) and ϕ_π in the range of 2 to 6 (Figure 4).

The following general observations can be made on Figure 3 and Figure 4:

- When R is small (i.e. control is aggressive) it has a strong effect on ϕ_y and ϕ_π.
- The value $R = 0$ results in large values of ϕ_y and ϕ_π, i.e. aggressive policy.
- When R is small, the inflation coefficient ϕ_π is more sensitive to the choice of λ than ϕ_y is.

- After approximately $R > 1$, further increase in R has very small effect on ϕ_y and ϕ_π.

For the economy model under consideration, the nearest point to the original Taylor rule is found at $\phi_y = 1$, $\phi_\pi = 2.4$ for $R = 0.55$ and $\lambda = 0.05$**.** These values are close to the original Taylor rule and other Taylor-like rules [8, 32].

3.1.5 Original Taylor rule in MPC framework

Even though the specific ϕ_y and ϕ_π values of the original Taylor rule were not recovered in the preceding section for the value of β used mostly in literature, such values can be obtained if a different value of β is considered. It turns out that the original Taylor rule can be recovered for $\beta \leq 0.96$, for which expressions for ϕ_y and ϕ_π similar to eqns. (25) and (26) can be derived in the same way. As shown in Figure 5 and Figure 6, the original Taylor rule values for ϕ_y and ϕ_π can be derived when $\beta = 0.96$ for $R = 1.06$ and $\lambda = 0.36$ in eqn. (11).

In general, determining values of MPC weights that would correspond to specific values of ϕ_y and ϕ_π is an instance of the inverse linear quadratic regulator problem. An infinite number of solutions generally exist for that problem. Feasibility and characterization of these solutions can be obtained in terms of linear matrix inequality algorithms [33, section 10. 6, p. 147]. This issue will be explored elsewhere.

3.1.6 Taylor rules and resulting closed-loop stability

For any rule proposed, it is important to determine, at the very least, whether such a rule results in a stable closed loop. Combination of the Taylor rule in eqn. (1) with the simple economy model, eqn. (4), yields (Appendix B) the closed loop structure

$$\mathbf{x}_{t+1} = \mathbf{A}_{\mathrm{CL}}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}, \tag{27}$$

where

$$\mathbf{A}_{\mathrm{CL}} \triangleq \mathbf{A} + \mathbf{B}\mathbf{c}^T = \begin{bmatrix} \rho - \xi\phi_y & \xi - \xi\phi_\pi \\ \alpha & 1 \end{bmatrix}. \tag{28}$$

It can be shown (Appendix B) that both eigenvalues of $\mathbf{A}_{\mathrm{CL}}$ are inside the unit disk, i.e. the closed-loop system is stable, if and only if

$$\phi_\pi > 1, \tag{29}$$

$$-2.1 + 0.12\phi_\pi < \phi_y < 8.5 + 0.06\phi_\pi. \tag{30}$$

as illustrated in Figure 7. This is in agreement with the well established *Taylor principle* that the central bank should raise its interest rate *more than one-for-one* with increase in inflation [34, 35]. Figure 4 shows that this requirement is satisfied for all combinations of the MPC weighting parameters R and λ. In fact, Figure 8 illustrates that the stability conditions, eqns. (29) and (30), are satisfied for all choices of R and λ when $\beta = 0.99$. However, this is not the case for $\beta \le 0.95$, as illustrated in Figure 9, which shows that as the value of β is reduced, the value of R should not be too small, to avoid closed-loop instability.

It is interesting to note that as $R \to \infty$, namely high values of interest rate are heavily penalized, the closed loop remains stable, due to the stabilizing equality constraint, eqn. (14). For $R \to \infty$, eqns. (25) and (26) suggest that $\phi_y = \dfrac{q_{y,3}}{p_3} = 0.70$ and

$\phi_\pi = \dfrac{q_{\pi,3}}{p_3} = 2.5$.

Following the preceding observations, it should be noted that the widespread practice of using a discount factor β may be more problematic than realized, in the sense that it may not result in robustly stabilizing strategies. This situation, namely the need to shape weights of the terms in the MPC objective in an *increasing* rather than decreasing fashion in order to ensure robustness, has been rigorously analyzed in the past [30, 31] and should be explored further.

3.2 Taylor rules from MPC with zero lower bound

When the interest rate must satisfy a ZLB constraint, the optimization problem to be solved by MPC entails the objective in eqn. (11), the equality constraints in eqns. (12)-(15), and the inequality constraint in eqn. (16). It can be shown (see Appendix C) that for $S = 0$, the entire optimization problem can be cast in the form

$$\min_{\mathbf{z}} \frac{1}{2} \mathbf{z}^T \mathbf{H} \mathbf{z}, \tag{31}$$

subject to

$$\mathbf{G}\mathbf{z} \le \mathbf{w} + \mathbf{D}\mathbf{x}_t, \tag{32}$$

where $\mathbf{z} \mathrel{\hat{=}} \mathbf{u}_m + \mathbf{H}^{-1}\mathbf{F}^T\mathbf{x}_t$, $\mathbf{D} \triangleq \mathbf{E} + \mathbf{G}\mathbf{H}^{-1}\mathbf{F}^T$, and $\mathbf{G}$, $\mathbf{w}$, $\mathbf{E}$ are defined in Appendix C.

Eqns. (31) and (32) suggest that the optimization problems solved by MPC at successive points in time differ only by the right-hand side of eqn. (32), which is affine in the state $\mathbf{x}_t$. No single formula exists for the explicit solution of all of these problems. However, the optimal solution can be expressed explicitly at each point as

$$\mathbf{z}_{t|t}^{\text{opt}} = \mathbf{H}^{-1}\mathbf{G}_A^T\left(\mathbf{G}_A\mathbf{H}^{-1}\mathbf{G}_A^T\right)^{-1}\left(\mathbf{w}_A + \mathbf{D}_A\mathbf{x}_t\right), \tag{33}$$

where $\mathbf{G}_A$, $\mathbf{w}_A$, $\mathbf{D}_A$ correspond to the set of active inequality constraints in eqn. (32), and are finite in number. Which inequality constraints in eqn. (32) will be active (i.e. equalities) at any time point t depends only on $\mathbf{x}_t$ and this can be shown [19] to be easily determined by checking the conditions

$$\mathbf{G}\mathbf{H}^{-1}\mathbf{G}_A^T\left(\mathbf{G}_A\mathbf{H}^{-1}\mathbf{G}_A^T\right)^{-1}\left(\mathbf{w}_A + \mathbf{D}_A\mathbf{x}_t\right) < \mathbf{w} + \mathbf{D}\mathbf{x}_t \tag{34}$$

and

$$-\left(\mathbf{G}_A\mathbf{H}^{-1}\mathbf{G}_A^T\right)^{-1}\left(\mathbf{w}_A + \mathbf{D}_A\mathbf{x}_t\right) \geq 0, \tag{35}$$

for each of the possible choices of {$\mathbf{G}_A$, $\mathbf{w}_A$, $\mathbf{D}_A$}. While the number of combinations of active/inactive inequality constraints may be generally large, we show in the sequel that this number is fairly small for the problem at hand, resulting in a small set of explicit rules in the form of eqn. (33), which are shown to be Taylor-like.

More specifically, for a certain {$\mathbf{G}_A$, $\mathbf{w}_A$, $\mathbf{D}_A$}, the inequalities in eqn. (34) and (35) define a linear polytope, for which the same sets of constraints remain active or inactive, and the same formula, eqn. (33), can be used to express the optimal solution for any $\mathbf{x}_t$ in that polytope. The collection of all polytopes, which are finite in number, spans the entire set in which $\mathbf{x}_t$ lies and which is bounded for a stable closed loop. Therefore, determining the active and inactive constraints in eqn. (32), and consequently the corresponding $\mathbf{G}_A$, $\mathbf{w}_A$, $\mathbf{D}_A$, is a simple matter of using a look-up table, to determine in which polytope $\mathbf{x}_t$ lies, i.e. for which of the possible { $\mathbf{G}_A$, $\mathbf{w}_A$, $\mathbf{D}_A$} eqns. (34) and (35) are satisfied. Then, eqn. (33) can be used to determine the optimal interest rate either as

$$\begin{aligned} i_t &= -[\underbrace{1 \quad 0 \quad \cdots \quad 0}_{m-1}]\mathbf{H}^{-1}\left(\mathbf{G}_A^T\left(\mathbf{G}_A\mathbf{H}^{-1}\mathbf{G}_A^T\right)^{-1}\left(\mathbf{w_A}+\mathbf{D_A}\mathbf{x}_t\right)-\mathbf{F}^T\mathbf{x}_t\right)+r^*+\pi^* \\ &= \phi_y(y_t-y^*)+\phi_\pi(\pi-\pi^*)+r^*+\pi^* \end{aligned} , \qquad (36)$$

which is a Taylor-like rule, or as

$$i_t = 0 \qquad (37)$$

namely at the ZLB value.

To our knowledge, the above development is the first rigorous derivation of an explicit Taylor-like rule that satisfies the ZLB without resorting to either ad hoc clipping of the interest rate value produced by a Taylor rule [12-14] *or numerical simulation* [8, 15-18].

3.3 Taylor rules with inertia from MPC

A simple form of a Taylor-like rule with an inertia term is

$$i_t = \phi_y(y_t-y^*)+\phi_\pi(\pi_t-\pi^*)+\phi_i(i_{t-1}-i^*)+r^*+\pi^* . \qquad (38)$$

Rules such as the above have been proposed based on empirical arguments and simulation studies, in efforts to reduce large interest rate fluctuations [2, and references therein, 4]. We explain below that such rules result naturally from appropriate tailoring of the MPC objective function to include terms that penalize the rate of change of interest rate.

To illustrate this, consider again the MPC optimization problem formulated in eqn. (11) with $R=0$ and $S>0$, namely no penalty on the interest rate itself, but a penalty on its rate of change. As in section 3.1, it can be shown (Appendix D), that the resulting MPC optimization in this case becomes

$$\min_{\mathbf{u}_m}\left[\frac{1}{2}\mathbf{u}_m^T\tilde{\mathbf{H}}\mathbf{u}_m+\tilde{\mathbf{x}}_t^T\tilde{\mathbf{F}}\mathbf{u}_m+\frac{1}{2}\tilde{\mathbf{x}}_t^T\tilde{\mathbf{Y}}\tilde{\mathbf{x}}_t\right], \tag{39}$$

where $\tilde{\mathbf{H}}\in\Re^{(m-1)\times(m-1)}$, $\tilde{\mathbf{F}}\in\Re^{3\times(m-1)}$, $\tilde{\mathbf{Y}}\in\Re^{3\times3}$ are functions of $\mathbf{A}$, $\mathbf{B}$, S, and λ; and the vector $\tilde{\mathbf{x}}$ is defined as

$$\tilde{\mathbf{x}}_t \triangleq \begin{bmatrix}\Delta y_t\\ \Delta\pi_t\\ \Delta u_{t-1}\end{bmatrix} \triangleq \begin{bmatrix}y_t-y^*\\ \pi_t-\pi^*\\ u_{t-1}-u^*\end{bmatrix}. \tag{40}$$

In the absence of a ZLB, the minimum in the optimization problem in eqn. (39) is attained at $\mathbf{u}_m^{\text{opt}}=-\tilde{\mathbf{H}}^{-1}\tilde{\mathbf{F}}^T\tilde{\mathbf{x}}_t$, resulting in the optimal interest rate

$$\begin{aligned} i_t &= -\underbrace{\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}}_{m-1}\tilde{\mathbf{H}}^{-1}\tilde{\mathbf{F}}^T\tilde{\mathbf{x}}_t+r^*+\pi^* \\ &= \phi_y(y_t-y^*)+\phi_\pi(\pi-\pi^*)+\phi_i(i_{t-1}-i^*)+r^*+\pi^* \end{aligned} \tag{41}$$

which in exactly eqn. (38).

A parametric analysis similar to that in section 3.1.4 can be performed again to assess the effect of the MPC weights S and λ on the parameters ϕ_y, ϕ_π, and ϕ_i. Similar choices of $N=80$, $m=4$ and $\beta=0.99$ as before yield

$$\phi_y=\frac{\tilde{q}_{y,3}S^6+\tilde{q}_{y,2}(\lambda)S^4+\tilde{q}_{y,1}(\lambda)S^2+\tilde{q}_{y,0}(\lambda)}{\tilde{p}_3S^6+\tilde{p}_2(\lambda)S^4+\tilde{p}_1(\lambda)S^2+\tilde{p}_0(\lambda)}, \tag{42}$$

$$\phi_\pi=\frac{\tilde{q}_{\pi,3}S^6+\tilde{q}_{\pi,2}(\lambda)S^4+\tilde{q}_{\pi,1}(\lambda)S^2+\tilde{q}_{\pi,0}(\lambda)}{\tilde{p}_3S^6+\tilde{p}_2(\lambda)S^4+\tilde{p}_1(\lambda)S^2+\tilde{p}_0(\lambda)}, \tag{43}$$

$$\phi_i = \frac{S^2\left(\tilde{q}_{i,3}S^4 + \tilde{q}_{i,2}(\lambda)S^2 + \tilde{q}_{i,1}(\lambda)\right)}{\tilde{p}_3S^6 + \tilde{p}_2(\lambda)S^4 + \tilde{p}_1(\lambda)S^2 + \tilde{p}_0(\lambda)}, \tag{44}$$

respectively, where the values of the corresponding parameters are shown in Table 5. From eqn. (44) it is clear that the inertial term ϕ_i is zero for $S = 0$. Use of eqns. (42), (43), and (44) yields the patterns shown in Figure 10, Figure 11, and Figure 12 for the coefficients ϕ_y, ϕ_π, ϕ_i as functions of λ and S. The following trends can be observed.

- The policy coefficients ϕ_y and ϕ_π decrease with increase in S.
- When S is small the effect of λ on ϕ_π is dominant compared to the effect on ϕ_y.
- After approximately $S > 2$ further increase on S does not change the policy coefficients by much.
- The inertial term ϕ_i increases with increase in S and eventually converges to 0.7. This result can be explained on the basis of the stabilizing policy criterion. If ϕ_i is large compared to ϕ_π and ϕ_y, the closed loop will behave like an open loop and due to the unstable nature of the open-loop economy model, related policies will not stabilize the economy. These results are consistent with prior literature observations [2, and references therein].

3.3.1 Inertia-based rules and resulting closed-loop stability

For Taylor rules with inertia as in eqn. (38) the corresponding closed-loop is

$$\begin{bmatrix} \mathbf{x} \\ \psi \end{bmatrix}_{t+1} = \tilde{\mathbf{A}}_{\mathrm{CL}} \begin{bmatrix} \mathbf{x} \\ \psi \end{bmatrix}_t, \tag{45}$$

where

$$\tilde{\mathbf{A}}_{\mathrm{CL}} \hat{=} \begin{bmatrix} \mathbf{A}+\mathbf{B}\mathbf{c}^T & \mathbf{B} \\ \phi_i \mathbf{c}^T & \phi_i \end{bmatrix} \hat{=} \left[\begin{array}{cc:c} \rho-\xi\phi_y & \xi-\xi\phi_\pi & -\xi \\ \alpha & 1 & 0 \\ \hdashline \phi_i\phi_y & \phi_i\phi_\pi & \phi_i \end{array}\right], \tag{46}$$

and $\psi_t = u_t - \mathbf{c}^T \mathbf{x}_t$. It can be shown (Appendix E) that all eigenvalues of $\tilde{\mathbf{A}}_{CL}$ are inside the unit disk if and only if

$$\phi_i + \phi_\pi > 1 \tag{47}$$

$$\phi_\pi > -142 - 142\phi_i + 16.7\phi_y \tag{48}$$

$$176 - 108\phi_i > \phi_\pi \tag{49}$$

$$33.5 - 35.5\phi_i + 16.7\phi_y > \phi_\pi \tag{50}$$

$$17.2 + 10.5\phi_i^2 + 8.33\phi_y + \phi_i(-28.1 - 5.06\phi_y) > \phi_\pi \tag{51}$$

as shown in Figure 13. As in section 3.1.4, it is also found that all combinations of S and λ result in stabilizing monetary policies. Eqn. (47) is the counterpart of eqn. (29) and has been derived before in a different setting, using a rational expectations approach [36].

