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The Rate of Change of the Social Cost of Carbon and the Social Planner's Hotelling Rule

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Abstract This paper derives the social cost of carbon (SCC) and its rate of change. It does so in a deterministic Ramsey model of optimal economic growth with carbon emissions from burning fossil fuels. It is shown that the determinants of the rate of change of the SCC are substantially almost identical to the determinants in the social planner's Hotelling rule if a unit of fossil fuel use leads to exactly one unit of carbon emission, while otherwise these formulas differ substantially. As is also shown in this paper, in the special case in which the two formulas are substantially almost identical, a Pigovian tax on fossil fuel use and a Pigovian tax on carbon emissions are both equal to the SCC, while otherwise only a Pigovian tax on carbon emissions equals the SCC.

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This paper is a revision of Kögel (2012) submitted to this journal´s special issue on The Social Cost of Carbon. The co-editor decided that the author should place his paper better into the context of existing literature and that this would necessarily constitute a new submission to the journal. The author hereby followed this advice by a slight change of his model assumptions, which allowed him to better contrast his paper results to the results in the existing literature.

1 Introduction

The social cost of carbon (SCC) is defined as the present value of the marginal damage from carbon emission, where the damage is caused through climate change. It represents an externality that is not considered by market agents in their decision making process. The externality can however be corrected with a Pigovian carbon tax. Complete internalization of the externality requires the tax rate on carbon emissions to equal the SCC at the optimal carbon emission level.[1] As a consequence, using Pigovian taxation or alternative climate change policies requires understanding of the determinants of the SSC at the optimal carbon emission level. Moreover, Dasgupta and Heal (1995) suggests that only the time path and not the level of a carbon tax affects the resource extraction path. This suggests that knowledge of the rate of change of the SCC is more important than knowledge of the level of the SCC. In turn, the theoretically correct rate of change of the SCC at the optimal carbon emission level can be derived from the optimality conditions of a social planner.

The SCC is usually estimated in applied work in integrated assessment models (IAMs), i.e. in simulation models that integrate economic and scientific models of global warming. The first step in calculating the SCC is to estimate the stream of future relative marginal damages of carbon. The second step in calculating the SCC is then to employ a discount rate (sometimes labeled consumption discount rate) to convert this stream of future relative marginal damages into a present value.[2] To choose the discount rate, IAMs usually employ a Ramsey rule, i.e. an optimality condition that must be fulfilled on the consumption path that maximises life-time utility of a representative household (e.g. in the Ramsey model). The Ramsey rule relates the discount rate to the income growth rate. In turn, since a well-known stylized fact of modern growth of Kaldor is a constant average income growth rate in industrialised countries over periods of at least hundred years (see e.g. Sørensen and Whitta-Jacobsen, 2010), it is standard in IAMs to employ a constant discount rate. It appears that this can also be motivated by the fact that historical data show trendless market rates of return on physical capital (more precisely, returns on risky stocks or risk-free government bonds). As

[1] See e.g. Nordhaus (2011, p. 2).

[2] See, among others, the articles in the special edition of this Economics E-Journal on "The Social Cost of Carbon" at www.economics-ejournal.org/special-areas/special-issues/the-social-cost-of-carbon.

a consequence, IAMs also usually use in their numerical simulations the historical average market rates of return as the value of the discount rate and use parameter values of the utility function that make the historical average market rates of return consistent with the Ramsey rule, given the historical average income growth rate.

Recently, Sinn (2008) shows that environmental policies that aim to decelerate climate change actually do in general just the opposite and accelerate climate change and labels this phenomenon a Green paradox. He acknowledges however that a Pigovian carbon tax would do the job to reduce the market economy's rate of carbon emissions to the social optimal level (cf. page 383). He nevertheless brushes off the usefulness of an optimal carbon tax, by arguing to the effect that a Pigovian carbon tax must be equal to the SCC on the optimal carbon emission path and that in practise it would be difficult to calculate the theoretically correct time path of the SCC. In contrast to this, Tol (2009, page 7) assesses quantitatively the importance of climate policy by assuming the carbon tax rate (and therefore the SCC) to grow with the discount rate. If appropriate, then one could employ the Ramsey rule to also estimate the growth rate of the SCC. For this reason, calculating the theoretically correct time path of the optimal carbon tax would then not be that difficult. In turn, Kuik (2009, page 9) finds it reasonable to assume the growth rate of the SCC to be equal to the discount rate, in light of the Hotelling rule of optimal resource extraction. Newbold et al. (2009) argue however that the growth rate of the SCC should be lower than the discount rate.[3]

The first contribution of this paper is a derivation of the rate of change of the SCC. More precisely, the predecessor version of this paper - Kögel (2012) - is the first paper that derived this rate of change in a Ramsey model with carbon emissions from burning fossil fuels. It has later also been derived in van der Ploeg and Withagen (2012b), which is an updated version of van der Ploeg and Withagen (2011).[4] However, contrary to this paper version, Kögel

[3]The applied climate change literature that employs IAMs uses the wording discount rate and calibrates it with historical average market rates of return. The present paper solves a social planner model. As a consequence, henceforth the paper only uses the wording social discount rate, which is defined to be equal to the social marginal product of capital.

[4]The rate of change of the SCC in van der Ploeg and Withagen (2012b) replaced the rate of change of the Hotelling rent that has been derived in van der Ploeg and Withagen (2011).

and van der Ploeg and Withagen (2012b) derived the rate of change of the SCC for the special case in which a unit of fossil fuel use leads to exactly one unit of carbon emission. This simplification does however matter, as this paper version shows. This paper provides also intuitive reasoning for the possible size and sign of the growth rate of the SCC. As I found out after the writing of this paper's predecessor version, the first contribution of this paper is also very closely related to a result in Grimaud et al. (2011), who derive the growth rate of the optimal carbon tax in an endogenous growth model with carbon emissions from burning fossil fuels. Their formula looks much more complicated than this paper's growth rate of the SCC. However, this paper shows that nevertheless their growth rate of the optimal carbon tax is substantially identical to the present paper's growth rate of the SCC if the social planner's optimality conditions are used to simplify their formula. A somewhat similar growth rate of the optimal carbon tax has also been derived in partial equilibrium models, such as Ulph and Ulph (1994).[5] As this paper shows, the latter literature's growth rate of the optimal carbon tax can be seen as a special case of the present paper's growth rate of the optimal carbon tax if instantaneous utility from consumption is linear and additively separable from instantaneous disutility from carbon emissions. Hence, the paper shows that the optimal carbon tax in partial equilibrium models is not too unrealistic, but is also not substantially identical to the one in general equilibrium models, since whether or not instantaneous utility is linear in consumption makes a difference.

