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Inequality Measurement with Subgroup Decomposability and Level-Sensitivity

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Abstract Subgroup decomposability is a very useful property in an inequality measure, and level-sensitivity, which requires a given level of inequality to acquire a greater significance the poorer a population is, is a distributionally appealing axiom for an inequality index to satisfy. In this paper, which is largely in the nature of a recollection of important results on the characterization of subgroup decomposable inequality measures, the mutual compatibility of subgroup decomposability and level-sensitivity is examined, with specific reference to a classification of inequality measures into relative, absolute, centrist, and unit-consistent types. Arguably, the most appealing combination of properties for a symmetric, continuous, normalized, transfer-preferring and replication-invariant (S-C-N-T-R) inequality measure to satisfy is that of subgroup decomposability, centrism, unit-consistency and level-sensitivity. The existence of such an inequality index is (as far as this author is aware) yet to be established. However, it can be shown, as is done in this paper, that there does exist an S-C-N-T-R measure satisfying the (plausibly) next-best combination of properties—those of decomposability, centrism, unit-consistency and level-neutrality.

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INEQUALITY MEASUREMENT WITH SUBGROUP DECOMPOSABILITY AND LEVEL-SENSITIVITY

1. Introduction

In the axiomatic approach to the measurement of inequality, a number of desirable properties of inequality indices have been advanced. In this article, we consider two specific properties – those of 'decomposability' and 'level-sensitivity' – and check for their mutual compatibility in the presence of other specified properties. The points made in this essay draw on a number of important results which have already been established in the literature: it is then mainly a matter of putting these results together in order to present a set of observations on the prospects of simultaneously meeting the requirements of decomposability and level-sensitivity. The outcome is arguably useful, insofar as taxonomies (in this case of inequality measures) are generally useful; the outcome is also inarguably dependent on a great deal of important prior work that has been done on the subject of decomposable inequality measures.

Subgroup decomposability (see Bourguignon 1979, Cowell 1980, Cowell and Kuga 1981, Shorrocks 1980, 1984, 1988) is the property that an inequality measure be expressible as an exact sum of a 'between-group component' (obtained by imagining that each person in any subgroup receives the subgroup's mean income) and a 'within-group component' (obtained as a weighted sum of subgroup inequality levels, the weights depending on the subgroups' income shares or population shares or some combination of the two shares).

Level-sensitivity can be thought of as a group-related egalitarian requirement that arises when a population is partitioned into non-overlapping income groups of the same size: it postulates that in this circumstance, and other things remaining the same, a given increase in subgroup inequality should cause overall inequality to rise by more the poorer (in terms of subgroup mean income) the subgroup is. This property has a strong affinity to a concern expressed in an early contribution by Amartya Sen (1973), and relating to the question of how our view on inequality ought to vary with the general level of a society's prosperity. As observed by Sen (1973: 36):

> Can it be asserted that our judgment of the extent of inequality will not vary according to whether the people involved are generally poor or generally rich? Some have taken the view that our concern with inequality increases as a society gets prosperous since the society can 'afford' to be inequality-conscious. Others have asserted that the poorer an economy, the more 'disastrous' the consequences of inequality, so that inequality measures should be sharper for low average income. This is a fairly complex question and is bedeviled by a mixture of positive and normative considerations. The view that for poorer economies inequality measures must be themselves sharper can be contrasted with the view that greater *importance* must be attached to any given inequality measure if the economy is poorer. The former incorporates the value in question into the measure of inequality itself, while the latter brings it in through the evaluation of the relative importance of a given measure at different levels of average income.

It is the former of the two views asserted by Sen at the conclusion of the preceding quote that is upheld by the level-sensitivity axiom.

In this essay, we examine the mutual compatibility of subgroup decomposability and level-sensitivity for certain broad classes of inequality measures, taxonomised according to their invariance to multiplicative or additive transformations of an income distribution. In terms of this classification, inequality measures can be relative or absolute. A relative inequality measure is 'scale-invariant', while an absolute inequality measure (see Blackorby and Donaldson 1980) is 'translation-invariant'. Scale-invariance is the property that the value of an inequality measure should remain unchanged if all persons' incomes were to be uniformly multiplied by any positive scalar, while translation-invariance requires such constancy in the value of an inequality measure when all persons' incomes are increased (or decreased) by the addition (or subtraction) of a fixed amount.

The invariance requirements just considered have both purely 'analytical' and 'normative' implications. At the analytical level, scale-invariance ensures that the value of an inequality index does not change with the units in which income is measured, while translation-invariance violates this property of neutrality with respect to the units of measurement. From this 'analytical' perspective, scale-invariance would appear to possess an attractive advantage over translation-invariance. However, from a 'normative' perspective, scale-invariance can be seen to uphold a 'right-wing' view of inequality and translation-invariance to uphold a 'left-wing' view, as pointed out by Serge-Christophe Kolm (1976a, 1976b). Notice that, given a two-person ordered income distribution $\mathbf{x} = (1,100)$, a doubling of each person's income would lead to the distribution $\mathbf{y} = (2,200)$: a scale-invariant index would uphold the (typically right-wing) judgment that the extent of inequality is the same in both distributions, despite the fact that out of the additional total income of 101 units in **y** vis-à-vis **x**, 100 units of income have gone to the richer person and only 1 unit to the poorer person. In contrast, if **z** were to be derived from **x** by the addition of 100 units of income to each person, so that **z** = (101, 200), a translation-invariant index would uphold the (typically left-wing) judgment that the extent of inequality is the same in both distributions, despite the fact that in the transition from **x** to **z**, the poorer person's income has risen by a factor of 10,000 per cent and the richer person's income by a factor of just 100 per cent.