It is again interesting to note that as $S \to \infty$, namely as aggressive changes in the value of interest rate are heavily penalized, the closed loop remains stable, due to the stabilizing equality constraint, eqn. (14). For $S \to \infty$, eqns. (42)-(44) suggest that $\tilde{\phi}_y = \frac{\tilde{q}_{y,3}}{\tilde{p}_3} = 0.095$, $\tilde{\phi}_\pi = \frac{\tilde{q}_{\pi,3}}{\tilde{p}_3} = 0.34$, and $\tilde{\phi}_i = \frac{\tilde{q}_{i,3}}{\tilde{p}_3} = 0.71$, which satisfy the inequalities in eqns. (48)-(51).

3.3.2 Taylor rules with inertia from MPC with zero lower bound

Following the same approach as in section 3.2, the optimization problem with eqn. (11) with $R = 0$, $S > 0$, subject to the equality constraints in eqns. (12)-(15), and the inequality constraint in eqn. (16) can be cast in the form

$$\min_{\tilde{\mathbf{z}}} \frac{1}{2} \tilde{\mathbf{z}}^T \tilde{\mathbf{H}} \tilde{\mathbf{z}}, \tag{52}$$

subject to

$$\mathbf{G}\tilde{\mathbf{z}} \le \mathbf{w} + \tilde{\mathbf{D}}\tilde{\mathbf{x}}_t, \tag{53}$$

where $\tilde{\mathbf{z}} \triangleq \mathbf{u}_m + \tilde{\mathbf{H}}^{-1}\tilde{\mathbf{F}}^T\tilde{\mathbf{x}}_t$, $\tilde{\mathbf{D}} \triangleq \tilde{\mathbf{E}} + \mathbf{G}\tilde{\mathbf{H}}^{-1}\tilde{\mathbf{F}}^T$ (see Appendix D). Again, an explicit solution through Taylor-like formulas can be obtained by applying the mpMPC solution to get direct counterparts of eqns. (33) through (35).

4 Numerical Simulations

The objective of this section is to illustrate the interest rate rules resulting from application of the methodology we outlined in the previous section. Emphasis is placed on directly including the ZLB constraint in the development of explicit rules.

4.1 Taylor rules form MPC with ZLB

The optimization problem defined by eqn. (31) with inequality constraints given by eqn. (32) is solved with the help of the mpMPC framework presented in section 3.2, to find the optimal interest rate rule. For the economic model discussed in section 2, the solution to the optimization problem depends on the weights λ and R in eqn. (11), for selected values of N, m and β (sections 3.1.1-3.1.3) and with $S = 0$. For each combination of

λ and R, a small number of Taylor-like rules emerge, depending on the linear polytope in which the inflation and output gap lie, as presented in Table 6 through Table 11. The corresponding linear polytopes are illustrated in Figure 14 through Figure 19.

Comparison of these tables and corresponding figures shows that the following four classes of rules emerge:

- Similar in nature to the standard Taylor rule, eqn. (1) (polytope 1),
- Setting the interest rate at its ZLB while maintaining closed-loop stability (polytope 2),
- Setting the interest rate at its ZLB *but with loss of* closed-loop stability (polytope 3) – a case of liquidity trap [12] – and
- Piecewise linear rules that are more aggressive than the Taylor-like rules that would result from optimization without anticipation of ZLB activation in the future (remaining polytopes).

Of these tables, Table 7, corresponding to Figure 15, suggests a rule in polytope 1 closest to the standard Taylor rule, in terms of both the values of $\{\phi_y, \phi_\pi\}$ ($\{1.0, 2.4\}$ vs. $\{0.5, 1.5\}$) and the closed-loop eigenvalues ($\{0.50, 0.94\}$ vs. $\{0.56, 0.97\}$).

Further comparison of these figures reveals that the optimal rules follow an *asymmetric* pattern for small values of R (Figure 14, Figure 16, Figure 18), as has also been observed in a number of numerical studies with $R = 0$ [8, 13, 17, 37]. However, this asymmetry practically disappears (i.e. it would be observable only for unrealistically large output gaps) for large values of R (Figure 17, Figure 19), namely for very sluggish policies.

Specifically, for *negative output gap*, the resulting interest rate value is equal to either what a single corresponding Taylor-like rule would produce, if that value were positive, or zero when that same Taylor rule would produce a negative value. While it is obvious that a negative interest rate value produced by a Taylor rule cannot be implemented, what is shown from the preceding analysis is that *a zero value resulting from clipping the unconstrained Taylor rule value is optimal*. In addition, for *negative output gap*, when the interest rate is close to zero and future violations of the ZLB are anticipated, *no more aggressive action is needed; the same Taylor rule remains optimal.*

On the other hand, for *positive output gap* and *low inflation*, more interesting behavior is observed, namely a small number of piecewise linear rules result, corresponding to the linear polytopes numbered 4 and above. These rules become more aggressive as the interest rate approaches the ZLB. This behavior (*pre-emptiveness*) has also been observed in numerical studies [2, and references therein, 17]. However, in contrast to these numerical simulation studies, explicit rules are derived here, and these rules are (piecewise) linear rather than nonlinear.

4.2 Taylor rules with inertia form MPC with ZLB

For $S = 0.55$ and $\lambda = 0.5$, the resulting piece-wise linear policies and corresponding polytopes are shown in Table 12. The parameter space of mpMPC, which is now three-dimensional, is partitioned in 6 polytopes shown in Figure 20. Polytope 1 corresponds to no constraint being active and hence it produces a rule as in eqn. (41). In polytope 2 the ZLB is active, i.e. the optimal policy is at zero. Polytopes 4, 5 and 6 entail rules that are different from the Taylor-like rule of polytope 1, in anticipation of future ZLB activation. The infeasibility polytope remains the same. From Table 12 and Figure 20 it can be concluded that in polytopes of low inflation and negative output gap, if the lagged interest rate i_{t-1} is high (polytopes 4 and 6), the optimal rule becomes less aggressive than the rule in the unconstrained case. However, for low i_{t-1}, the optimal rule is just a truncation to zero of the unconstrained case, eqn. (41). Also, in polytope 5, characterized by low inflation, high output gap, and high i_{t-1}, the optimal rule is more aggressive than the rule in the unconstrained case, eqn. (41). Therefore, an important conclusion is that *for rules with inertia ($S > 0$), the optimal policy becomes asymmetrical with respect to both lagged interest rate and output gap for low inflation economic conditions*.

4.3 Remarks on rules from MPC

The following can be observed in the results of sections 4.1 and 4.2.

- Polytope 1, where no constraint is active, grows in size with increasing R or S.
- The policy becomes sluggish and the size of polytopes 2, 4 and higher decreases as R or S increase.
- For any MPC formulation, situations may arise in which either a negative interest rate would be optimal (when the ZLB is not explicitly included in the optimization) or a stabilizing interest rate at or above the ZLB is not feasible (when the ZLB is explicitly included in the optimization). It can be shown (Appendix F) that for an economy model such as described by eqns. (4)-(7) the infeasibility polytope is characterized as the set of state values $\mathbf{x}_t$ that satisfy the inequality

$$\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{x}_t > \frac{\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{B}}{J_u - 1} i^* \tag{54}$$

It is clear that the state $\mathbf{x}_t$ may satisfy eqn. (54) fairly easily for economies with low i^*, i.e. such economies at corresponding conditions run the risk of falling into the infeasibility polytope where a stabilizing interest rate above the ZLB may not exist. This situation has also been studied in literature numerically [e.g., 38].

- For $\mathbf{x}_t$ in a polytope such that a feasible MPC solution exists but not all of the corresponding closed-loop eignevalues are inside the unit disk, the state will definitely escape from that polytope and will enter one where stability is guaranteed. By contrast, for $\mathbf{x}_t$ in a polytope such that no feasible MPC solution exists and not all of the corresponding closed-loop eigenvalues are inside the unit disk, instability will persist. This is illustrated further in Figure 23, discussed below.
- It should be noted that entering into the polytope 2, where the ZLB is active, is an alarming situation, as the infeasibility polytope 3 seats next to this polytope. The longer the economy stays at ZLB, the higher the chance of getting into the infeasibility polytope (a case of liquidity trap) as a result of sudden adverse fluctuations in the economy. Similar observations have been made through numerical simulation [12].
- In Figure 14 through Figure 19 real-time economy data are plotted for 2008Q1:2011Q1. It is clear that from Figure 14, Figure 16 and Figure 18 ($R = 0.07$) that clipping to zero is optimal interest rate for nearly all economic points while in Figure 15, Figure 17 and Figure 19 ($R = 0.55$) more of the economic data indicate non-zero interest rate due to the policy rule being sluggish.

4.4 Closed-loop Simulations

4.4.1 Illustration of proposed approach

The first set of simulations serves to simply illustrate the effects of ZLB on the closed-loop system. Simulations are shown using the rules presented in Table 6 through Table 11, as well as the rules with inertia shown in Table 12 along with five additional rules with similar structure but different MPC weights R and S (not shown in Table 12 for brevity). For this set of simulations the economy is considered to be at $y = -3.7$ and $\pi = 1.9$ in year 1, corresponding to 2009Q1. The results are summarized in Figure 21 and Figure 22. The resulting sums of squared errors (discrepancies between actual and desired values) are summarized in Table 13 and Table 14.

Based on these simulation results, it is clear that for small values of R or S, optimal interest rate rules are aggressive and more likely to produce interest rate values at the ZLB when corresponding conditions arise. Conversely, increase in the values of R or S results in sluggish response, as expected.

The second set of simulations illustrates a liquidity trap case. Figure 23, shows state-space partition for $R = 0.07$ and $\lambda = 0.5$. Two different initial conditions of the economy are considered. For the first case we let the initial point be $y_1 = -7.1$, $\pi_1 = 1.5$ (2009Q3), which lies in polytope 2 in Figure 23 and hence the corresponding optimal interest rate is zero. For the second case we let $y_1 = -7.1$, $\pi_1 = 0$, which lies inside the infeasibility polytope 3, namely no non-negative interest rate can stabilize the economy at that point. A zero interest rate alone results in an unstable closed loop. The only way to stabilize the closed loop would be through additional external stimulus. Given the fact that it is practically difficult to exactly quantify the polytope of liquidity trap, the central back should focus on external stimulus as soon as the ZLB is reached. Closed-loop simulations, the results of which are shown in Figure 24, confirm the preceding assertions for both cases. It is also interesting to note that even though the interest rate in the first case is stabilizing, recovery of the economy is very slow due to the effect of ZLB (inflation stabilization, in particular, takes many years).

4.4.2 Comparison with historical data

We use real-time data available to the central bank at the time of making a decision on the interest rate, for the period 1987Q4:2008Q4. For output gap we use Greenbook data over the period 1987Q4:2005Q4; for the remaining period we consider CBO data [39]. The real-time inflation data is also taken from the same publication.

We focus on the interest rate rule with inertia, eqn. (38), with $r^* = 1.9$ and $\pi^* = 2$. Since the coefficients ϕ_y, ϕ_π and ϕ_i are functions of the weights S and λ as given by eqns. (42)-(44), these weights and corresponding coefficients are estimated using regression to fit the historical data. Estimated values over the entire period of data are shown in Table 15. Figure 25 compares the interest rate resulting from fitting eqn. (38) to the interest rate implemented, as well as to the interest rate suggested by the standard Taylor rule (eqn. (1) with $\phi_y = 0.5$, $\phi_\pi = 1.5$), and by the Taylor rule with values fitted over the entire period of data examined (eqn. (1) with $\phi_y = 0.77$, $\phi_\pi = 2.0$). It is clear that the inertial rule captures the central bank decisions better, as also demonstrated by the residuals shown in Figure 27.

It is also interesting to examine whether additional insight may be gained by fitting data over short periods for which large residuals result from fitting the entire data set. One such period with large residuals is 2000Q1:2004Q4. Table 15 (line 2a) suggests that this period may be problematic, in that the corresponding inertial rule, if applicable, is not stabilizing, i.e. the fitted value of $\phi_\pi + \phi_i$ is greater than 1, thus violating the closed-loop stability condition in eqn. (47). In fact, it is dubious whether the same objective as on the average was used over that period, since the value of λ fitted over that period is negative, hence unacceptable. Constrained fitting (i.e. enforcing $0 \le \lambda \le 1$) produces parameter values that do correspond to a stabilizing rule (Table 15, line 2b) but nonetheless places all emphasis on output gap (growth). The actual policy implemented over that period and its role on stimulating over-expansion of the economy has been the subject of intense discussion [40].

5 Conclusions and Future Work

The main issue addressed in this work is the effect of zero lower bound on the optimal interest rate determined by a central bank. We address this issue in a multi-parametric model predictive control framework, which allows the derivation of explicit feedback rules even when inequality constraints are present. Application of this framework to a simple model of the US economy produced a number of Taylor-like rules, depending on the form and parameter values in the objective function employed by MPC. The results suggest that a small number of simple Taylor-like rules can be applied at each time, depending on the state of the economy. However, it was also shown that simply setting to zero negative interest rates produced by unconstrained Taylor rules is optimal in situations of negative output gap, as happened recently. Furthermore, it was observed, as has been noted elsewhere, that rules with inertia appear to better capture past decisions by the central bank. Such rules have been systematically derived here by considering penalties on the rate of interest rate change in the MPC objective function.

A number of issues touched in this work warrant further investigation, such as the following:

- The inverse problem: Given a suggested Taylor-like rule, what objective function, as in eqn. (11), is minimized? A promising approach is suggested in section 3.1.5.
- Robust stability and performance: There is a vast body of work in the automatic control community addressing the robustness issue, namely how a controller performs when the model assumed in controller design has quantifiable uncertainty.
- Modeling and selection of controlled variables: Should the pair output gap and inflation be the main focus or could variables such unemployment [5] be central in controlling an economy?
- Policy adaptation: The main attractiveness of a fixed rule is its simplicity and predictability [38]. However, such a rule may become sub-optimal over time, as the economy or disturbance models change [10]. Can a fixed rule be replaced by a fixed rule adaptation policy that maintains robustness?

We hope to address the above issues in forthcoming publications.

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7 References

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8 Appendices

Appendix A. mpMPC formulation for Taylor rules

Based on the optimization function in eqn. (11) and the method discussed in Muske and Rawlings [25] with discount factor β, the terminal penalty weight matrix $\bar{\mathbf{Q}}$ is

$$\bar{\mathbf{Q}} = \sum_{i=0}^{\infty} \mathbf{A}^{T^i} \beta^i \mathbf{Q} \mathbf{A}^i . \tag{55}$$

Since the unstable mode is constrained to be zero at time $k+N$, it follows that

$$\bar{\mathbf{Q}} = \tilde{\mathbf{v}}_s \sum \tilde{\mathbf{v}}_s^T . \tag{56}$$

where

$$\sum = \frac{\mathbf{v}_s^T Q \mathbf{v}_s}{1-\beta J_s^2} . \tag{57}$$

From eqns. (56) and (57) it follows that

$$\bar{\mathbf{Q}} = \frac{\mathbf{v}_s^T \mathbf{Q} \mathbf{v}_s}{1-\beta J_s^2} \tilde{\mathbf{v}}_s \tilde{\mathbf{v}}_s^T . \tag{58}$$

Further, eqn. (14) along with eqn. (15) results in

$$u_{t+m-1|t} = \mathbf{a}_m^T \mathbf{x}_t + \mathbf{b}_m^T \mathbf{u}_m , \tag{59}$$

where

$$\mathbf{a}_m^T = \frac{-\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{A}^N}{\tilde{\mathbf{v}}_{\mathrm{u}}^T \left(\mathbf{A}^{N-m}\mathbf{B} + ... + \mathbf{AB} + \mathbf{B} \right)} ,\ \mathbf{b}_m^T = \frac{-\tilde{\mathbf{v}}_{\mathrm{u}}^T \left[\mathbf{A}^{m-2}\mathbf{B} ..., \mathbf{AB}, \mathbf{B} \right]}{\tilde{\mathbf{v}}_{\mathrm{u}}^T \left(\mathbf{A}^{N-m}\mathbf{B} + ... + \mathbf{AB} + \mathbf{B} \right)} \tag{60}$$

and the optimization variable $\mathbf{u}_m$ contains the first $m-1$ elements of $\mathbf{u}$.