The second contribution of this paper is to show how the rate of change of the SCC is related to the social planner's Hotelling rule (i.e. the formula describing the social planner's optimal resource extraction path). An understanding of this relation is important because the social planner's Hotelling rule has been derived in earlier work (e.g. in Groth and Schou, 2007, and Sinn, 2008) and is therefore well-known within the literature. The present paper shows that the determinants of the rate of change of the SCC are substantially almost identical to the determinants in the social planner's Hotelling rule if a unit of fossil fuel use leads to exactly one unit of carbon emission. Otherwise those formulas differ substantially. As is also shown in this paper, in the special case in which the two formulas are substantially

[5] See also, among others, Goulder and Mathai (2000, Section 3), who extend such a model with R&D based technical progress in carbon abatement and van der Ploeg and Withagen (2012a), who extend such a model with backstops.

almost identical, a Pigovian tax on fossil fuel use and a Pigovian tax on carbon emissions are both equal to the SCC, while otherwise only a Pigovian tax on carbon emissions equals the SCC.

For its derivation, the present paper employs a model that can be viewed as a version that combines elements of, on the one hand, the model of van der Ploeg and Withagen (2012b) with those of, on the other hand, the models of Groth and Schou (2007) and Grimaud et al. (2011).[6] It shares with van der Ploeg and Withhagen the assumption that there is disutility from carbon emissions and shares with Groth and Schou and Grimaud et al. that there is productivity loss from carbon emsissions. Van der Ploeg and Withagen and Groth and Schou make, contrary to the present paper, the simplifying assumption that a unit of fossil fuel use leads to exactly one unit of carbon emission. On the other hand, van der Ploeg and Withagen is richer than the present paper's model because it allows for renewable backstops and resource extraction costs, while Groth and Schou is richer than the present paper's model because it allows for endogenous growth from productive externalities. Grimaud et al. is richer than the present model because it allows for R&D based endogenous growth and for a more realistic climate damage specification. Moreover, it allows for resource extraction costs, dissipation of the carbon stock in the atmosphere, a renewable backstop and so-called carbon capture and storage (CCS) (i.e. the option to pump carbon in the underground to store it away from the atmosphere). The richer model assumptions in these papers are important for the research questions of those papers. Since these assumptions are however not very relevant for the present paper's research question, they are dropped from the present paper's model. In my view, the model of van der Ploeg and Withagen builds on Krautkrämer (1985, second part), in which households enjoy utility from the remaining stock of a non-renewable resources. Krautkrämer does however not addressed the issue of global warming, not to speak about that his model allows for renewable backstops. Groth and Schou and Grimaud et al. in turn build on Stiglitz (1974) and in my view also on Dasgupta and Heal (1974) and Solow (1974, Appendix C), which introduced non-renewable resources into the Ramsey model, but did not allow for a negative external effect from pollution. The present paper's model is also related to Golosov et al. (2011). Golosov et al. make however specific assumptions on the utility function, the production function and on the carbon accumulation process that deliver a constant

[6] See also Sinn (2008) for a model within the second group of models.

consumption–output ratio, consistent with stylized fact of modern growth. Since the latter issue is not addressed in the present paper, a discussion of the assumption of Golosov et al. lies outside of the scope of this paper.

Section 2 of the present paper presents the social planner model structure. Section 3 derives the social planner model results. Section 4 derives Pigovian tax rates in a regulated market economy. Section 5 shows how this paper's growth rate of the optimal carbon tax is related to the one in Grimaud et al. and in partial equilibrium models, such as Ulph and Ulph. Finally, section 6 concludes.

2 The Model

To derive the SSC analytically, the paper assumes a social planner with perfect foresight, who maximizes life-time utility, W(O), of an infinitely lived representative household subject to the economy's resource constraints. The social planner solution can be replicated in a regulated market economy with climate policy such as Pigovian taxation. Following Krautkrämer (1985), but adapted to climate damage, life-time utility is assumed to be:[7]

$$W(0) = \int_0^\infty U(C, P) e^{-\rho t} dt, \tag{1}$$

where $U(C, P)$ represents instantaneous utility, C denotes consumption and P denotes the stock of carbon in the atmosphere.[8] It is assumed that $U_C > 0$, $U_P < 0$ and that $U_{CC} < 0$. No assumptions are made on the signs of U_{CP} and U_{PP}.[9] Literature often assumes $U_{CP} = 0$ and $U_{PP} \leq 0$, i.e. marginal damages from pollution to be non-decreasing in the pollution stock (e.g. Stokey, 1998). However, as emphasised in Hoel and Kverndokk (1996), the carbon stock in the atmosphere affects utility only indirectly by increasing the world temperature. As literature assumes the world temperature to be logarithmic in the carbon stock in the atmosphere, while marginal damages from rising world temperature might be non-decreasing, climate damage might be decreasing in the stock of carbon in the atmosphere (see also van der Ploeg

[7] The time index t is for most part of this paper omitted.

[8] Van der Ploeg and Withagen (2012b) assume a similar life-time utility function with $U(C, P) = V(C) - D(P)$, i.e. instantaneous utility from consumption, $V(C)$, to be additively separable from instantaneous disutility from the stock of carbon, $D(P)$.

[9] Note that symmetry requires that $U_{PC} = U_{CP}$.

and Withagen, 2012b). Moreover, ρ denotes the pure rate of time preference. For simplicity, the number of household in the economy is normalised to one. Note that it is therefore abstracted from population growth, which seems not to be too unrealistic for the very long-run, as population growth in industrialised countries is low and world population growth is predicted to slow down in the distant future. For simplicity, it is also abstracted from uncertainty, leaving its consideration to future research. As is standard in growth models with a non-renewable resource, the stock of fossil fuel in the ground evolves according to:[10]

$$S(t) = S(0) - \int_0^\infty R(t)dt \quad \Rightarrow \quad \dot{S} = -R, \tag{2}$$

where S denotes the stock of fossil fuel left in the ground and R denotes the use of fossil fuel in output production. Moreover, the stock of carbon in the atmosphere is assumed to evolve according to:[11]

$$\dot{P} = M(R), \quad \text{with} \quad M_R > 0, \tag{3}$$

where M denotes the flow of carbon emissions. As mentioned in the introduction, the specification in (3) differs from van der Ploeg and Withagen (2012b) and Groth and Schou (2007), who assume $P = R$, i.e. who assume that a unit of fossil fuel use leads to exactly one unit of carbon emission. The exact functional form of $M(R)$ is left open. We could follow Ulph and Ulph (1994) and assume $M(R) = \xi R$, where ξ is a constant. Alternatively, we could for example assume that $M(R) = R^\xi$, where $M_R = \xi R^{\xi-1}$, which is rising in R if $\xi > 1$ and is declining in R if $\xi < 1$. Finally, we could follow Grimaud et al. (2009) and assume that $M(R) = R - Q$, with $Q = R^\xi$, where Q represents abatement and $0 < \xi < 1$.[12] Following van der Ploeg and Withagen (2012b) and Groth and Schou (2007), it is in (3) for simplicity abstracted from dissipation of the carbon stock in the atmosphere.[13]