Using eqns. (12) and (15) for the case when $k > m$ yields

$$\hat{\mathbf{x}}_{t+k|t} = \mathbf{A}^k x_t + \sum_{\ell=1}^{m-1} \mathbf{A}^{k-\ell} \mathbf{B} u_{t+\ell-1|t} + \left(\sum_{\ell=m}^{k} \mathbf{A}^{k-\ell} \mathbf{B} \right) u_{t+m-1|t} . \tag{61}$$

Using eqns. (59) and (61) yields

$$\hat{\mathbf{x}}_{t+k|t} = \mathbf{f}_{k,\ell} \mathbf{x}_t + \sum_{\ell=1}^{m-1} \mathbf{h}_{k,\ell} u_{t+\ell-1|t} \ , \tag{62}$$

where

$$\mathbf{h}_{k,\ell} = \begin{cases} \left(\mathbf{A}^{k-\ell} + b_\ell \left(\sum_{\ell=m}^{k} \mathbf{A}^{k-\ell} \right) \right) \mathbf{B} & k \geq m, \ell \leq k \\ \mathbf{A}^{k-\ell} \mathbf{B} & k < m, \ell \leq k \\ 0 & k < m, \ell > k \end{cases} , \tag{63}$$

$$\begin{aligned} \mathbf{f}_k &= \left[\mathbf{f}_{k,1} ...... \mathbf{f}_{k,m-1} \right] \in \Re^{2\times(m-1)} \\ \mathbf{f}_{k,\ell} &= \begin{cases} \mathbf{A}^k + \sum_{\ell=m}^{k} \mathbf{A}^{k-l} \mathbf{B} \mathbf{a}^T & \text{for } k \geq m \\ \mathbf{A}^k & \text{for } k < m \end{cases} , \end{aligned} \tag{64}$$

$$\mathbf{h}_k = \left[\mathbf{h}_{k,1} ...... \mathbf{h}_{k,m-1} \right] \in \Re^{(m-1)\times(m-1)} . \tag{65}$$

Substituting eqns (62)-(65) into eqn. (11) with $S = 0$ yields eqn. (22).

The solution to eqn. (22) is

$$\mathbf{u}_m = -\mathbf{H}^{-1} \mathbf{F}^T \mathbf{x}_t . \tag{66}$$

where

$$\mathbf{H} = \sum_{k=1}^{N-1} \mathbf{h}_k^T \beta^k \mathbf{Q} \mathbf{h}_k + \mathbf{h}_N^T \beta^N \bar{\mathbf{Q}} \mathbf{h}_N + R^2 \left(\frac{\beta^{m-1} - \beta^N}{1-\beta} \mathbf{b}\mathbf{b}^T + \mathbf{D}_R \right), \tag{67}$$

$$\mathbf{D}_R \hat{=} \operatorname{diag}\left[1 \quad \beta \quad . \quad . \quad \beta^{m-2} \right] \tag{68}$$

$$\mathbf{F} = \left(\sum_{k=1}^{N-1} \mathbf{f}_k^T \beta^k \mathbf{Q} \mathbf{h}_k \right) + \mathbf{f}_N^T \beta^N \bar{\mathbf{Q}} \mathbf{h}_N + R^2 \frac{\beta^{m-1} - \beta^N}{1-\beta} \mathbf{a} \mathbf{b}^T . \tag{69}$$

Appendix B. Closed-loop stability for Taylor rule

The standard Taylor rule can be written as

$$u_t = \mathbf{c}^T \mathbf{x}_t \, , \tag{70}$$

where $\mathbf{c}^T \triangleq \begin{bmatrix} \phi_y & \phi_\pi \end{bmatrix}$.

The characteristic equation for the matrix $\mathbf{A}_{\mathrm{CL}}$ in eqn. (28) is given by

$$f(\mu) \triangleq \mu^2 - \left(1 + \rho + \alpha\xi - \xi\phi_y - \alpha\xi\phi_\pi\right)\mu + \left(\rho - \xi\phi_y\right) , \tag{71}$$

where μ is an eigenvalue of the matrix $\mathbf{A}_{\mathrm{CL}}$. For closed-loop stability the eigenvalues of the matrix $\mathbf{A}_{\mathrm{CL}}$ should lie inside the unit disk, which is guaranteed (by the Jury-Routh–Hurwitz stability criterion) if and only if

$$2 + 2\rho - 2\xi\phi_y + \alpha\xi(\phi_\pi - 1) > 0 , \tag{72}$$

$$1 - \rho + \xi\phi_y - \alpha\xi(\phi_\pi - 1) > 0 , \tag{73}$$

$$\alpha\xi(\phi_\pi - 1) > 0 . \tag{74}$$

Given that $\alpha\xi > 0$, eqn. (74) is satisfied if and only if $\phi_\pi > 1$.

Appendix C. mpMPC formulation for Taylor rules with inertia

Using the equality constraints in Appendix A, the ZLB constraint given in eqn. (16) can be written as,

$$\mathbf{G}\mathbf{u}_m \leq \mathbf{w} + \mathbf{E}\mathbf{x}_t\,, \tag{75}$$

where $\mathbf{G} \hat{=} \begin{bmatrix} -\mathbf{I} \\ -\mathbf{b}^T \end{bmatrix}$; $\mathbf{I}$ is the identity matrix in $\Re^{(m-1)\times(m-1)}$; $\mathbf{w} \hat{=} \begin{bmatrix} i^* & \cdots & i^* \end{bmatrix}^T \in \Re^m$;

$\mathbf{E} = \begin{bmatrix} \mathbf{\Theta} \\ \mathbf{a}^T \end{bmatrix}$; $\mathbf{\Theta} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}^T \in \Re^{(m-1)\times 2}$. Therefore, the optimization problem eqn. (22) subject to the constraint eqn. (75) can be formulated as

$$\min_{\mathbf{z}} \frac{1}{2}\mathbf{z}^T\mathbf{H}\mathbf{z} \tag{76}$$

$$\mathbf{G}\mathbf{z} \leq \mathbf{w} + \mathbf{D}\mathbf{x}_t \tag{77}$$

where $\mathbf{z} \hat{=} \mathbf{u}_m + \mathbf{H}^{-1}\mathbf{F}^T\mathbf{x}_t$, $\mathbf{D} \hat{=} \mathbf{E} + \mathbf{G}\mathbf{H}^{-1}\mathbf{F}^T$.

Appendix D. mpMPC formulation for Taylor rules with inertia

Adopting the same approach as shown in Appendix A, a similar kind of expression for the optimization problem set-up in eqn. (11) can be derived when $S > 0$ as

$$\min_{\mathbf{u}_m} \left[\frac{1}{2}\mathbf{u}_m^T \tilde{\mathbf{H}} \mathbf{u}_m + \tilde{\mathbf{x}}_t^T \tilde{\mathbf{F}} \mathbf{u}_m + \frac{1}{2} \tilde{\mathbf{x}}_t^T \tilde{\mathbf{Y}} \tilde{\mathbf{x}}_t \right], \tag{78}$$

where,

$$\tilde{\mathbf{x}}_t \triangleq \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta u_{t-1} \end{bmatrix}, \tag{79}$$

$$\begin{aligned} \tilde{\mathbf{H}}_{m-1\times m-1} = \sum_{k=1}^{N-1} \mathbf{h}_k^T \beta^k \mathbf{Q} \mathbf{h}_k + \mathbf{h}_N^T \beta^N \bar{\mathbf{Q}} \mathbf{h}_N + \\ S^2 \left(\beta^{m-1} \left(\mathbf{b}-\mathbf{b}_0\right)\left(\mathbf{b}-\mathbf{b}_0\right)^T + \beta^{N-1} \mathbf{b}\mathbf{b}^T + \mathbf{S}_0 \right) \end{aligned}, \tag{80}$$

where $\mathbf{b}_0 = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T \in \Re^{m-1}$, $\mathbf{S}_0 \in \Re^{(m-1)\times(m-1)}$ is given by,

$$\mathbf{S}_0 \triangleq \left[s_{i,j} \right] \begin{cases} s_{i,j} = \beta^{i-1}\left(1+\beta\right), i = j, i \neq m-1 \\ s_{i,j} = \beta^{m-2}, i = j, i = m-1 \\ s_{i,j} = -\beta, \left|i-j\right| = 1 \\ s_{i,j} = 0, \left|i-j\right| > 1 \end{cases}, \tag{81}$$

and

$$\tilde{\mathbf{F}}_{3\times m-1} = \begin{bmatrix} \left(\sum_{k=1}^{N-1} \mathbf{f}_k^T \mathbf{Q} \mathbf{h}_k \right) + \mathbf{f}_N^T \bar{\mathbf{Q}} \mathbf{h}_N + S^2 \left[\beta^{m-1} \mathbf{a} \left(\left(\mathbf{b}-\mathbf{b}_0\right)^T + \beta^{N-1} \mathbf{b}^T \right) \right] \\ -S^2, \underbrace{0.......0}_{m-2} \end{bmatrix}. \tag{82}$$

When there is no inequality constraint, the solution to eqn. (78) is given by

$$\mathbf{u}_m = -\tilde{\mathbf{H}}^{-1} \tilde{\mathbf{F}}^T \tilde{\mathbf{x}}_t . \tag{83}$$

ZLB constraint given by eqn. (16) is equivalent to,

$$\mathbf{G}\mathbf{u}_m \le \mathbf{w} + \tilde{\mathbf{E}}\tilde{\mathbf{x}}_t \,, \tag{84}$$

where $\tilde{\mathbf{E}} = \begin{bmatrix} \mathbf{E} & \mathbf{E}_0 \end{bmatrix}$ and $\mathbf{E}_0 = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}^T \in \Re^m$. Eqns. (78) and (84) can be formulated as,

$$\min_{\tilde{\mathbf{z}}} \frac{1}{2} \tilde{\mathbf{z}}^T \tilde{\mathbf{H}} \tilde{\mathbf{z}} \tag{85}$$

$$\mathbf{G}\tilde{\mathbf{z}} \le \mathbf{w} + \tilde{\mathbf{D}}\tilde{\mathbf{x}}_t \tag{86}$$

where $\tilde{\mathbf{z}} \mathrel{\hat{=}} \mathbf{u}_m + \tilde{\mathbf{H}}^{-1}\tilde{\mathbf{F}}^T\tilde{\mathbf{x}}_t$, $\tilde{\mathbf{D}} \mathrel{\hat{=}} \tilde{\mathbf{E}} + \mathbf{G}\tilde{\mathbf{H}}^{-1}\tilde{\mathbf{F}}^T$. Eqn. (85) and inequality constraints eqn. (86) are used for mpMPC formulation to derive explicit inertia-based Taylor rules with ZLB constraints.

Appendix E. Closed-loop stability for inertial Taylor-like rule

The interest rate rule is

$$u_t = \phi_i u_{t-1} + \mathbf{c}^T \mathbf{x}_t \,, \tag{87}$$

The characteristic equation for the matrix $\tilde{\mathbf{A}}_{CL}$ is given by

$$\begin{aligned} \tilde{f}(\mu) &\hat{=} \mu^3 - \left(1+\rho-\xi\phi_y+\phi_i\right)\mu^2 \\ &+\left(\rho-\xi\phi_y+\left(1+\rho\right)\phi_i-\alpha\xi\left(1-\phi_\pi\right)\right)\mu-\left(\rho-\alpha\xi\right)\phi_i \end{aligned}. \tag{88}$$

Closed-loop stability is guaranteed (by the Jury-Routh–Hurwitz stability criterion) if and only if

$$2+2\phi_i+2\rho(1+\phi_i)-2\xi\phi_y-\alpha\xi(1+\phi_i-\phi_\pi)>0\,, \tag{89}$$

$$4-4\rho\phi_i+\alpha\xi(1+3\phi_i-\phi_\pi)>0\,, \tag{90}$$

$$2-2\phi_i+2\rho(-1+\phi_i)+2\xi\phi_y+\alpha\xi(1-3\phi_i-\phi_\pi)>0\,, \tag{91}$$

$$\alpha\xi(\phi_\pi+\phi_i-1)>0\,, \tag{92}$$

$$\begin{aligned} &-8\left(\left(\alpha\xi\phi_i\right)^2+\left(\rho\phi_i-1\right)\left(1-\rho+\rho\phi_i\right)-\phi_i+\xi\phi_y\right) \\ &+\alpha\xi\left(\left(1-2\rho\right)\phi_i^2+\phi_i\left(1+\rho-\xi\phi_y\right)+\phi_\pi-1\right)>0 \end{aligned}, \tag{93}$$

Appendix F. Infeasibility polytope

The model decomposition of $\mathbf{A}$ is represented by,

$$\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1} = \left[\mathbf{v}_{\mathrm{u}} \;\vdots\; \mathbf{v}_{\mathrm{s}} \right] \begin{bmatrix} J_{\mathrm{u}} & 0 \\ 0 & J_{\mathrm{s}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{v}}_{\mathrm{u}}^{T} \\ \hdashline \tilde{\mathbf{v}}_{\mathrm{s}}^{T} \end{bmatrix} \tag{94}$$

where

$$J_u = \frac{1+\rho+\sqrt{(1-\rho)^2+4\alpha\xi}}{2} > 1 \tag{95}$$

$$J_{\mathrm{s}} = \frac{1+\rho-\sqrt{(1-\rho)^2+4\alpha\xi}}{2} < 1 \tag{96}$$

Eqns. (94) and (8) imply

$$\mathbf{V}^{-1}\hat{\mathbf{x}}_{t+k|t} = \sum_{\ell=0}^{k-1} \mathbf{J}^{\ell}\mathbf{V}^{-1}\mathbf{B}u_{t+k-\ell-1|t} + \mathbf{J}^{k}\mathbf{V}^{-1}\mathbf{x}_t \tag{97}$$

From eqn. (97) stable and unstable modes can be treated separately. In terms of the unstable mode

$$\tilde{\mathbf{v}}_{\mathrm{u}}^{T}\hat{\mathbf{x}}_{t+k|t} = \sum_{\ell=0}^{k-1} J_u^{\ell}\, \tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{B}u_{t+k-\ell-1|t} + J_u^{k}\, \tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{x}_t \tag{98}$$

If $\mathbf{x}_t$ lies in the polytope of attraction, then

$$\lim_{k\to\infty} \tilde{\mathbf{v}}_{\mathrm{u}}^{T}\hat{\mathbf{x}}_{t+k|t} = 0 \tag{99}$$

and

$$\tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{x}_t = -J_u^{-k}\sum_{\ell=0}^{k-1} J_u^{\ell}\, \tilde{\mathbf{v}}_{\mathrm{u}}^{T}\mathbf{B}u_{t+k-\ell-1|t} \tag{100}$$

since $-u_{t+k-\ell-1|t} \le i^*$.

The polytope of attraction is given by

$$\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{x}_t \le \lim_{k\to\infty} \left(\sum_{\ell=0}^{\ell=k-1} J_u^{\ell-k} \right) \tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{B} i^* \Rightarrow \tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{x}_t \le \frac{\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{B}}{J_u - 1} i^* . \quad (101)$$

Hence the infeasibility polytope is characterized by,

$$\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{x}_t > \frac{\tilde{\mathbf{v}}_{\mathrm{u}}^T \mathbf{B}}{J_u - 1} i^* . \quad (102)$$

Similarly, in the case of inertial policy the above exercise can be repeated and the counterpart of eqn. (101) can be derived.

Table 1. Parameter estimates of US economy model

Parameter	Estimate	Standard Error
ρ	0.63	0.06
ξ	0.19	0.05
$r*$	1.9	0.74
α	0.12	0.06
σ_{e^y}	1.4	
σ_{e^π}	0.93	

Table 2. Closed-loop eigenvalues for Taylor-like rules derived from unconstrained MPC for $\lambda = 0.05$ and $R = 0.07$

m	N							
	20		40		60		80	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
2	0.05	0.95	0.05	0.95	0.05	0.95	0.05	0.94
3	0.07	0.95	0.07	0.97	0.07	0.96	0.07	0.96
4	0.07	0.95	0.07	0.97	0.07	0.97	0.07	0.96
8	0.07	0.95	0.07	0.97	0.07	0.97	0.07	0.97
12	0.07	0.95	0.07	0.97	0.07	0.97	0.07	0.97
16	0.07	0.95	0.07	0.97	0.07	0.97	0.07	0.97

Table 3. Output gap and inflation coefficients in Taylor-like rules (eqn. (1)) derived from unconstrained MPC for $\lambda = 0.05$ and $R = 0.07$

m	N							
	20		40		60		80	
	ϕ_y	ϕ_π	ϕ_y	ϕ_π	ϕ_y	ϕ_π	ϕ_y	ϕ_π
2	3.2	2.9	3.1	2.4	3.1	2.4	3.1	2.5
3	3.2	2.9	3.1	2.4	3.1	2.4	3.1	2.5
4	3.2	2.9	3.1	2.4	3.1	2.4	3.1	2.5
8	3.2	2.9	3.1	2.3	3.1	2.3	3.1	2.3
12	3.2	2.9	3.1	2.2	3.1	2.2	3.1	2.2
16	3.2	2.9	3.1	2.2	3.1	2.1	3.1	2.2

Table 4. Polynomial coefficients in eqns. (25) and (26) as functions of λ

$$q_{y,3} = 1.04$$
$$q_{y,2} = 0.297 + 0.444\lambda$$
$$q_{y,1} = -0.04(-1.04 + \lambda)(0.420 + \lambda)$$
$$q_{y,0} = 6.11 \times 10^{-4} (-1.06 + \lambda)(-1.01 + \lambda)(0.365 + \lambda)$$
$$q_{\pi,3} = 3.67$$
$$q_{\pi,3} = 0.37 + 2.28\lambda$$
$$q_{\pi,1} = -0.124(-1.09 + \lambda)(0.084 + \lambda)$$
$$q_{\pi,0} = 1.59 \times 10^{-3} (-1.12 + \lambda)(-1.01 + \lambda)(0.059 + \lambda)$$
$$p_3 = 1.48$$
$$p_2 = 0.197 + 0.157\lambda$$
$$p_1 = -0.0108(-1.02 + \lambda)(0.641 + \lambda)$$
$$p_0 = 1.32 \times 10^{-4} (-1.03 + \lambda)(-1.01 + \lambda)(0.512 + \lambda)$$

Table 5. Polynomial coefficients in eqns. (42) -(44) as functions of λ

$$\tilde{q}_{y,3} = 0.428$$
$$\tilde{q}_{y,2} = 1.03 + 3.17\lambda$$
$$\tilde{q}_{y,1} = -0.117(-1.05 + \lambda)(0.372 + \lambda)$$
$$\tilde{q}_{y,0} = 6.11\times10^{-4}(-1.06 + \lambda)(-1.01 + \lambda)(0.365 + \lambda)$$
$$\tilde{q}_{\pi,3} = 1.51$$
$$\tilde{q}_{\pi,2} = 0.911 + 14.0\lambda$$
$$\tilde{q}_{\pi,1} = -0.323(-1.11 + \lambda)(0.0653 + \lambda)$$
$$\tilde{q}_{\pi,0} = 1.59\times10^{-3}(-1.12 + \lambda)(-1.01 + \lambda)(0.0588 + \lambda)$$
$$\tilde{q}_{i,3} = 3.19$$
$$\tilde{q}_{i,2} = 0.294 + 0.370\lambda$$
$$\tilde{q}_{i,1} = -0.00278(-1.02 + \lambda)(0.690 + \lambda)$$
$$\tilde{p}_3 = 4.50$$
$$\tilde{p}_2 = 0.920 + 1.42\lambda$$
$$\tilde{p}_1 = -0.0326(-1.03 + \lambda)(0.553 + \lambda)$$
$$\tilde{p}_0 = 1.32\times10^{-4}(-1.03 + \lambda)(-1.01 + \lambda)(0.512 + \lambda)$$

Table 6. mpMPC solution and state space partition for $R = 0.07$, $\lambda = 0.05$

No.	Polytope bounds	Interest rate Δi_t	Closed-loop Eigenvalues
1	$\begin{bmatrix} -0.78 & -0.62 \\ -0.14 & -0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.98 \\ 1.71 \end{bmatrix}$	$\begin{bmatrix} 3.12 & 2.49 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$	0.07 0.96
2	$\begin{bmatrix} 0.78 & 0.62 \\ -0.27 & -0.96 \\ 0.76 & 0.65 \\ 0.62 & 0.79 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} -0.98 \\ 3.70 \\ -1.03 \\ -1.39 \end{bmatrix}$	-3.9	0.58 1.05
3	$\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le -3.70$	-3.9 (Infeasible)	0.58 1.05
4	$\begin{bmatrix} -0.76 & -0.65 \\ -0.20 & -0.98 \\ 0.14 & 0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 1.03 \\ 2.02 \\ -1.71 \end{bmatrix}$	$\begin{bmatrix} 3.15 & 2.70 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 0.36$	0.07 0.96
5	$\begin{bmatrix} -0.62 & -0.79 \\ -0.13 & -0.99 \\ 0.20 & 0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 1.39 \\ 4.42 \\ -2.02 \end{bmatrix}$	$\begin{bmatrix} 3.52 & 4.49 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 4.05$	0.05 0.92
6	$\begin{bmatrix} -0.27 & -0.96 \\ 0.13 & 0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 3.70 \\ -4.42 \end{bmatrix}$	$\begin{bmatrix} 5.55 & 19.6 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 71.3$	0.00 0.58

Table 7. mpMPC solution and state space partition for $R=0.55$**,** $\lambda=0.05$

No.	Polytope bounds	Interest rate Δi_t	Closed-loop Eigenvalues
1	$\begin{bmatrix}-0.39 & -0.92\\ -0.28 & -0.96\end{bmatrix}\begin{bmatrix}\Delta y_t\\ \Delta\pi_t\end{bmatrix}\le\begin{bmatrix}1.47\\ 1.67\end{bmatrix}$	$\begin{bmatrix}1.03 & 2.44\end{bmatrix}\begin{bmatrix}\Delta y_t\\ \Delta\pi_t\end{bmatrix}$	0.50 0.93
2	$\begin{bmatrix}0.39 & 0.92\\ -0.27 & -0.96\\ 0.37 & 0.93\end{bmatrix}\begin{bmatrix}\Delta y_t\\ \Delta\pi_t\end{bmatrix}\le\begin{bmatrix}-1.47\\ 3.70\\ -1.52\end{bmatrix}$	-3.9	0.58 1.05
3	$\begin{bmatrix}0.27 & 0.96\end{bmatrix}\begin{bmatrix}\Delta y_t\\ \Delta\pi_t\end{bmatrix}\le -3.70$	-3.9 (Infeasible)	0.58 1.05
4	$\begin{bmatrix}-0.38 & -0.93\\ -0.32 & -0.95\\ 0.28 & 0.96\end{bmatrix}\begin{bmatrix}\Delta y_t\\ \Delta\pi_t\end{bmatrix}\le\begin{bmatrix}1.50\\ 1.63\\ -1.67\end{bmatrix}$	$\begin{bmatrix}1.13 & 2.77\end{bmatrix}\begin{bmatrix}\Delta y_t\\ \Delta\pi_t\end{bmatrix}+0.59$	0.50 0.92
5	$\begin{bmatrix}-0.37 & -0.93\\ 0.32 & 0.95\end{bmatrix}\begin{bmatrix}\Delta y_t\\ \Delta\pi_t\end{bmatrix}\le\begin{bmatrix}1.52\\ -1.63\end{bmatrix}$	$\begin{bmatrix}1.34 & 3.39\end{bmatrix}\begin{bmatrix}\Delta y_t\\ \Delta\pi_t\end{bmatrix}+1.65$	0.48 0.89

Table 8. mpMPC solution and state space partition for $R = 0.07 \quad \lambda = 0.8$

No.	Polytope bounds	Interest rate Δi_t	Closed-loop Eigenvalues
1	$\begin{bmatrix} -0.35 & -0.94 \\ -0.17 & -0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.40 \\ 0.74 \end{bmatrix}$	$\begin{bmatrix} 3.39 & 9.09 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$	0.22 0.76
2	$\begin{bmatrix} 0.35 & 0.94 \\ -0.27 & -0.96 \\ 0.28 & 0.96 \\ 0.33 & 0.95 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} -0.98 \\ 3.70 \\ -0.57 \\ -0.45 \end{bmatrix}$	-3.9	0.58 1.05
3	$\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le -3.70$	-3.9 (Infeasible)	0.58 1.05
4	$\begin{bmatrix} -0.33 & -0.95 \\ -0.21 & -0.98 \\ 0.17 & 0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.45 \\ 0.76 \\ -0.74 \end{bmatrix}$	$\begin{bmatrix} 3.65 & 10.6 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 1.13$	0.21 0.72
5	$\begin{bmatrix} -0.28 & -0.96 \\ 0.21 & 0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.57 \\ -0.76 \end{bmatrix}$	$\begin{bmatrix} 5.17 & 17.5 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 6.48$	0.04 0.61

Table 9. mpMPC solution and state space partition for $R = 0.55$**,** $\lambda = 0.8$

No.	Polytope bounds	Interest rate Δi_t	Closed-loop Eigenvalues
1	$\begin{bmatrix}-0.28 & -0.96\end{bmatrix}\begin{bmatrix}\Delta y_t \\ \Delta\pi_t\end{bmatrix} \le 0.86$	$\begin{bmatrix}1.29 & 4.36\end{bmatrix}\begin{bmatrix}\Delta y_t \\ \Delta\pi_t\end{bmatrix}$	0.56 0.83
2	$\begin{bmatrix}0.28 & 0.96 \\ -0.27 & -0.96\end{bmatrix}\begin{bmatrix}\Delta y_t \\ \Delta\pi_t\end{bmatrix} \le \begin{bmatrix}-0.86 \\ 3.70\end{bmatrix}$	-3.9	0.58 1.05
3	$\begin{bmatrix}0.27 & 0.96\end{bmatrix}\begin{bmatrix}\Delta y_t \\ \Delta\pi_t\end{bmatrix} \le -3.70$	-3.9 (Infeasible)	0.58 1.05

Table 10. mpMPC solution and state space partition for $R = 0.07$, $\lambda = 0.5$

No.	Polytope bounds	Interest rate Δi_t	Closed-loop Eigenvalues
1	$\begin{bmatrix} -0.44 & -0.90 \\ 0.14 & -0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.49 \\ 0.72 \end{bmatrix}$	$\begin{bmatrix} 3.51 & 7.11 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$	0.12 0.84
2	$\begin{bmatrix} 0.44 & 0.90 \\ -0.27 & -0.96 \\ 0.41 & 0.91 \\ 0.32 & 0.95 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} -0.49 \\ 3.70 \\ -0.52 \\ -0.66 \end{bmatrix}$	-3.9	0.58 1.05
3	$\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le -3.70$	-3.9 (Infeasible)	0.58 1.05
4	$\begin{bmatrix} -0.44 & -0.91 \\ -0.18 & -0.98 \\ 0.14 & 0.99 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.52 \\ 0.86 \\ -0.72 \end{bmatrix}$	$\begin{bmatrix} 3.67 & 8.26 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 0.84$	0.12 0.81
5	$\begin{bmatrix} -0.32 & -0.95 \\ 0.18 & -0.98 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 0.66 \\ -0.86 \end{bmatrix}$	$\begin{bmatrix} 4.72 & 13.95 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 5.82$	0.04 0.69

Table 11. mpMPC solution and state space partition for $R = 0.55$, $\lambda = 0.5$

No.	Polytope bounds	Interest rate Δi_t	Closed-loop Eigenvalues
1	$\begin{bmatrix} -0.31 & -0.95 \\ -0.29 & -0.96 \\ -0.27 & -0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 1.0 \\ 1.09 \\ 1.14 \end{bmatrix}$	$\begin{bmatrix} 1.21 & 3.71 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix}$	0.53 0.87
2	$\begin{bmatrix} 0.31 & 0.95 \\ -0.27 & -0.96 \\ 0.30 & 0.95 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} -1.00 \\ 3.70 \\ -1.04 \end{bmatrix}$	-3.9	0.58 1.05
3	$\begin{bmatrix} 0.27 & 0.96 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le -3.70$	-3.9 (Infeasible)	0.58 1.05
4	$\begin{bmatrix} -0.61 & -0.79 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 2.03 \\ 4.51 \\ -2.99 \end{bmatrix}$	$\begin{bmatrix} 1.4 & 4.38 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 0.8$	0.53 0.84
5	$\begin{bmatrix} -0.61 & -0.79 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} \le \begin{bmatrix} 2.03 \\ 4.51 \\ -2.99 \end{bmatrix}$	$\begin{bmatrix} 1.77 & 5.63 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \end{bmatrix} + 2.23$	0.51 0.79

Table 12. mpMPC solution and state space partition for $S = 0.55$, $\lambda = 0.5$

No.	Polytope bounds	Interest rate Δi_t	Closed-loop Eigenvalues
1	$\begin{bmatrix} -0.31 & -0.94 & -0.16 \\ -0.33 & -0.95 & -0.04 \\ -0.29 & -0.96 & 0.01 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} 1.27 \\ 0.97 \\ 1.21 \end{bmatrix}$	$\begin{bmatrix} 0.96 & 2.88 & 0.48 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix}$	0.74 $0.59 + 0.20i$ $0.59 - 0.20i$
2	$\begin{bmatrix} 0.31 & 0.94 & 0.16 \\ -0.27 & -0.96 & 0 \\ 0.31 & 0.93 & 0.19 \\ 0.32 & 0.92 & 0.22 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} -1.27 \\ 3.70 \\ -1.42 \\ -1.43 \end{bmatrix}$	-3.9	0.58 1.05 0
3	$\begin{bmatrix} 0.27 & 0.96 & 0 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le -3.7$	-3.9 (Infeasible)	0.58 1.05 0
4	$\begin{bmatrix} -0.32 & -0.92 & -0.22 \\ -0.28 & -0.96 & 0.03 \\ 0.30 & 0.95 & 0.04 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} 1.43 \\ 1.33 \\ -0.97 \end{bmatrix}$	$\begin{bmatrix} 0.63 & 1.88 & 0.44 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} - 1.06$	0.87 $0.54 + 0.13i$ $0.54 - 0.13i$
5	$\begin{bmatrix} -0.31 & -0.95 & -0.05 \\ 0.29 & 0.96 & -0.01 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} 0.89 \\ -1.21 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 & 0.48 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} + 0.16$	0.73 $0.59 + 0.21i$ $0.59 - 0.21i$
6	$\begin{bmatrix} -0.31 & -0.93 & -0.19 \\ -0.27 & -0.96 & 0.02 \\ 0.31 & 0.95 & 0.05 \\ 0.28 & 0.96 & -0.03 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} \le \begin{bmatrix} 1.42 \\ 3.97 \\ -0.89 \\ -1.33 \end{bmatrix}$	$\begin{bmatrix} 0.71 & 2.1 & 0.43 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta \pi_t \\ \Delta i_{t-1} \end{bmatrix} - 0.69$	0.85 $0.54 + 0.13i$ $0.54 - 0.13i$

Table 13. Sum of squared errors for closed-loop simulations with $\lambda = 0.05$

	$S=0, R=0.07$	$S=0, R=0.55$	$S=0.07, R=0$	$S=0.55, R=0$
$\sum_{t=2}^{20} y_t^2$	3.30	4.12	3.29	3.65
$\sum_{t=2}^{20} (\pi_t - 2)^2$	7.21	6.69	7.22	6.52
$\sum_{t=1}^{19} (i_t - 3.9)^2$	54.5	53.2	54.4	54.3
$\sum_{t=1}^{19} (i_t - i_{t-1})^2$	3.63	1.52	3.32	1.54

Table 14. Sum of squared errors for closed-loop simulations with $\lambda = 0.8$

	$S = 0, R = 0.07$	$S = 0, R = 0.55$	$S = 0.07, R = 0$	$S = 0.55, R = 0$
$\sum_{t=2}^{20} y_t^2$	7.14	5.75	7.43	7.99
$\sum_{t=2}^{20} (\pi_t - 2)^2$	3.02	3.72	2.97	2.93
$\sum_{t=1}^{19} (i_t - 3.9)^2$	84.0	71.2	85.8	88.4
$\sum_{t=1}^{19} (i_t - i_{t-1})^2$	4.27	1.43	3.59	2.47

Table 15: Inertial Policy estimation for US interest rate rule based on real-time data. Standard deviations are reported in brackets.