[10] Cf. Perman et al. (2003, page 489). A hat on a variable represents the rate of change (or in other words the time derivative) of that variable.

[11] See also Perman et al. (2003, Chapter 16) for such a possibly non-linear effect from fossil fuel use on accumulation of the carbon stock.

[12] More precisely, Grimaud et al. actually assume that $M(R) = hR - Q$ and $Q = (hR)^\xi (L_Q)^{1-\xi}$, if $L_Q < hR$, and $Q = hR$, if $L_Q \geq hR$, where h is constant and L_Q denotes labour used for abatement.

[13] Sinclair (1994) argues that the speed by which the stock of carbon is dissipated is slow enough to be ignored as a first-order approximation.

Similar to Groth and Schou (2007) and Grimaud et al. (2011) production of output, Y, takes place according to the following aggregate production function:[14]

$$Y = F(K, R, P, t), \tag{4}$$

where K represents the stock of physical capital and F depends on t to allow for exogenous technical progress.[15] For simplicity it is abstracted from use of labor in output production. It is assumed that $F_K > 0$, $F_R > 0$ and $F_P < 0$. Moreover, $F_{KK} < 0$ and $F_{RR} < 0$. For the same reasons as in case of the utility function, no assumptions are made on the signs of F_{KR}, F_{KP} and F_{RP}, as well as on the sign of F_{PP}.[16] Finally, capital accumulation is assumed to evolve according to the following differential equation:

$$\dot{K} = Y - C, \tag{5}$$

where it is for simplicity abstracted from capital depreciation. More importantly, in (5) it is also abstracted from costs for extraction of fossil fuel (see similarly, Groth and Schou, 2007).[17]

3 Results

Combining (1)-(5) the present value Hamiltonian that the social planner maximises is:[18]

[14]Contrary to this production function, Groth and Schou assume emissions to reduce environmental quality, where environmental quality is an input into output production. The authors also assunme a Cobb-Douglas production function. As mentioned in the introduction, this paper's production function abstracts, for simplicity but without loss of generality, from various rich elements in the production functions of Groth and Schou and Grimaud et al.

[15]Sinn (2008) suggests interpreting productivity loss from the carbon stock to include output loss from devoting output to mitigate climate change that is therefore not available for consumption or capital accumulation.

[16]The latter is in contrast to Sinn (2008), who assumes to the effect that output loss from the carbon stock is rising in the stock of carbon.

[17]Sinclair (1994), who apparently identifies fossil fuels mainly with oil, justifies abstracting from extraction costs with the argument that marginal extraction costs for oil represent a very modest fraction of the price of oil.

[18]In the Hamiltonian, the minus in front of μ_P ensures μ_P to be positive, which allows it to be interpreted as a (positive) shadow price.

$$H = U(c,P)e^{-\rho t} + \lambda[F(K,R,P,t) - C] + \mu_S[-R] + (-\mu_P)[M(R)].$$

The first order conditions from maximisation of H are:

$$\frac{\partial H}{\partial C} = 0 \quad \Rightarrow \quad U_C e^{-\rho t} = \lambda, \tag{6}$$
$$\frac{\partial H}{\partial R} = 0 \quad \Rightarrow \quad \lambda F_R = \mu_S + \mu_P M_R, \tag{7}$$
$$\frac{\partial H}{\partial K} = -\dot{\lambda} \quad \Rightarrow \quad \dot{\lambda} = -\lambda F_K, \tag{8}$$
$$\frac{\partial H}{\partial S} = -\dot{\mu}_S \quad \Rightarrow \quad \dot{\mu}_S = 0, \tag{9}$$
$$\frac{\partial H}{\partial P} = -(-\dot{\mu}_P) \quad \Rightarrow \quad \dot{\mu}_P = U_P e^{-\rho t} + \lambda F_P. \tag{10}$$

As is shown in Appendix A, if we define the social discount rate, r, to be equal to the social marginal product of capital, then from (6) and (8) we find, similarly to e.g. Weikard and Zhu (2005), the following modified Ramsey rule:[19]

$$r = \rho + \eta_{CC}\hat{C} + \eta_{CP}\hat{P}, \tag{11}$$
$$\text{where} \quad \eta_{CC} \equiv -\frac{U_{CC}C}{U_C} \quad \text{and} \quad \eta_{CP} \equiv -\frac{U_{CP}P}{U_C}$$

Moreover, as is shown in Appendix B, using (6)-(10), we can derive the social planner's Hotelling rule as:[20]

$$\dot{F_R} = rF_R + \left(\frac{U_P}{U_C} + F_P\right) M_R + \omega_P \dot{M}_R. \tag{12}$$

[19] A hat on a variable represents the growth rate of that variable.