	Period	S	λ	ϕ_y	ϕ_π	ϕ_i	$\phi_\pi + \phi_i$
	1987Q4:2008Q4	0.83(0.23)	0.09(0.03)	0.29	0.71	0.62	1.33
1	1987Q4:1999Q4	1.1(0.43)	0.10(0.06)	0.24	0.67	0.64	1.31
2a	2000Q1:2004Q4	0.15(0.08)	-0.07(0.03)	0.66	0.13	0.47	0.60
2b	2000Q1:2004Q4	0.3	0	0.48	0.60	0.55	1.15
3	2005Q1:2008Q4	0.44(0.26)	0.16(0.1)	0.53	1.25	0.55	1.80

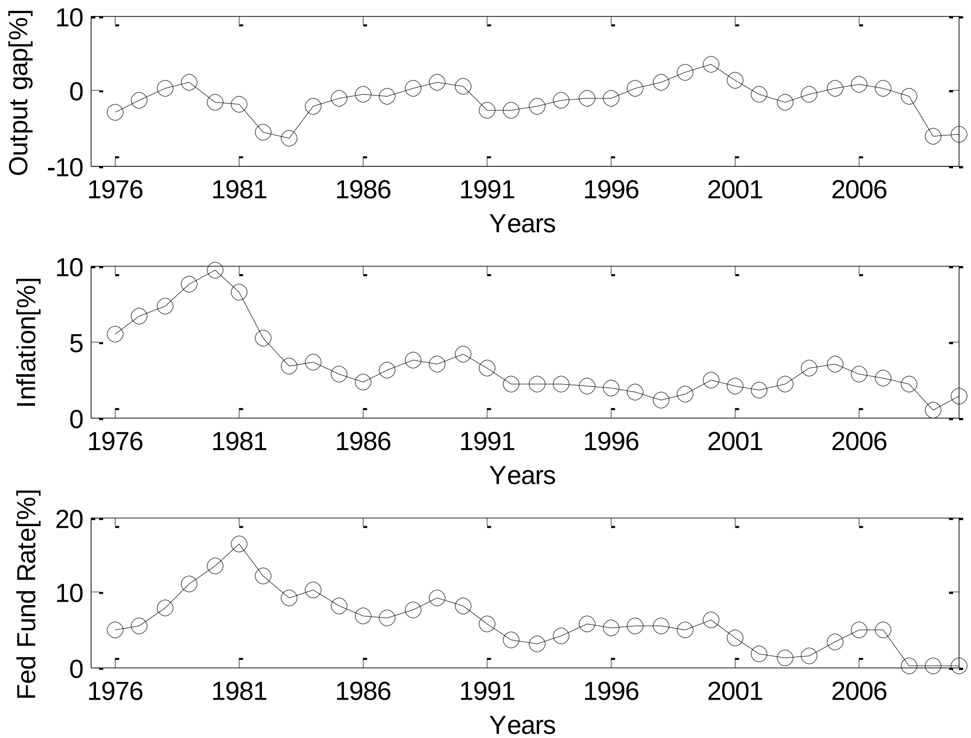

Figure 1. Revised data for US output gap, GDP deflator inflation rate and federal fund rates in annual percentage for year 1976-2010.[24]

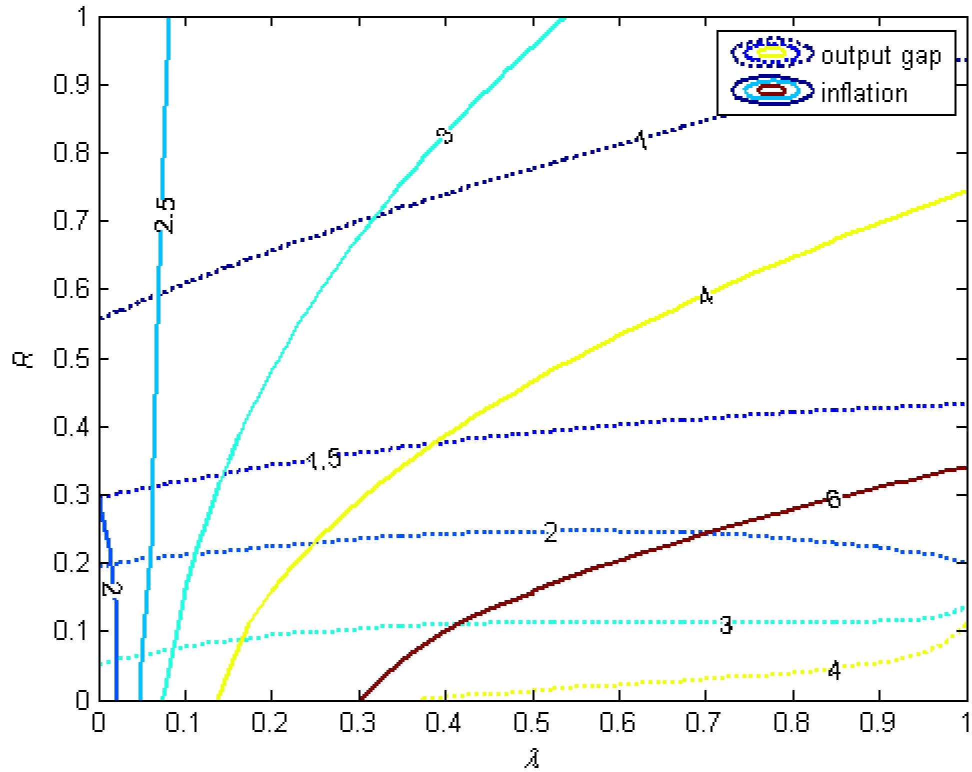

Figure 2. Taylor-like interest rate rule for when there is no constraint on interest rate for various values of tuning parameters R and λ. Solid and dotted lines represent inflation and output gap coefficient respectively based on eqn. (25)-(26). This solution is also valid when no constraint is active in case of constrained MPC.

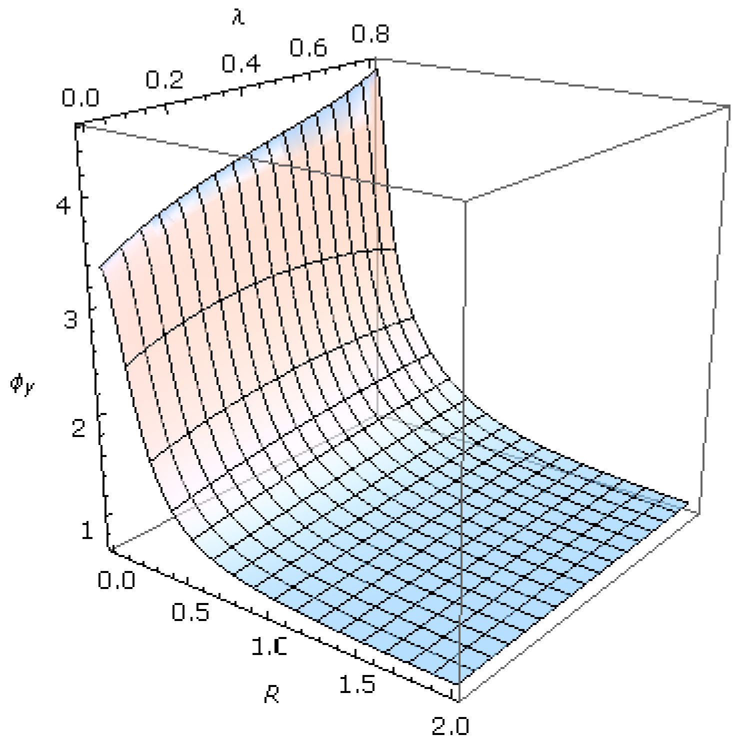

Figure 3. Output gap coefficient ϕ_y for Taylor rule when $\beta = 0.99$

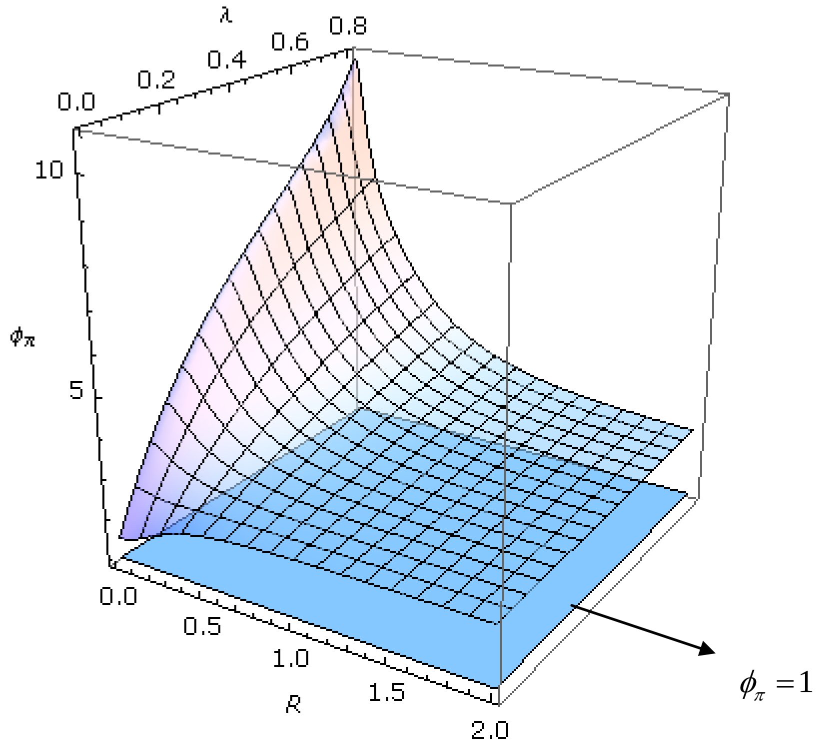

Figure 4. Inflation coefficient ϕ_π for Taylor rule when $\beta = 0.99$

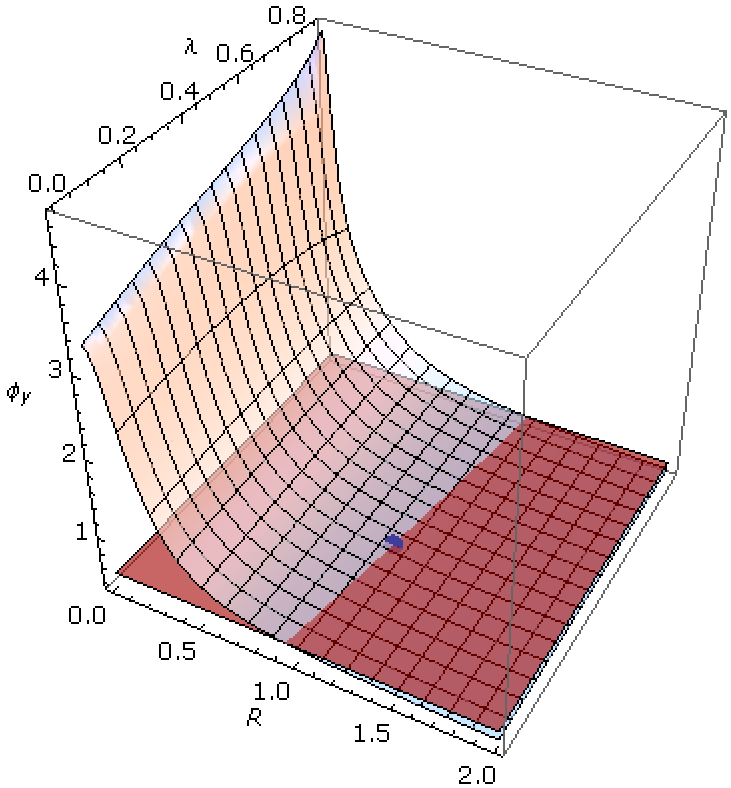

Figure 5. Output gap coefficient ϕ_y for Taylor rule when $\beta = 0.96$. The location of Taylor coefficient $\phi_y = 0.5$ is shown by the circle.

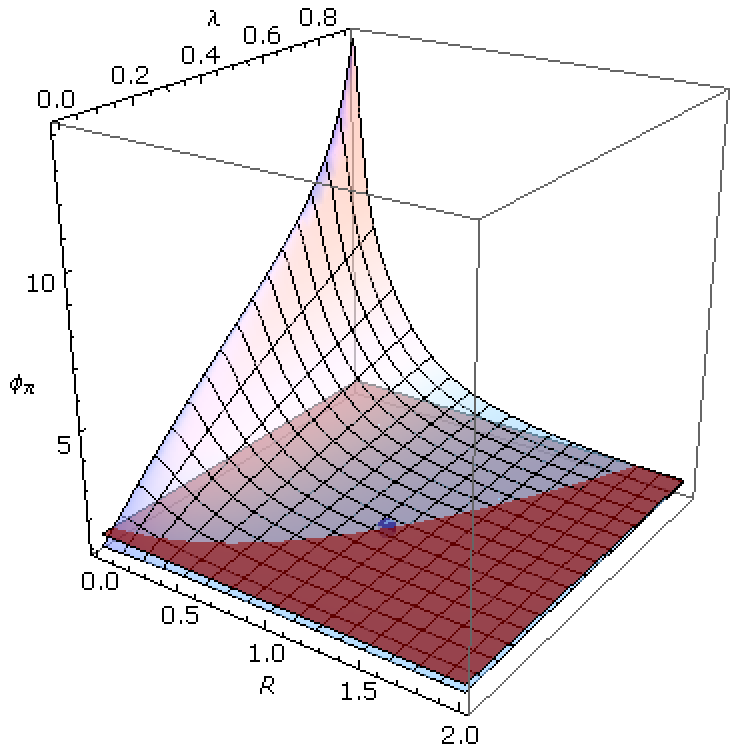

Figure 6. Inflation rate coefficient ϕ_π for Taylor rule when $\beta = 0.96$. The location of Taylor coefficient $\phi_\pi = 1.5$ is shown by the circle.

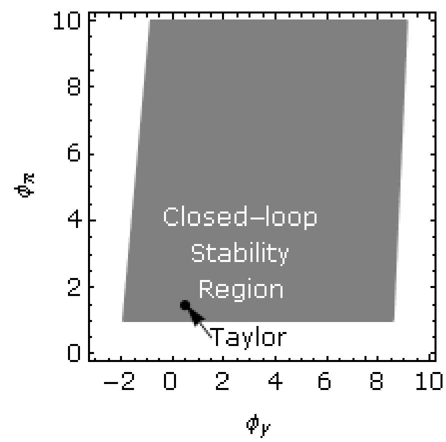

Figure 7. Closed-loop stability region for the US economy model in terms of Taylor rule coefficients ϕ_y and ϕ_π when $\beta = 0.99$.

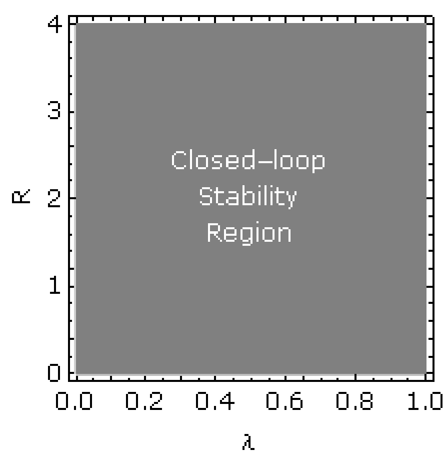

Figure 8. Closed-loop stability region in terms of MPC tuning parameters R and λ for $\beta = 0.99$.

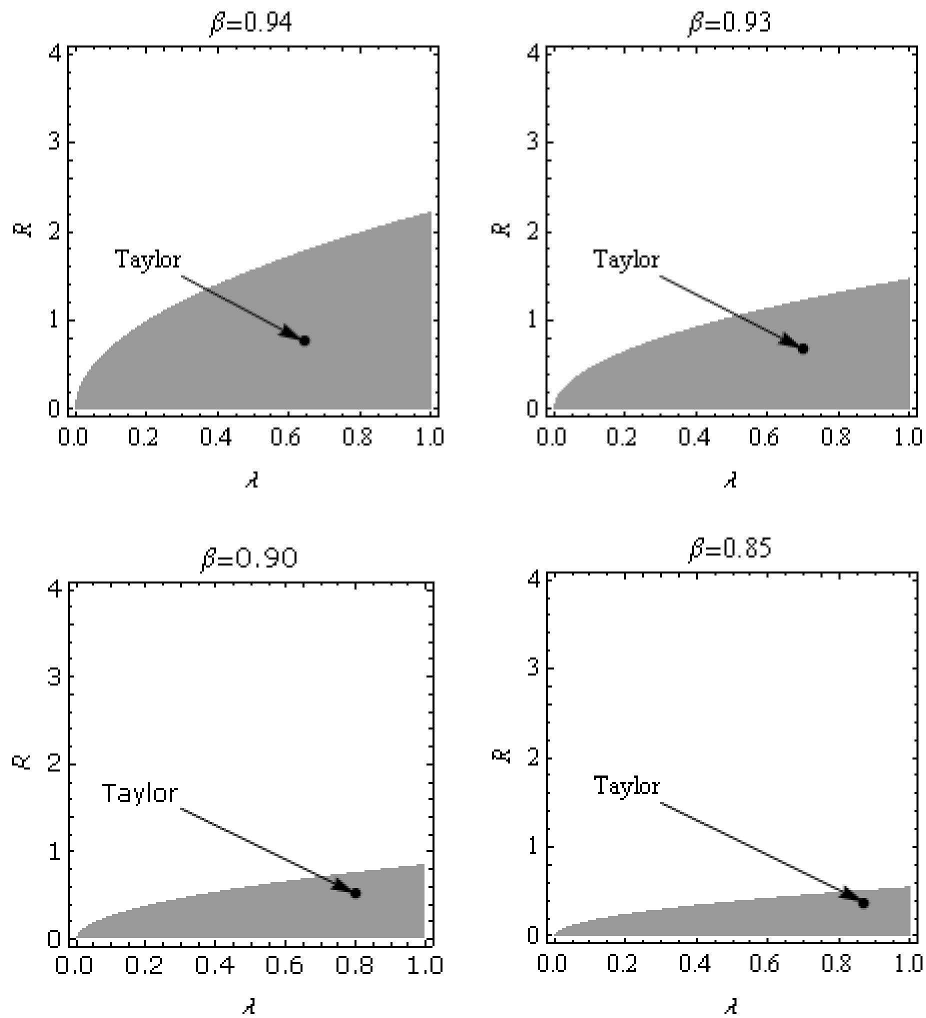

Figure 9. Closed-loop stability region (shaded) in terms of MPC weight parameters R and λ for various values of $\beta < 0.95$. The location of original Taylor rule is shown by circle.

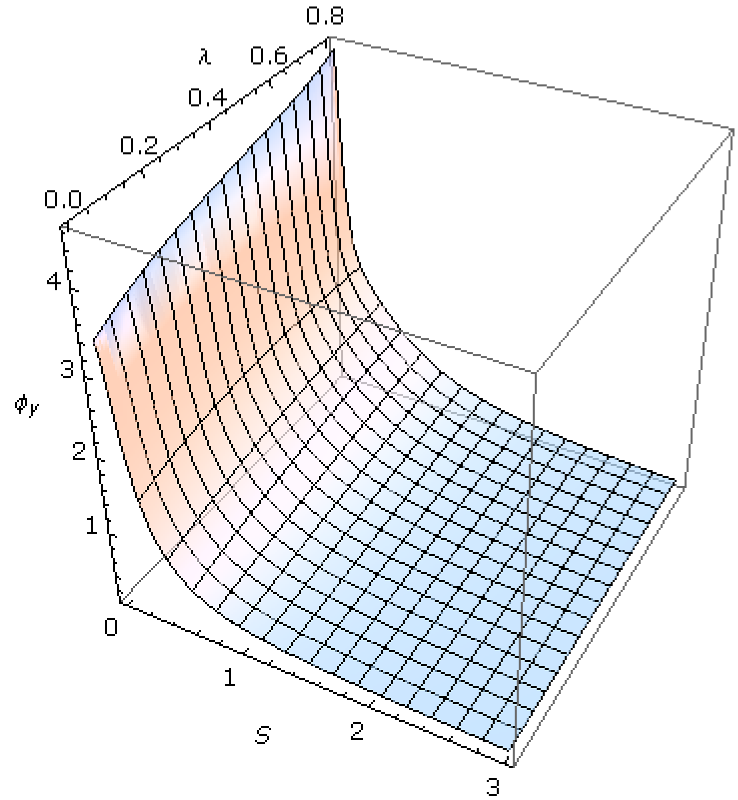

Figure 10. Output gap coefficient ϕ_y for Taylor rules with inertia

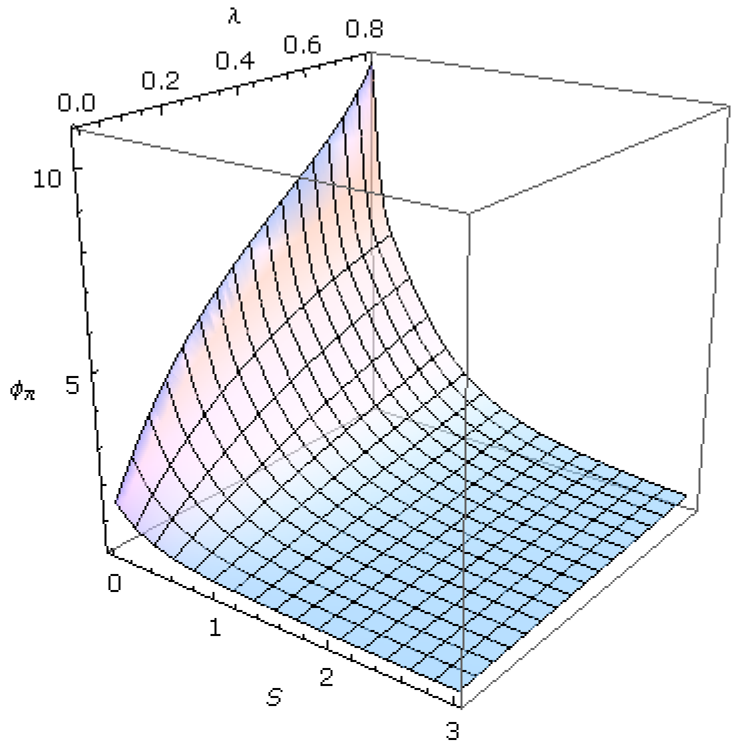

Figure 11. Inflation coefficient ϕ_π for Taylor rules with inertia

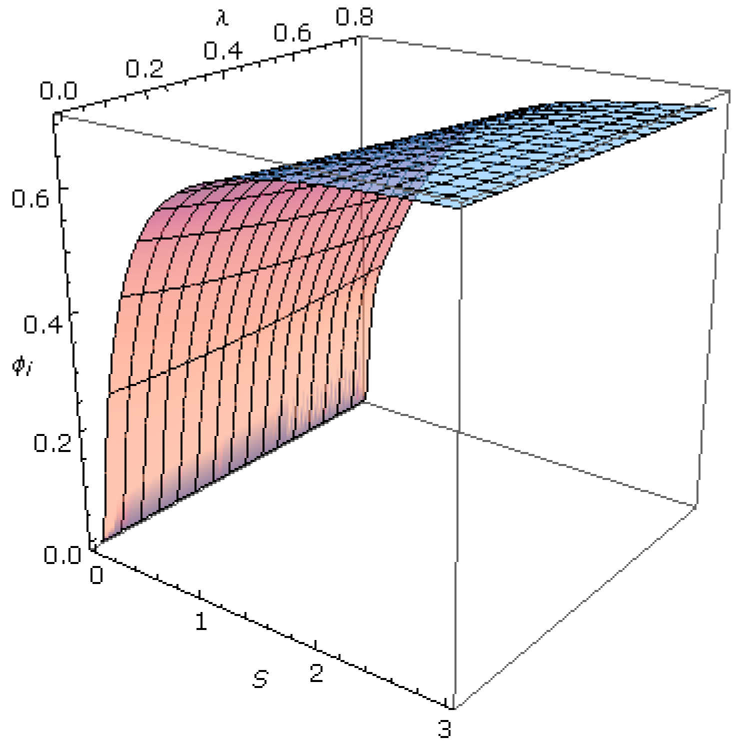

Figure 12. Lagged inertest rate coefficient ϕ_i for Taylor rules with inertia

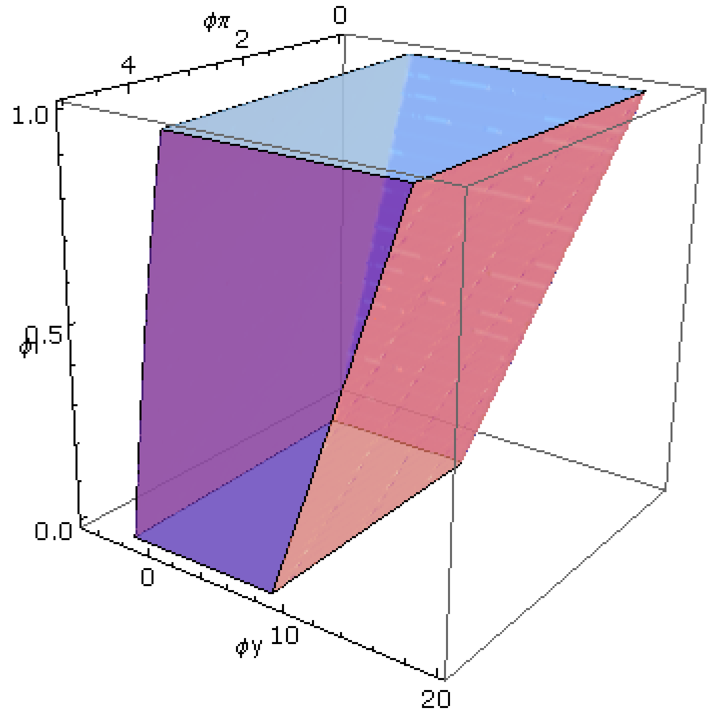

Figure 13. Closed-loop stability region for the US economy model in terms of coefficients ϕ_y, ϕ_π and ϕ_i for Taylor rule with inertia.

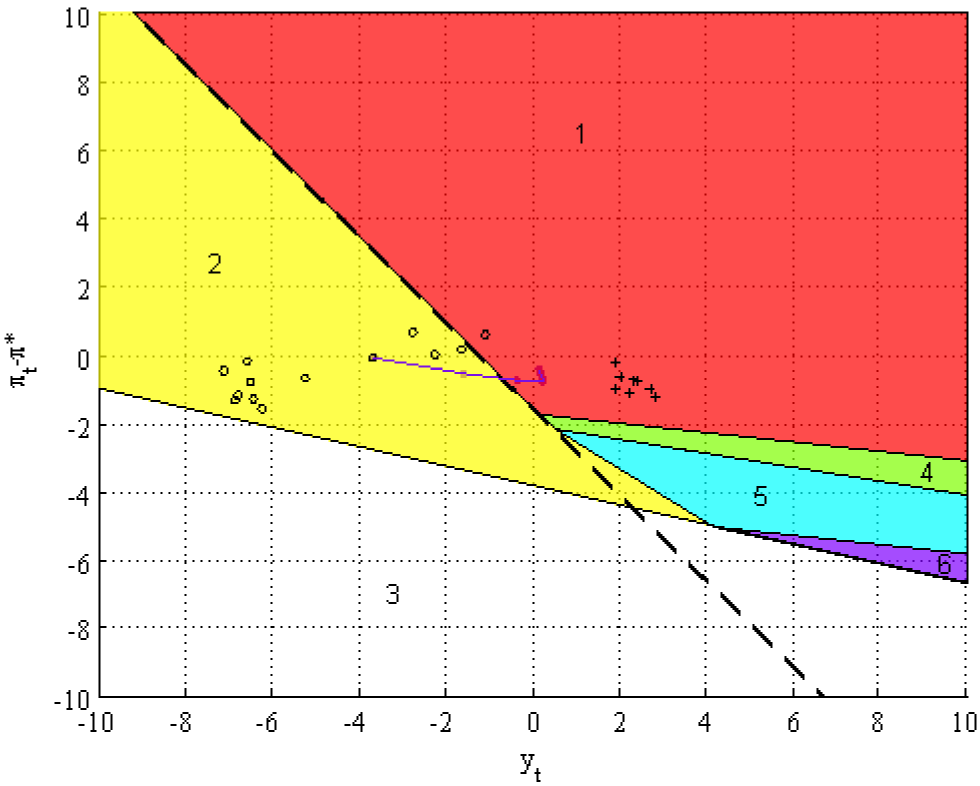

Figure 14. State space partition for $R = 0.07$ and $\lambda = 0.05$, corresponding rules are in Table 6, o represents actual economy data points for period 08Q1-11Q1, + represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (-3.7, 1.9), dashed line represents truncated solution of unconstrained case.

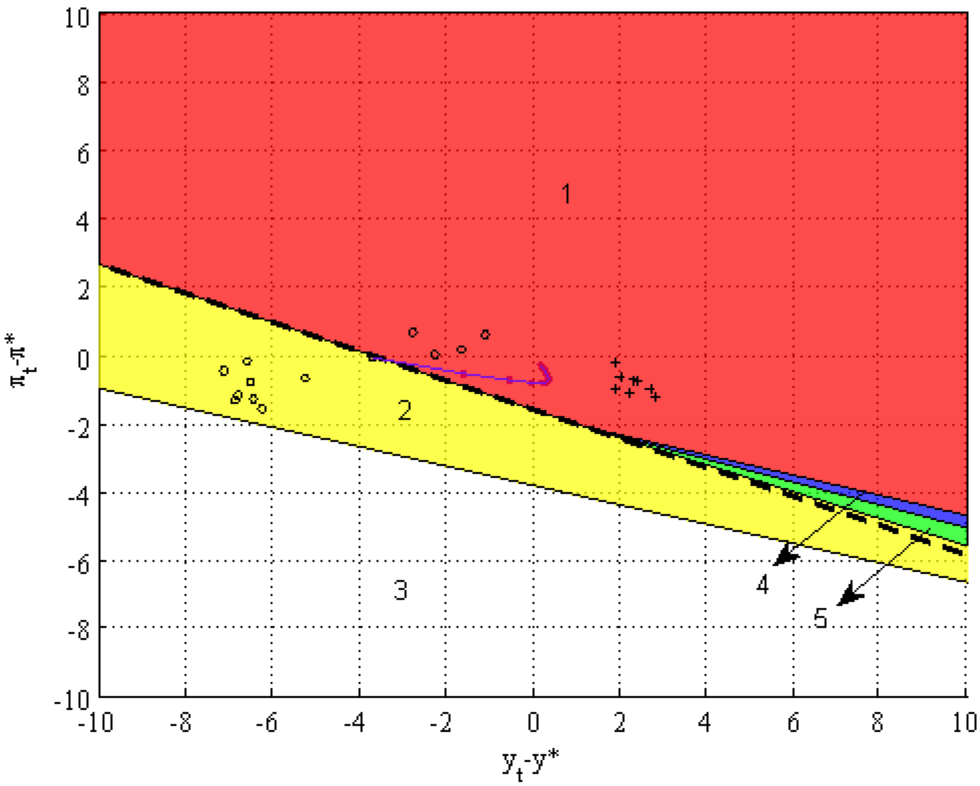

Figure 15. State space partition for $R = 0.55$ $\lambda = 0.05$**, corresponding rules are in Table 7,** o **represents actual economy data points for period 08Q1-11Q1,** + **represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (-3.7, 1.9), dashed line represents truncated solution of unconstrained case.**