[20] See similar formulas in van der Ploeg and Witthagen (2012b), Groth and Schou (2007) and Sinn (2007, 2008). Contrary to the social planner's Hotelling rule in van der Ploeg and Witthagen (2012b), (12) contains also the marginal output loss from emission, F_P. Contrary to the social planner's Hotelling rule in Groth and Schou (2007) and Sinn (2007, 2008), (12) contains also.the marginal rate of substitution between consumption and the carbon stock, U_P/U_C. None of the just mentioned papers includes M_R and the third term on the right hand side of (12), $\omega_P \dot{M}_R$.

where $\omega_P \equiv \mu_p/\lambda$. Equation (12) allows for an interpretation along the line of reasoning in van der Ploeg and Withagen (2012b). The left hand side of (12) represents the social return from leaving a marginal unit of fossil fuel in the ground. In the social optimum this social return must be equal to the right hand side of (12). The first term on the right hand side of (12) is the social return from extracting a marginal unit of fossil fuel, allocating the marginal unit of fossil fuel to output production for the return r and investing the proceeds, F_R, in capital to be used in output production for the return F_K. Due to climate change, we have to add to this return the second term on the right hand side of (12). The second term is the negative marginal climate damage from a unit of fossil fuel use. This latter negative term represents an instantaneous climate externality, which a social planner considers in his allocation decision, while it is not considered by market agents in an unregulated market economy. Therefore, in an unregulated market economy, the right hand side of (12) is higher than would be socially optimal and therefore in such an unregulated market economy the extraction rate of fossil fuel is higher than the socially optimal extraction rate.[21] Contrary to most of the existing literature, with the exception of Perman et al. (2003, Chapter 16), (12) contains on the right hand side also the third term $\omega_P \dot{M}_R$. This third term represents the marginal change of the carbon stock damage from a unit of fossil fuel use (see a related interpretation in Perman et al., 2003, page 543). This is also an externality that is not considered by market agents in an unregulated market economy. If burning of fossil fuels would not cause pollution, then (12) would reduce to $\dot{F_R} = rF_R$.

As is shown in Appendix C, the SCC, ω_P, i.e. the present value of the future relative marginal damage from a unit of carbon emission, can be derived to be:[22]

$$\omega_P = -\int_t^\infty \left[\frac{U_P(z)}{U_C(z)} + F_p(z)\right] e^{-\int_t^z r(u)du} dz. \tag{13}$$

In turn, (13) can be used to derive the following proposition:

PROPOSITION 1: *The rate of change of the SCC equals:*

$$\dot{\omega_P} = r\omega_P + \left(\frac{U_P}{U_C} + F_P\right). \tag{14}$$

[21] See also Sinn (2007).

[22] See van der Ploeg and Withagen (2012b) for the case $F_P = 0$.

PROOF: We can take time derivatives of (13) upon application of the Leibniz rule (see Sydsaetter and Hammond, 1985, page 548-549, example 16.8), which gives:

$$\dot{\omega_P} = \left(\frac{U_P}{U_C} + F_P\right) - \int_t^\infty \left[\frac{U_P(z)}{U_C(z)} + F_p(z)\right] r(z) e^{-\int_t^z r(u)du} dz \qquad (15)$$

Using (13) in (15) and rearranging gives (14).

□

Since U_P and F_P are both negative, it follows that the growth rate of the SCC, $\dot{\omega_P}/\omega_P$, falls short of the social discount rate. As (14) reveals, for the growth rate of the SCC to be insignificantly different from the social discount rate (and to remain so over time), we need fulfillment of all or many of the following conditions:

(i) Climate damages must be moderate (i.e. the absolute values of U_P and F_P must be small and we need marginal damages from emissions to be non-increasing in the carbon stock, i.e. $U_{PP} \geq 0$ and $F_{PP} \geq 0$, to ensure that the absolute values of U_P and F_P remain small, as world temperature rises).

(ii) If we examine a country or a group of countries rather than the world, then the country or the group of countries must be at a low development stage (i.e. U_C must be large due to a low value of C, since $U_{CC} < 0$, see van der Ploeg and Withagen, 2012b).

(iii) Consumption, physical capital and technnology must be good substitutes for not temperate climate (i.e. along the intuition of Sterner and Persson, 2008, and Neumayer, 1999, growth of consumption and capital, and technical progress should buffer the utility and productivity loss from rising world temperature).[23]

[23] Suppose for example that $U(C,P) = [1/(1-\frac{1}{\varepsilon})]\tilde{C}(C,P)^{1-\frac{1}{\varepsilon}}$, with $\tilde{C} = [(1-\gamma)C^{\frac{\sigma-1}{\sigma}} + \gamma(P^{-1})^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$, where $\tilde{C}(C,P)$ represents a composite consumption index of C and P, ε denotes the intertemporal elasticity of substitution and σ denotes the intratemporal elasticity of substitution between C and P^{-1} (see such specifications in Kögel, 2009, and Hoel and Sterner, 2007, for the case of environmental quality rather than pollution). In this case we can derive that $-(U_p/U_C) = (CP^{1-2\sigma})^{\frac{1}{\sigma}}$. This implies that if σ is very large

(iv) Preference for temperate climate must not be akin to a "luxury good" (i.e. U_P must not be rising as development leads to rising per-capita income, see e.g. Anthoff and Tol, 2012).

In contrast, drastic violation of all or most of these conditions is required for the growth rate of the SCC to even be (or to even become) negative. Following the intuition of Withagen (1994) in a simpler model, compared with a model without climate damage, it is optimal to initially extract less fossil fuel and to extract later more fossil fuel. The optimality of little initial fossil fuel extraction is reflected in high initial SCC that might decline over time to make more fossil fuel extraction later optimal.

Provided $\mu_P \dot{M}_R \leq -\left(\frac{U_P}{U_C} + F_P\right) M_R$, then also the growth rate of F_R falls short of the social discount rate. Hence, the intuition of Kuik (2009, page 9) that the time path of the SCC should be related to the Hotelling rule of optimal resource extraction was correct, but the intuition of Newbold et al. (2009) that the growth rate of the SCC should be lower than the (social) discount rate was even more correct. More precisely, comparing (14) with (12) gives rise to the following corallary:

CORALLARY 1: *The determinants of the rate of change of the SCC are substantially almost identical to the determinants in the social planner's Hotelling rule if* $M_R = 1$ (and therefore also $\dot{M}_R = 0$), *i.e. if a unit of fossil fuel use leads to exactly one unit of carbon emission.*

A glance at (7), reveals that $\omega_P \neq F_R$ and therefore the right hand sides of (14) and (12) are not exactly identical, even if $M_R = 1$ (and therefore also $\dot{M}_R = 0$). However, then they are substantially almost identical.[24] In contrast, arguably, if $M_R \neq 1$, then (14) and (12) do differ substantially. That this is so is shown in the next section.

(i.e. if consumption growth is a good substitute for not temperate climate), then growth of C and P does not increase $-(U_p/U_C)$ by very much because then $1 - 2\sigma < 0$ and $(1/\sigma)$ is very small. Similar arguments can be made for the substitutability of physical capital and technology for not temperate climate in the ouput production function.