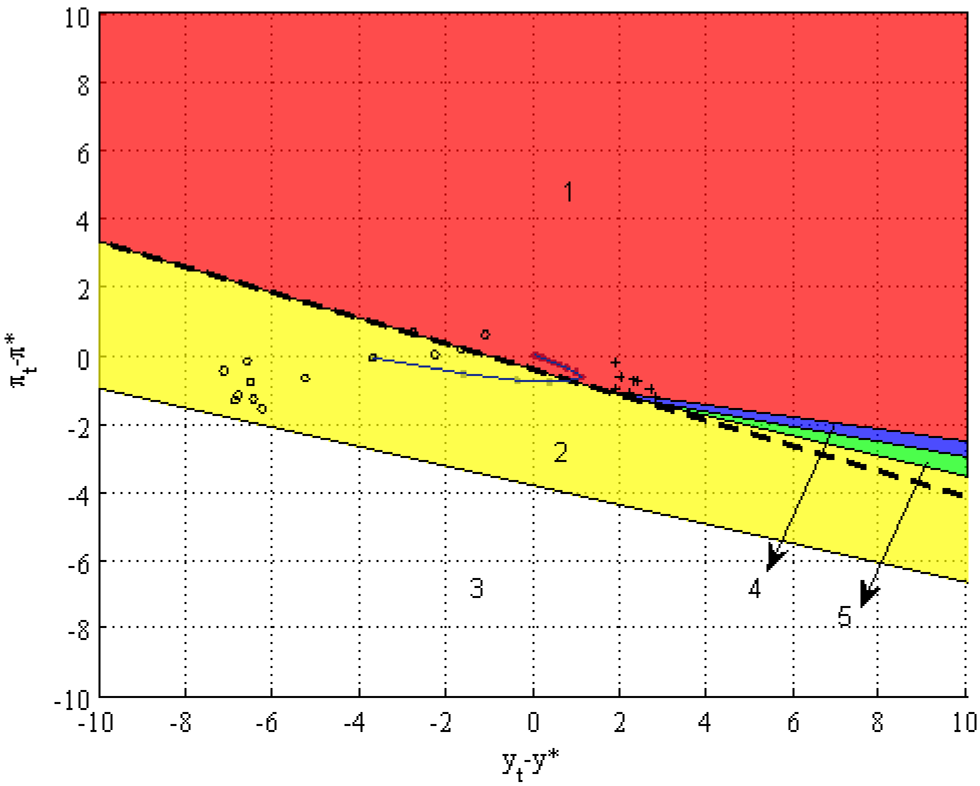

Figure 16. State space partition for $R = 0.07$**,** $\lambda = 0.8$**, corresponding rules are in Table 8,** $\circ$ **represents actual economy data points for period 08Q1-11Q1,** $+$ **represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (-3.7, 1.9), dashed line represents truncated solution of unconstrained case.**

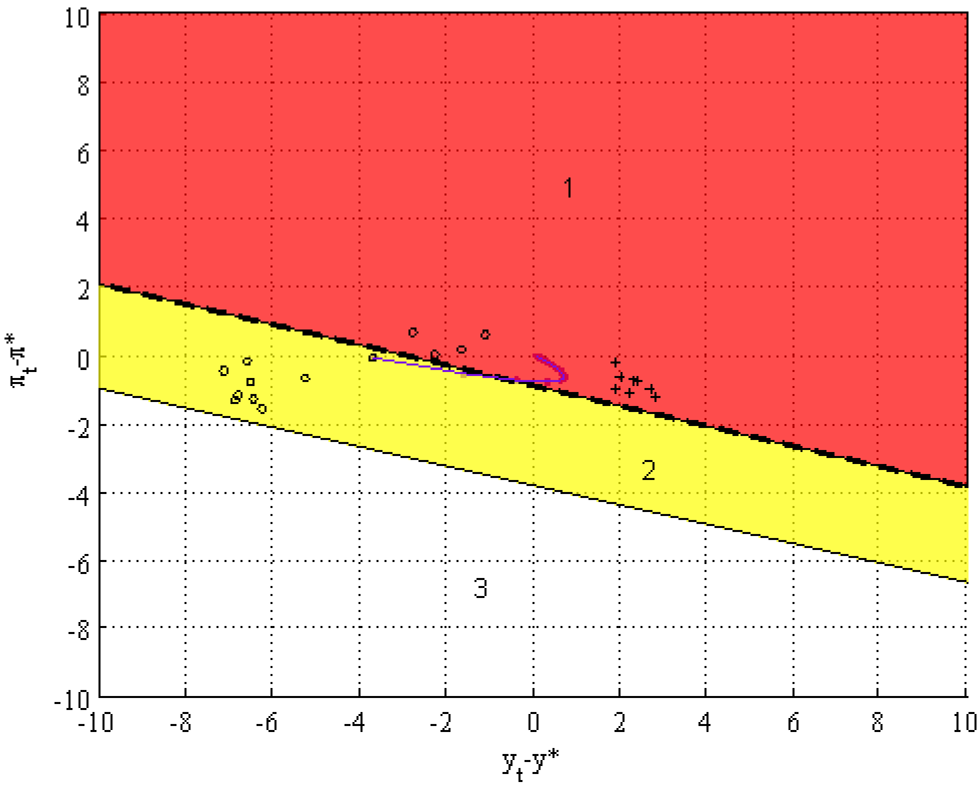

Figure 17. State space partition for $R = 0.55$, $\lambda = 0.8$, corresponding rules are in Table 9, $\circ$ represents actual economy data points for period 08Q1-11Q1, $+$ represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (-3.7, 1.9), dashed line represents truncated solution of unconstrained case.

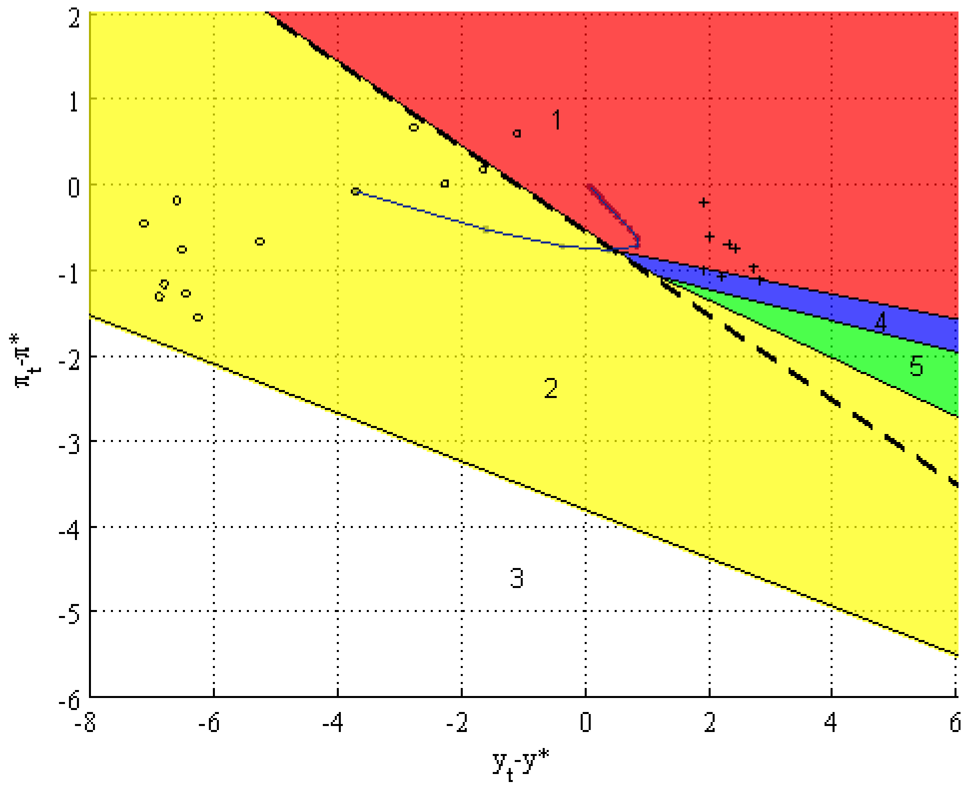

Figure 18. State space partition for $R = 0.07 \;\; \lambda = 0.5$**, corresponding rules are in Table 10,** $\circ$ **represents actual economy data points for period 08Q1-11Q1,** $+$ **represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (-3.7, 1.9), dashed line represents truncated solution of unconstrained case.**

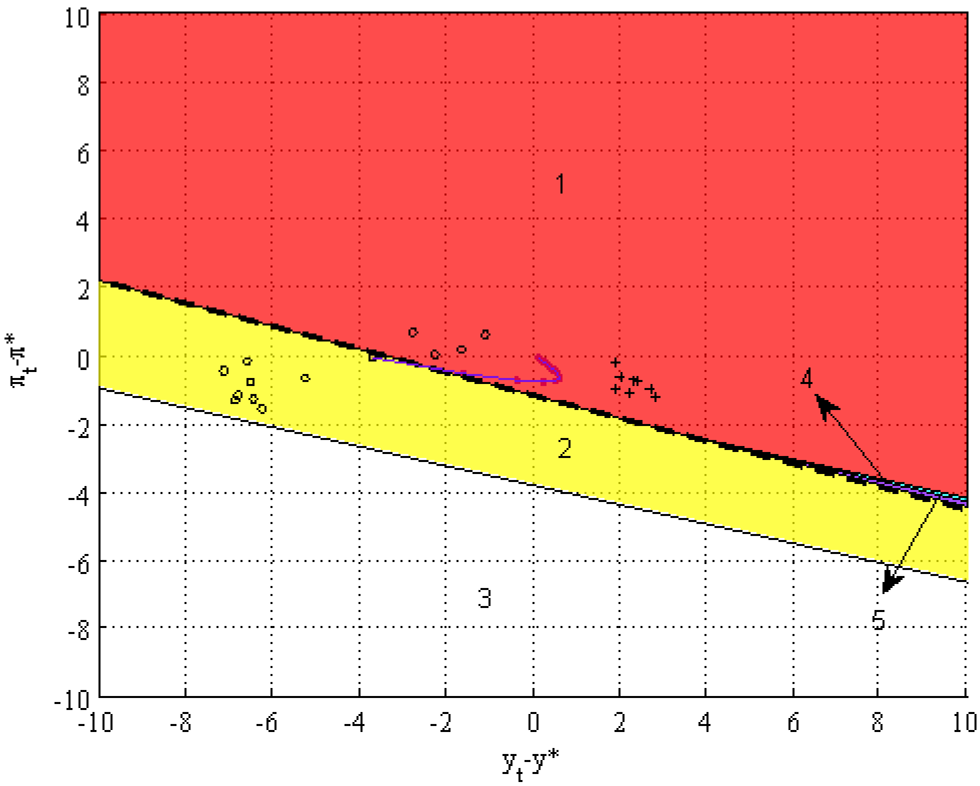

Figure 19. space partition for $R = 0.55$, $\lambda = 0.5$, corresponding rules are in Table 11, o represents actual economy data points for period 08Q1-11Q1, + represents actual economy data points for period 98Q1:99Q4, solid curve represent closed loop response from initial state (-3.7, 1.9), dashed line represents truncated solution of unconstrained case.

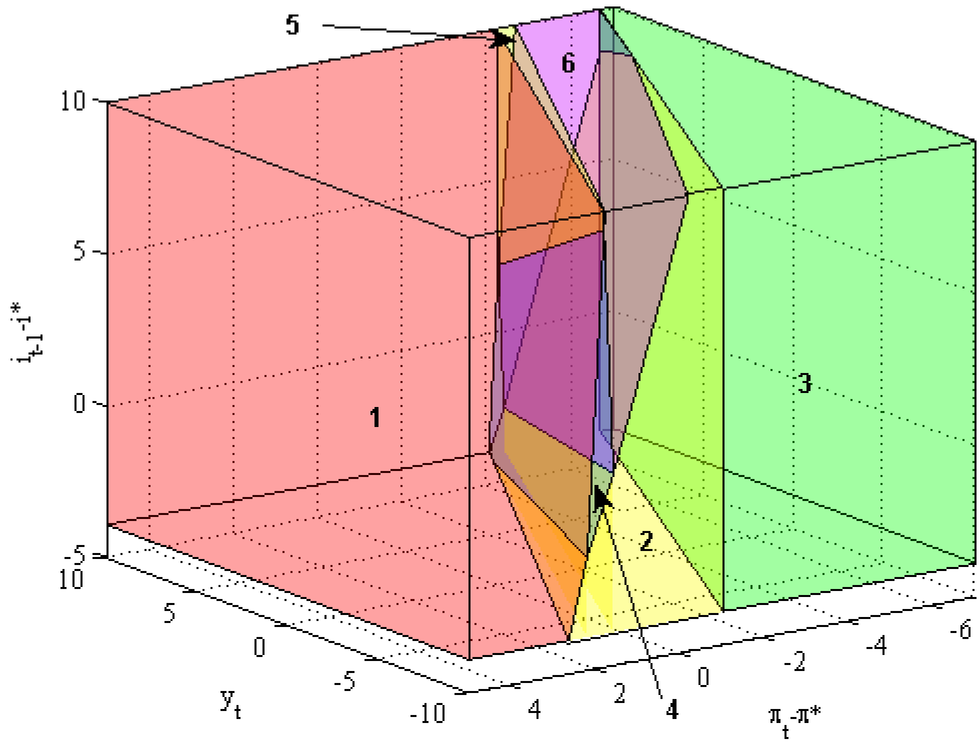

Figure 20. State-space partition for $S = 0.55$ and $\lambda = 0.5$. Corresponding rules are in Table 12.