[24] Had we allowed for extraction costs, then the extraction costs would only appear in the social planner's Hotelling rule and not in the formula for the rate of change of the SCC. This would constitute another difference between those two formulas. However, still those formulas were substantially almost identical (at least, this appears to be the opinion of a referee of the predecessor version of this paper, which I accept).

4 The Regulated Market Economy

This section derives the optimal carbon tax rate that allows replication of the social planner's optimality conditions in a regulated market economy. Two different forms of Pigovian taxation can do this job. One of these taxes is a (per volume) Pigovian tax on fossil fuel use. Another of these taxes is a (per volume) Pigovian tax on carbon emissions.[25] In an unregulated market economy, firms that burn fossil fuels do not consider the carbon emissions that arise from this, which lead to loss of utility and productivity. Therefore, carbon emission is a negative externality. A Pigovian tax rate equals the external effect at the socially optimal emission level and can therefore internalise the externality, i.e. can give output producers the incentive to use only the optimal amount of fossil fuels and therefore produce only the optimal amount of emissions that follows from the social planner's optimality conditions. Assuming a continuum of identical output producers with total mass one that rent physical capital from the households, the period profit of each output producer, Π, is:[26]

$$\Pi = Y - iK - (p_R + t_R)R - t_M M(R), \tag{16}$$

where the output price is normalised to one, i is the market rate of return to rent physical capital and p_R denotes the price of a unit of fossil fuel. Moreover, t_R denotes the tax on fossil fuel use and t_M denotes the tax on the flow of carbon emissions. Substituting (4) in (16) for Y implies the optimisation problem to be:

$$\underset{K,R}{max}\Pi = F(K, R, P, t) - iK - (p_R + t_R)R - t_M M(R), \tag{17}$$

Each output producer's first order conditions are:

$$\frac{\partial \Pi}{\partial K} = 0 \quad \Rightarrow \quad F_K = i. \tag{18}$$

$$\frac{\partial \Pi}{\partial R} = 0 \quad \Rightarrow \quad F_R = p_R + t_R + t_M M_R \tag{19}$$

[25] Literature often examines the optimal ad valorem tax (see e.g. Sinclair, 1994 and Grimaud et al, 2009). This paper examines instead the per volume tax because it aims to derive conditions for this tax to equal the SCC. Nevertheless, an optimal ad valorem tax and an optimal per volume tax both internalise the externality.

[26] If it is assumed that $M(R) = R - Q$, with $Q = R^{\xi}$, then it is assumed that output producers do the abatement of emissions themselves (cf. Grimaud et al., 2009).

In addition, Appendix D shows that the social planner's optimality conditions imply $\omega_S = \omega_S(0)\exp(\int_0^t r(z)dz)$, where $\omega_S \equiv \mu_S/\lambda$ and where ω_S represents the social scarcity rent of fossil fuel (see a similar labelling in van der Ploeg and Withagen, 2012b). It is also shown in Appendix D that in the regulated market economy each resource owner will choose a resource extraction path of R that implies $p_R = p_R(0)\exp(\int_0^t i(z)dz)$. Moreover, internalisation of the climate externality implies that market agents choose the socially optimal quantities of R and P and therefore the private marginal product of capital equals the social marginal product of capital. As a consequence, it follows from (18) that $i = r$. Since then also $p_R(0) = \omega_S(0)$, it follows that, in case of internalisation of the climate externality, we have:

$$p_R = \omega_S. \tag{20}$$

In turn, it is straightforward to see that (20) together with the first order condition (19) give rise to the following proposition:

PROPOSITION 2: *In a regulated market economy, the social planner's optimality conditions can be replicated upon use of one of the following optimal tax rate combinations:*

(i) $t_R = \omega_p M_R$ and $t_M = 0$,

(ii) $t_M = \omega_p$ and $t_R = 0$.

PROOF: Substituting (20) in (19) and use of the definitions $\omega_S \equiv \mu_S/\lambda$ and $\omega_P \equiv \mu_p/\lambda$ shows that the resulting equation is identical to (7) if either $t_R = \omega_p M_R$ and $t_M = 0$ or $t_M = \omega_p$ and $t_R = 0$.

□

It is also straightforward to see that Proposition 2 leads to the following corallary.

CORALLARY 2: *If $M_R = 1$, then a Pigovian tax on fossil fuel use and a Pigovian tax on carbon emissions are both equal to the SCC, while otherwise only a Pigovian tax on carbon emissions equals the SCC.*

As a consequence of Corallary 2, it matters whether or not $M_R = 1$. Therefore, if $M_R = 1$, then the determinants of the rate of change of the SCC are substantially almost identical to the determinants in the social planner's Hotelling rule, while otherwise they differ substantially.

5 Relation to Previous literature

As was mentioned in the introduction, Grimaud et al. (2011) derive the growth rate of the optimal carbon tax rate in an endogenous growth model with carbon emissions from burning fossil fuels. Their formula looks much more complicated than the present paper's growth rate of the SCC. It remains more complicated even if we abstract from their more realistic but also more complicated climate damage specification. Using the present paper's simpler climate damage specification and using the assumption $U_P = 0$, which Grimaud et al. make, then their growth rate of the optimal tax on carbon emissions becomes:[27]

$$\hat{\tau_M} = r + \left[\frac{F_P U_C}{\displaystyle\int_t^\infty F_P(z) U_C(z) e^{-\rho(z-t)} dz.} \right]. \tag{21}$$

It is shown in Appendix E that using the social planner's optimality conditions, then (21) reduces to the present paper's optimal tax on carbon emissions, which can be shown from combining Proposition 2(ii) with (14) and imposing $U_P = 0$ to be:

$$\hat{\tau_M} = r + \frac{F_P}{\tau_M}, \tag{22}$$

Equation (22) clearly looks much simpler than (21).

As was also mentioned in the introduction, a somewhat similar growth rate of the optimal carbon tax as in the present paper has been derived in partial equilibrium models, such as Ulph and Ulph (1994). In that model a social planner maximises life-time utility $W(0) = \displaystyle\int_0^\infty \left[B(R) - D(P) \right] e^{-rt} dt$ subject to the constraints $\dot{S} = -R$ and $\dot{P} = \xi R$, where $B'(R) > 0$ and $D'(P) > 0$ and the notation remains unchanged.[28] From this optimisation problem, Ulph and Ulph derive the growth rate of the optimal tax on carbon emissions as:[29]

[27] Cf. Grimaud et al (2011, equation (48)).

[28] In this summary of the results of Ulph and Ulph, I abstract for comparability from extraction costs and from dissipation of the carbon stock in the atmosphere, which are considered in Ulph and Ulph.

[29] Cf. Ulph and Ulph (1994, equation (8)).

$$\hat{\tau_M} = r - \frac{D'(P)}{\tau_M}. \tag{23}$$

Combining Proposition 2(ii) with (14) and imposing $F_P = 0$, we find the present paper's growth rate of the optimal tax on carbon emissions to be:

$$\hat{\tau_M} = r + \frac{U_P/U_C}{\tau_M}. \tag{24}$$

It is straightforward to see that (24) is identical to (23) if we assume this paper's instantaneous utility function to be:

$$U(C, P) = C - D(P), \tag{25}$$

that is, if we assume instantaneous utility from consumption to be linear and additively separable from instantaneous disutility from carbon emissions. Hence, (23) can be seen as a special case of (24). Therefore, on the one hand, the optimal carbon tax in partial equilibrium models is not too unrealistic. On the other hand, it is also not substantially identical to the one in general equilibrium models. This is so because whether or not instantaneous utility is linear in consumption makes a difference.

6 Conclusion

This paper derived the rate of change of the SCC in a Ramsey model with emissions from burning fossil fuel. It has been shown that the determinants of the rate of change of the SCC are substantially almost identical to the determinants in the social planner's Hotelling rule if a unit of fossil fuel use leads to exactly one unit of carbon emission, while otherwise those formulas differ substantially. The paper has also shown that in the special case in which the two formulas are substantially almost identical, then a Pigovian tax on fossil fuel use and a Pigovian tax on carbon emissions are both equal to the SCC. Otherwise only a Pigovian tax on carbon emissions equals the SCC.

Future research might examine the time path of the SCC and therefore of the optimal carbon tax. Grimaud et al. (2011) estimated their paper's growth rate of the optimal carbon tax with a cap on carbon and without it. They did the estimations with use of functional forms and calibrated parameters from the latest version of the DICE model (Nordhaus, 2008). In the case without a carbon cap, they found the growth rate of the optimal carbon tax to be insignificantly different from the social discount rate and to remain so over time. Climate change pessimists however might argue that the DICE model makes assumptions that ensure fulfillment of many of the conditions listed in section 3 of this paper and fulfillment of those conditions can be questioned. However, calibrated parameters for the case in which many of the conditions listed in section 3 are not fulfilled are not readily available because getting calibrated parameters in such a scenario requires to account for climate externalities, which is not an easy task. Clearly, this task lies outside of the scope of the present paper.

Appendix A: Derivation of the Ramsey Rule, i.e. of (11) in the Text

Taking time derivatives of (6) we obtain:

$$U_{CC}\dot{C}e^{-\rho t} + U_{CP}\dot{P}e^{-\rho t} - \rho U_C e^{-\rho t} = \dot{\lambda}. \tag{26}$$

Upon substituting (6) in (8) we get:

$$\dot{\lambda} = -U_C e^{-\rho t} F_K. \tag{27}$$

Substituting (27) in (26) yields:

$$U_{CC}\dot{C}e^{-\rho t} + U_{CP}\dot{P}e^{-\rho t} - \rho U_C e^{-\rho t} = -U_C e^{-\rho t} F_K \dot{\lambda}. \tag{28}$$

Rearranging (28) and using the definition $r \equiv F_K$ gives rise to equation (11) in the text.

Appendix B: Derivation of the Social Planner's Hotelling Rule, i.e. of (12) in the Text

Define $\omega \equiv \mu/\lambda$, where we used the further definition $\mu \equiv \mu_S + \mu_p M_R$. Taking natural logarithm and then time derivatives of the definition of ω, we find:

$$\hat{\omega} = \hat{\mu} - \hat{\lambda} = \frac{\dot{\mu_S}}{\mu} + \frac{\dot{\mu_p}}{\mu} M_R + \frac{\mu_p}{\mu}\dot{M}_R - \hat{\lambda}. \tag{29}$$

Upon substitution of (9) and (10), use of (8) and rearranging, (29) becomes:

$$\hat{\omega} = F_K + \left(\frac{U_P e^{-\rho t} + \lambda F_P}{\mu}\right) M_R + \frac{\mu_p}{\mu}\dot{M}_R. \tag{30}$$

Using the definition $\mu \equiv \mu_S + \mu_p M_R$ in (8) leads to:

$$\lambda F_R = \mu. \tag{31}$$

Substituting (31) in (30) and using the definition $\omega_P \equiv \mu_p/\lambda$, we obtain:

$$\hat{\omega} = F_K + \left(\frac{U_P e^{-\rho t}}{\lambda F_R} + \frac{F_P}{F_R}\right) M_R + \frac{\omega_p}{F_R}\dot{M}_R. \tag{32}$$

Upon combining (6) with (32), we get:

$$\hat{\omega} = F_K + \left(\frac{U_P/U_C + F_P}{F_R}\right) M_R + \frac{\omega_p}{F_R}\dot{M}_R. \tag{33}$$

Rearranging (31), using the definition $\omega \equiv \mu/\lambda$, taking natural logarithms and then time derivatives, gives rise to:

$$\hat{F_R} = \hat{\omega}. \tag{34}$$

Finally, substituting (34) in (33), using the definition $r \equiv F_K$ and multiplying both sides of the resulting expression by F_R yields (12) in the text.

Appendix C: Derivation of the SCC, i.e. of (13) in the Text

Taking natural logarithms and then time derivatives of the definition $\omega_p \equiv \mu_p/\lambda$, we obtain:

$$\hat{\omega_P} = \hat{\mu_p} - \hat{\lambda}. \tag{35}$$

Combining (10) and (8) with (35) and rearranging gives:

$$\hat{\omega_P} = F_K + \left(\frac{U_P e^{-\rho t} + \lambda F_P}{\mu_p}\right). \tag{36}$$

Upon use of $\omega_p \equiv \mu_p/\lambda$ (36) becomes:

$$\hat{\omega_P} = F_K + \left(\frac{U_P e^{-\rho t}}{\lambda \omega_p} + \frac{F_P}{\omega_p}\right). \tag{37}$$