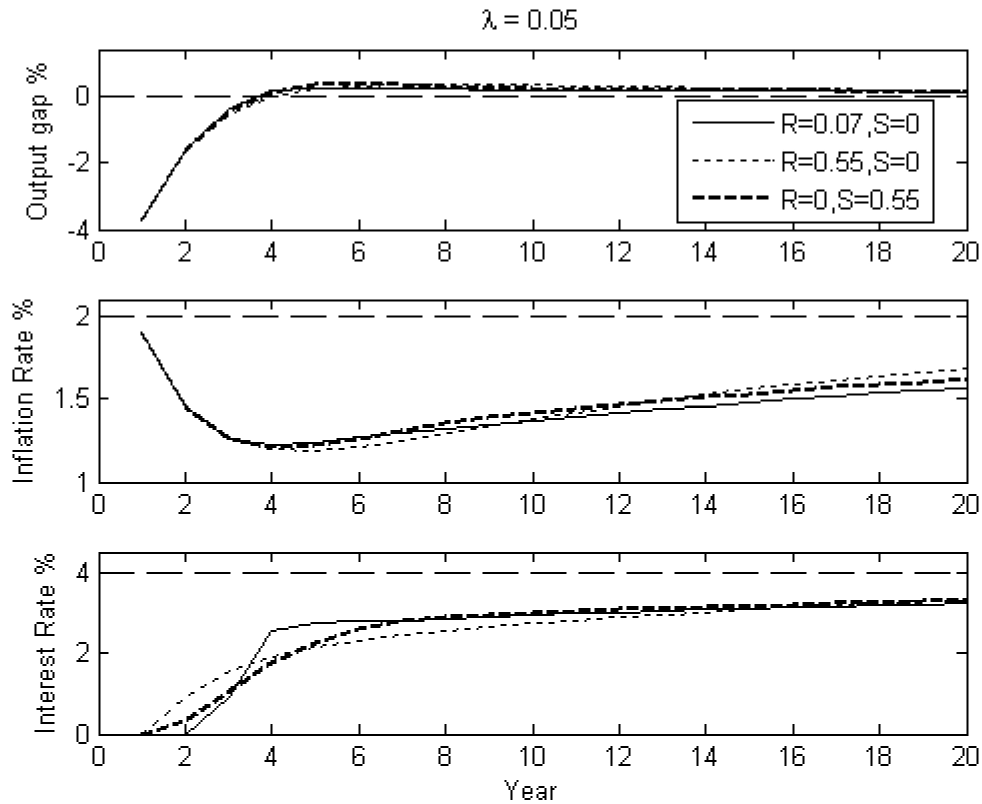

Figure 21. Closed-loop simulation for US economy (start point is 2009Q1) for

$\lambda = 0.05$

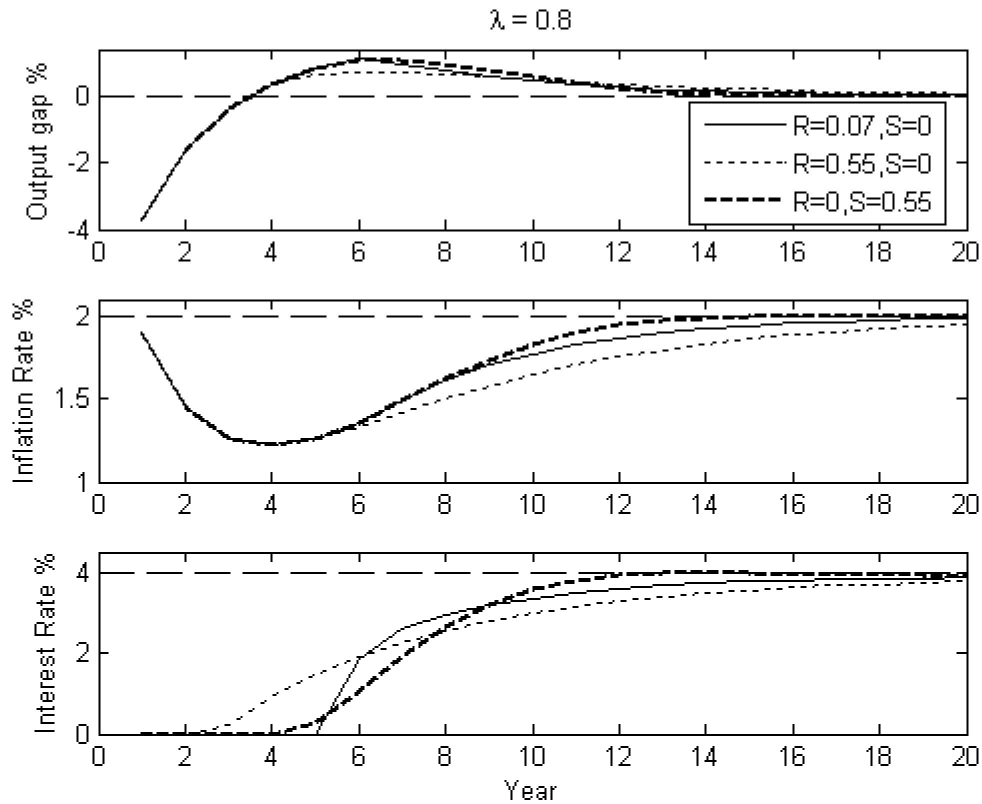

Figure 22. Closed-loop simulation for US economy (start point is 2009Q1) for

$\lambda = 0.8$

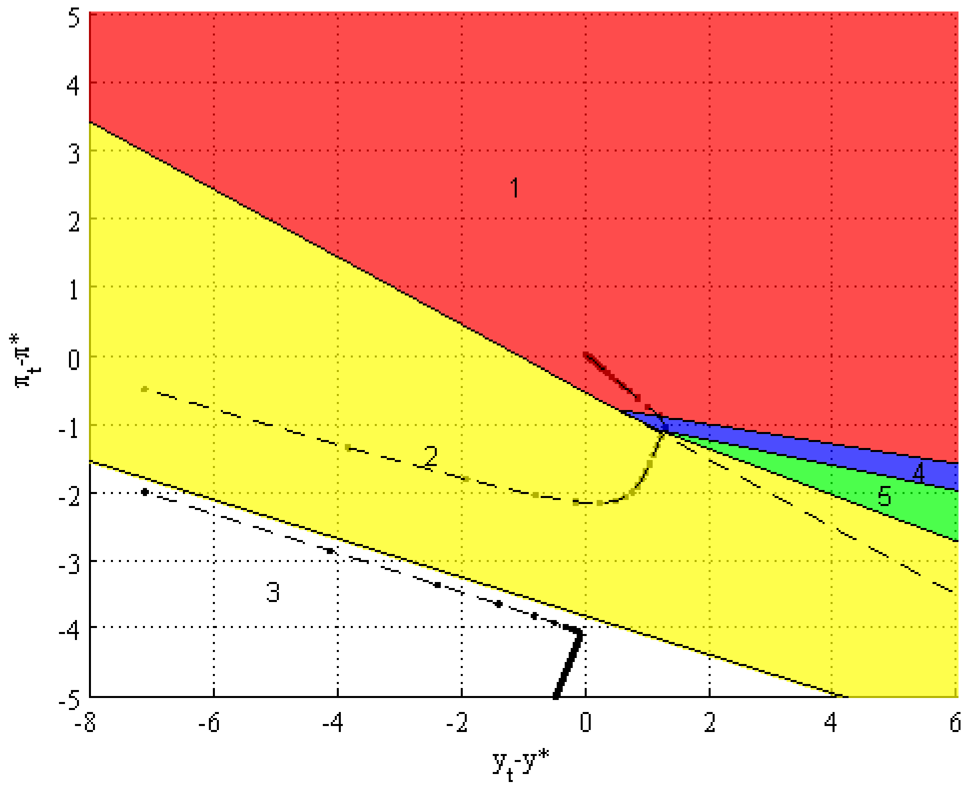

Figure 23. Closed-loop simulation for (y,π)=(-7.1, 1.5) 2009Q3 and (y,π)=(-7.0, 0) virtual point for $R=0.07$, $\lambda=0.5$. The later state lies in infeasibility polytope and no positive interest rate can stabilize the closed loop.

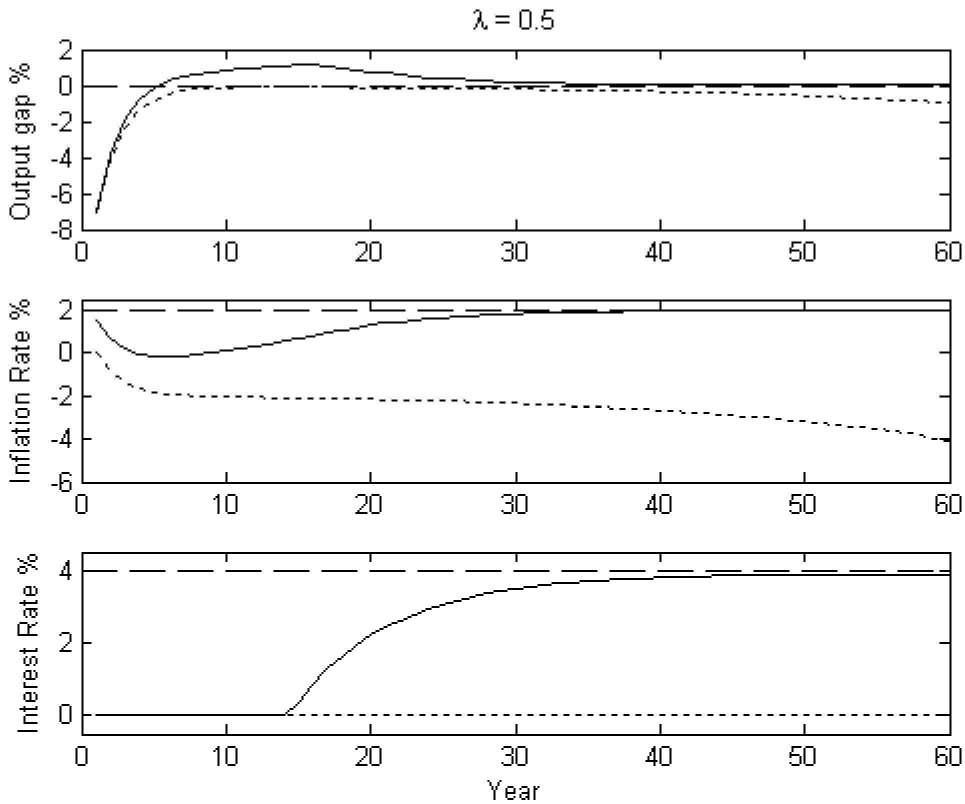

Figure 24. Closed-loop simulation for Figure 23.

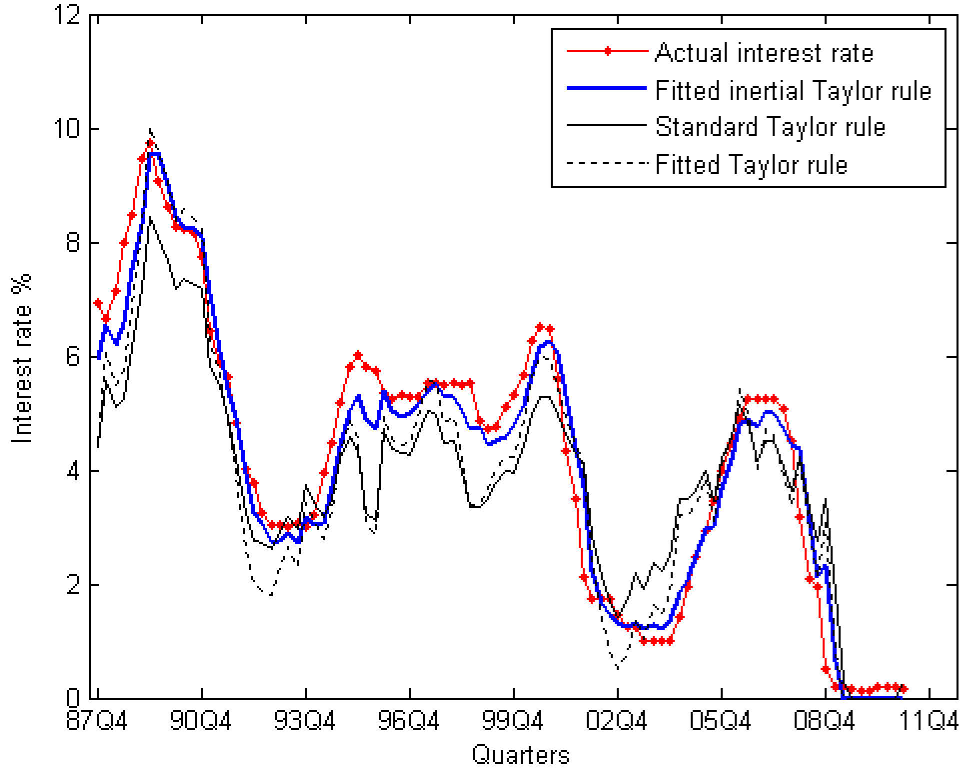

Figure 25 Federal funds rate, standard Taylor rule, fitted inertial and fitted Taylor rules (fitting period 1987Q4: 2008Q4) for period 1987Q4: 2011Q1. Note that the interest rate reduction in 2008 suggested by the inertial Taylor rule is more drastic than that suggested by the standard Taylor rule. Note also that the actual interest rate over the period 2002-2005 is captured fairly well by the inertial Taylor rule, while the standard Taylor rule produces significantly larger values, as has been studied extensively by Taylor [40].

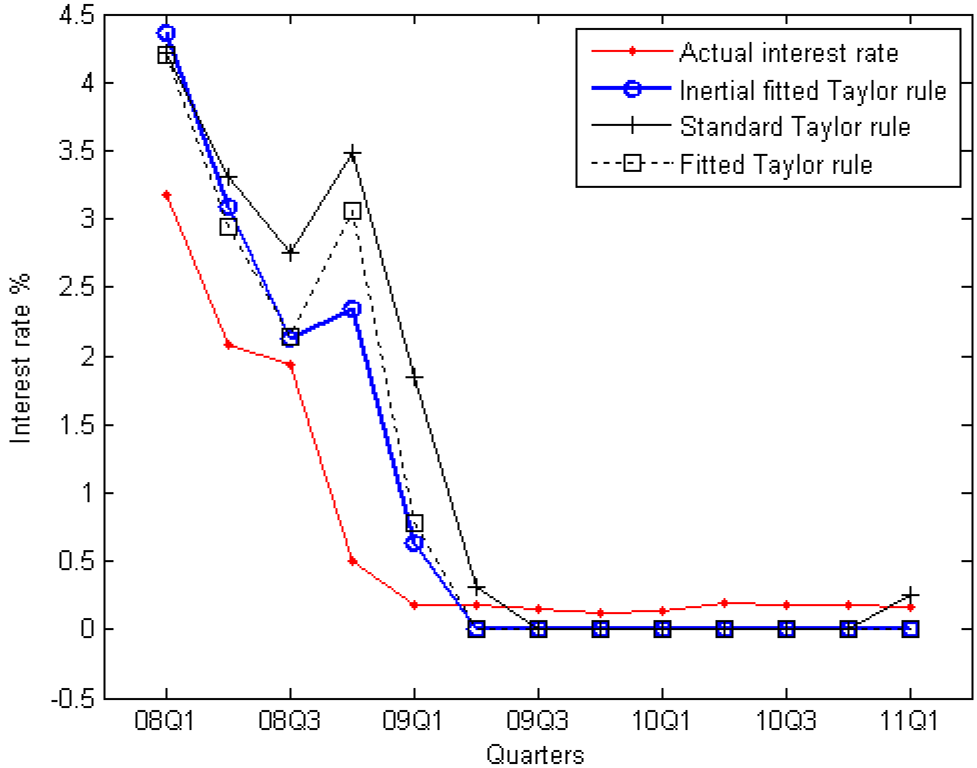

Figure 26. Magnified view of Figure 25 when interest rates are near zero.

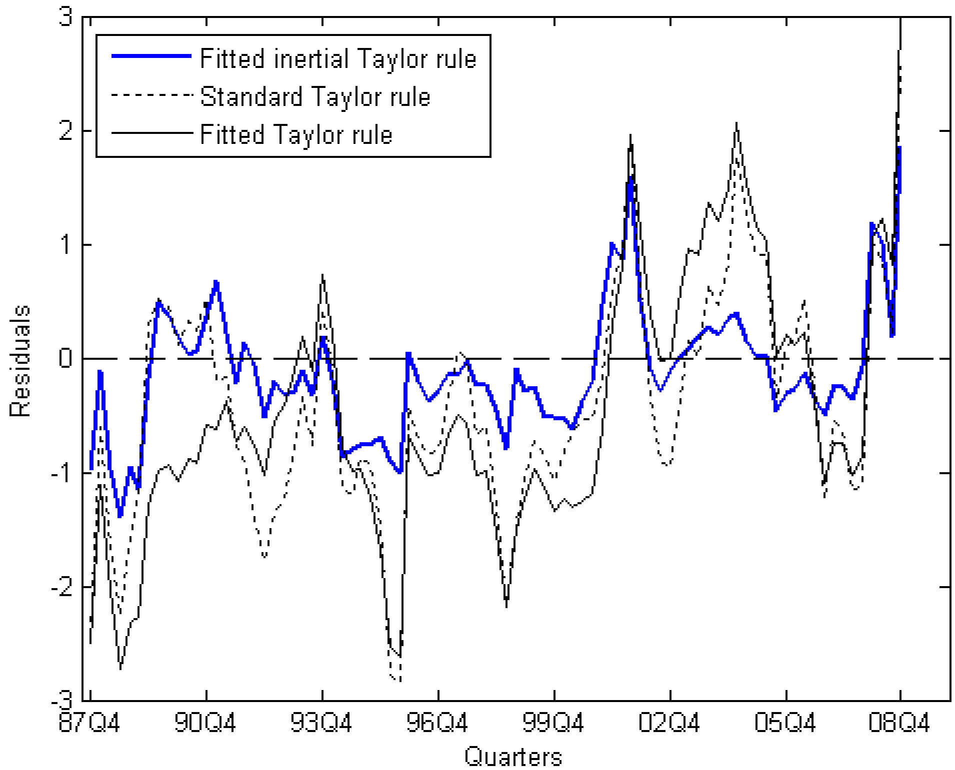

Figure 27. Residuals for policies in Figure 25 for fitting period 1987Q4: 2008Q4

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