Combining (6) with (37), using the definition $r \equiv F_K$ and multiplying both sides of the resulting expression by ω_p, we find:

$$\dot{\omega_P} = r\omega_P + \left(\frac{U_P}{U_C} + F_P\right). \tag{38}$$

Solving (38) gives the following general solution:[30]

$$\omega_P = e^{\int r(t)dt}\left(\tilde{\omega}_P + \int e^{-\int r(t)dt}\left[\frac{U_P(t)}{U_C(t)} + F_P(t)\right]dt\right), \tag{39}$$

where $\int f(x)dx$ is called an indefinite integral.[31] Solving (39) forward by fixing the terminal condition $\omega_P(T)$, we obtain the particular solution as:

$$\omega_P = e^{-\int_t^T r(z)dz}\omega_P(T) - \int_t^T e^{-\int_t^z r(u)du}\left[\frac{U_P(z)}{U_C(z)} + F_P(z)\right]dz. \tag{40}$$

If we let in (40) T go to infinite, then (40) becomes:[32]

$$\omega_P = -\int_t^\infty \left[\frac{U_P(z)}{U_C(z)} + F_P(z)\right] e^{-\int_t^z r(u)du}dz + \lim_{T\to\infty}\omega_P(T)e^{-\int_t^T r(z)dz} \tag{41}$$

A similar equation to (41) can be found in the asset pricing literature for the stock price.[33] Upon use of the analogy to that literature, one can say that (41) has an infinite number of solutions unless one imposes in (41) the restriction that $\lim_{T\to\infty}\omega_P(T)e^{-\int_t^T r(z)dz} = 0$, which implies that ω_P cannot indefinitely grow faster than r and become infinitely large according to (41) (cf. Sørensen and Whitta-Jacobsen, 2010, page 395). If one imposes this restriction, then (41) becomes (13) in the text and then ω_P equals its so-called "fundamental" part of (41) only.

Appendix D: Derivation of the Social Scarcity Rent of Fossil Fuel and of the Price of a Unit of Fossil Fuel

Defining $\omega_S \equiv \mu_S/\lambda$, taking natural logarithms and then time derivatives yields:

[30] See Wälde (2011, pages 94-95) or Sydsaeter et al. (2005, page 200) for the mathematical approach to solve (38).

[31] Cf. Sydsaetter and Hammond (1985, page 326).

[32] See Wälde (2011, pages 100-101) and Sydsaeter et al. (2005, pages 201-202) for the mathematical approaches to derive (40) and (41).

[33] See e.g. Naoui (2011, p. 125).

$$\frac{\dot{\omega_S}}{\omega_S} = \frac{\dot{\mu_S}}{\mu_S} - \frac{\dot{\lambda}}{\lambda}. \tag{42}$$

Upon combining (42) with (9) and (8) and using the definition $r \equiv F_K$ and rearranging, we find:

$$\dot{\omega_S} = r\omega_S. \tag{43}$$

Solving (43) gives the particular solution as:[34]

$$\omega_S = \omega_S(0)e^{\int_0^t r(z)dz}. \tag{44}$$

Next, assume a continuum of identical resource owners with total mass one, which maximise their discounted profits $V(0) = \int_0^\infty p_R(t)R(t)e^{-\int_0^t i(z)dz}dt$ subject to their constraint $\dot{S} = -R$. This optimisation problem gives rise to the following present value Hamiltonian:

$$\tilde{H} = p_R(t)R(t)e^{-\int_0^t i(z)dz} + \tilde{\lambda}[-R].$$

The first order conditions from maximisation of $\tilde{H}$ are:

$$\frac{\partial \tilde{H}}{\partial R} = 0 \quad \Rightarrow \quad p_R e^{-\int_0^t i(z)dz} = \tilde{\lambda}, \tag{45}$$

$$\frac{\partial \tilde{H}}{\partial S} = -\dot{\tilde{\lambda}} \quad \Rightarrow \quad \dot{\tilde{\lambda}} = 0. \tag{46}$$

Taking the time derivative of (45) gives:

$$\dot{\tilde{\lambda}} = \dot{p_R} e^{-\int_0^t i(z)dz} - p_R i e^{-\int_0^t i(z)dz}. \tag{47}$$

Upon substituting (46) in (47) and rearranging we obtain the familiar Hotelling rule:

$$\dot{p_R} = i p_R. \tag{48}$$

[34]See Sydsaetter and Hammond, 1985, page 767, example 21.6.

Solving (48) in the same way in which we solved (43) gives the particular solution as:[35]

$$p_R = p_R(0)e^{\int_0^t i(z)dz}. \tag{49}$$

Appendix E: Derivation that (21) reduces to (22)

Rearranging (21) yields:

$$\hat{\tau_M} = r + \left[\frac{F_P U_C e^{-\rho t}}{\int_t^\infty F_P(z) U_C(z) e^{-\rho z} dz.} \right]. \tag{50}$$

Setting in (6) $t = z$ implies:

$$U_C(z)e^{-\rho z} = \lambda(z). \tag{51}$$

Using (6) and (51) in (50), we find:

$$\hat{\tau_M} = r + \left[\frac{F_P \lambda}{\int_t^\infty \lambda(z) F_P(z) dz.} \right]. \tag{52}$$

Next, assuming $U_P = 0$, then (10) becomes:

$$\dot{\mu}_P = \lambda F_P. \tag{53}$$

Upon ordinary integration of (53) we obtain:

$$\mu_P = \int_t^\infty \lambda(z) F_P(z) dz. \tag{54}$$

Substituting (54) in (52) and using the definition $\omega_P \equiv \mu_p/\lambda$, gives rise to:

$$\hat{\tau_M} = r + \frac{F_P}{\omega_p}. \tag{55}$$

[35] The maximisation of the Hamiltonian and the derivation of (49) follows Faucheux and Noel (2001, page 149-150). Unfortunately, this publication is only available in German and French. However, similar derivations can be found in English in Perman et al. (2003, page 515) and Grimaud and Rouge (2005, page 119).

Using in (55) the fact that according to Proposiotion 2(ii) internalisation of the climate externality is achieved if $t_M = \omega_p$ and $t_R = 0$, we get (22) in the text.

References

Anthoff, D., and R.S.J. Tol (2012). Schelling's Conjecture on Climate and Development: A Test. In Robert W. Hahn and Alistair Ulph (Eds.), *Climate Change and Common Sense: Essays in Honour of Tom Schelling.* Robert W. Hahn and Alistair Ulph (eds.), Oxford University Press, 2012: 260-273.

Aronsson, T., and K.-G. Löfgren (1998). Green Accounting in Imperfect Market Economies - A Summary of Recent Research. *Environmental and Resource Economics* 11: 273-287.

Dasgupta, P., and G. Heal (1974). The Optimal Depletion of Exhaustible Resources. *Review of Economic Studies* 41: S. 3-28.

Dasgupta, P., and G. Heal (1995). *Economy Theory and Exhaustible Resources.* Cambridge University Press, Cambridge.

Faucheux, S., and J.-F.Noel (2001). *Ökonomie natürlicher Ressourcen und der Umwelt.* Metropolis Verlag, Marburg (Germany).

Goulder, L.H., and K. Mathai (2000). Optimal CO_2 Abatement in the Presence of Induced Technical Change. *Journal of Environmental Economics and Management* 39: 1-38.

Golosov, M., Hassler, J., Krusell, P., and A. Tsyvinski (2011). Optimal Taxes on Fossil Fuel in General Equilibrium. NBER Working Paper 17348.

Grimaud, A., Lafforgue, G., and B. Magne (2011). Climate Change Mitigation Options and Directed Technical Change. *Resource and Energy Economics* 33: 938-962.

Grimaud, A., Magne, B., and L. Rouge (2009). Polluting Non-Renewable Resources, Carbon Abatement and Climate Policy in a Romer Growth Model. IDEI Working Paper, No. 548.

Grimaud, A., and L. Rouge (2005). Polluting Non-Renewable Resources, Innovation and Growth: Welfare and Environmental Policy. *Resource and Energy Economics* 27: 109-129.

Groth, C., and P. Schou (2007). Growth and Non-Renewable Resources: The Different Roles of Capital and Resource Taxes. *Journal of Environmental Economics and Management* 53: 80-98.

Hoel, M., and S. Kverndokk (1996). Depletion of Fossil Fuels and the Impacts of Global Warming. *Resource and Energy Economics* 18: 115-136.

Hoel M., and T. Sterner (2007). Discounting and Relative Prices. *Climatic Change* 84: 1573-1480.

Kögel, T. (2009). On the Relation between Discounting of Climate Change and Edgeworth-Pareto Substitutability. *Economics: The Open-Access, Open-Assessment E-Journal,* Vol. 3, 2009-27 (Version 2).

Kögel, T. (2012). The Growth Rate of the Social Cost of Carbon in an Optimal Growth Model. Mimeo (uploaded on 11 January 2012), downloadable through typing paper title in scholar.google.com

Krautkrämer, J. A. (1985). Optimal Growth, Resource Amenities and the Reservation of Natural Environments, *Review of Economic Studies* 52: 153–170.

Kuik, O. (2009). A Perspective Paper on Mitigation as a Response to Climate Change. Copenhagen Consensus Center, Denmark.

Neumayer, E. (1999). Global Warming: Discounting is not the Issue, but Substitutability is. *Energy Policy* 27: 33–43.

Naoui, K. (2011). Intrinsic Bubbles in the American Stock Exchange: The Case of the S&P 500 Index. *International Journal of Economics and Finance* 3: 124-132.

Newbold, S., Griffiths, C., Moore, C. ,Wolverton, A., and E. Kopitz (2009). The "Social Cost of Carbon" made Simple. Presentation at the Annual Meeting of the Benefit Cost Analysis, Washington D.C.

Nordhaus, W.D. (2008). *A Question of Balance: Weighing the Options on Global Warming Policies.* Yale University Press, Yale (Connecticut).

Nordhaus, W.D. (2011). Estimates of the Social Cost of Carbon: Background and Results from the RICE-2011 Model. Cowles Foundation Discussion Paper No. 1826.

Perman, R., Ma, Y., McGilvray, J., and M. Common (2003). *Natural Resource and Environmental Economics*. 3rd Edition. Harlow, England: Pearson/Addison Wesley.

Ploeg van der, F., and C. Withagen (2011). Growth and the Optimal Carbon Tax: When to Switch from Exhaustible Resources to Renewables? OxCarre Research Paper 55 (revised version of 4 January 2011), downloadable through typing paper title in scholar.google.com

Ploeg van der, F. and C. Withagen (2012a). Too Much Coal, too Little Oil. *Journal of Public Economics* 96: 62-77.

Ploeg van der, F., and C. Withagen (2012b). Growth, Renewables and the Optimal Carbon Tax. OxCarre Research Paper 55 (revised version of 21 June 2012).

Sinclair, P.J.N. (1994). On the Optimum Trend of Fossil Fuel Taxation. *Oxford Economic Papers* 46: 869-877.

Sinn, H.-W. (2007). Pareto Optimality in the Extraction of Fossil Fuels and the Greenhouse Effect. CESifo working paper no. 2083.

Sinn, H.-W. (2008). Public Policies against Global Warming. *International Tax and Public Finance*, 15: 360-394.

Solow, R. M. (1974). Intergenerational Equity and Natural Resources. *Review of Economic Studies* 41: S. 29-45.

Sørensen, P.B., and H.J. Whitta-Jacobsen (2010). *Introducing Advanced Macroeconomics: Growth and Business Cycles*. 2nd edition, Harlow, England: Pearson/Addison Wesley.

Sterner, T., and U.M. Persson (2008). An Even Sterner Review: Introducing Relative Prices into the Discounting Debate. *Review of Environmental Economics and Policy* 2: 61–76.

Stiglitz, J.E. (1974) Growth with Exhaustible Natural Resources: Efficient and Optimal Growth Paths, *Review of Economic Studies* 41: S. 123–137.

Stokey, N., 1998. Are there Limits to Growth? *International Economic Review* 39: 1-39.

Sydsaetter, K., and P.J. Hammond (1985). *Mathematics for Economic Analysis.* Prentice Hall, New Jersey.

Sydsaeter K., Hammond, P., Seierstad, A., and A. Strom. (2005). *Further Mathematics for Economic Analysis.* Prentice Hall, Harlow (England).

Tol, R.S. J. (2009). An Analysis of Mitigation as a Response to Climate Change. Copenhagen Consensus Center, Denmark.

Ulph, A., and D. Ulph (1994). The Optimal Time Path of a Carbon Tax. *Oxford Economic Papers* 46: 857-868.

Wälde, K. (2011). *Applied Intertemporal Optimisation.* Edition 1.1. Mainz University, Gutenberg Press, Mainz (Germany).

Weikard, H.-P., Zhu, X., 2005. Discounting Environmental Quality: When should Dual Rates be used. *Economic Modelling* 22: 868-878.

Withagen, C. (1994). Pollution and Exhaustability of Fossil Fuels. *Resource and Energy Economics* 16: 235-242.