One can see now that one can have inequality measures which are a 'compromise' between absolute and relative measures. The compromise we effect would depend on whether we take a purely analytical or a normative view of the two classes of measures. Under a purely analytical interpretation, a compromise class of measures would be *unit-consistent* measures (Zheng 2007), namely inequality measures which satisfy the requirement that the inequality-*ranking* of distributions is invariant with respect to the choice of units in which income is measured. As it happens, all right-wing measures and some left-wing measures are unit-consistent. A different type of compromise is the normative one between right- and left-wing measures, which leads to a class of *centrist* or *intermediate* measures (see, for example, Zheng 2007): an intermediate measure is one which satisfies the property that (i) a uniform scaling-up of every individual's income should increase inequality and (ii) the addition of any given income to every person's income should reduce inequality. It should be noted that the two types of compromise we have just considered are mutually independent: unit-consistent inequality measures are not necessarily centrist measures, and similarly centrist inequality measures are not necessarily unit-consistent.

In examining subgroup decomposability and level-sensitivity of inequality measures for a classification of measures according to their disposition toward distributional values and unit-consistency, this article proceeds as follows. The following section introduces concepts and notation. This is followed by a section which advances a set of observations on subgroup decomposability and level-sensitivity for alternative types of inequality measures. The final section offers a summary and conclusions.

2. Basic Concepts

$\mathcal{N}$ is the set of positive integers, and $\mathcal{R}$ is the real line. For every $n \in \mathcal{N}$, $\mathbf{X}_n$ is the set of positive *n*-vectors $\mathbf{x} = (x_1, ..., x_i, ..., x_n)$, and each $\mathbf{x}$ is to be interpreted as an income vector whose typical element x_i is the income of individual i in a community of n individuals. $\mathbf{X}$ is the set $\cup_{n \in \mathcal{N}} \mathbf{X}_n$, and an inequality index is a mapping $I : \mathbf{X} \to \mathcal{R}$ such that, for every $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x})$ is a real number which is supposed to indicate the amount of inequality associated with the distribution **x**. For every income vector $\mathbf{x} \in \mathbf{X}$, $N(\mathbf{x})$ is the set of individuals represented in **x**, and $n(\mathbf{x}) \equiv \# N(\mathbf{x})$ is the dimensionality of **x**, while $\mu(\mathbf{x})$ is the mean income of **x**. If a population is partitioned into $K(\geq 1)$ subgroups $\{1, ..., j, ..., K\}$, then $\mathbf{x}_j$ is the income vector of the *j*th subgroup, $N(\mathbf{x}_j)$ is the set of individuals represented in $\mathbf{x}_j$, $n(\mathbf{x}_j)$ is the dimensionality of $\mathbf{x}_j$, $\mu(\mathbf{x}_j)$ is the mean income of $\mathbf{x}_j$, and $I(\mathbf{x}_j)$ is the extent of inequality associated with $\mathbf{x}_j$ ($j = 1, ..., K$). Where there is no ambiguity, we shall also write I for $I(\mathbf{x})$, n for $n(\mathbf{x})$, μ for $\mu(\mathbf{x})$, I_j for $I(\mathbf{x}_j)$, n_j for $n(\mathbf{x}_j)$, μ_j for $\mu(\mathbf{x}_j)$, and so on.

Let $\boldsymbol{I}^*$ be the set of inequality measures such that a typical member of this set, $I : \mathbf{X} \to \mathcal{R}$, satisfies the following properties:

Symmetry (Axiom S), which is the requirement that for all $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x}) = I(\mathbf{x\Pi})$ where $\mathbf{\Pi}$ is any appropriately dimensioned permutation matrix (so measured inequality is impervious to the personal identities of individuals);

Normalization (Axiom N), which is the requirement that for all $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x}^0) = 0$, where $\mathbf{x}^0$ is the vector obtained from $\mathbf{x}$ by setting $x_i^0 = \mu(\mathbf{x}) \forall i = 1, ..., n(\mathbf{x})$ (so that inequality is taken to be zero when all incomes are equalized);

Continuity (Axiom C), which is the requirement that I be continuous on $\mathbf{X}_n$ for all $n \in \mathcal{N}$ (so that 'similar income distributions have similar inequality values');

Schur-Concavity (Axiom SC), which is the requirement that for all $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x}) > I(\mathbf{xB})$ where $\mathbf{B}$ is any appropriately dimensioned bistochastic matrix which is not a permutation matrix (so that any movement toward equalization of the incomes in a distribution causes measured inequality to decline);

Replication Invariance (RI), which is the requirement that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $I(\mathbf{x}) = I(\mathbf{y})$ whenever $\mathbf{y}$ is a *q*-replication of $\mathbf{x}$, that is, $\mathbf{y} = (\mathbf{x},...,\mathbf{x})$, $n(\mathbf{y}) = qn(\mathbf{x})$, and q is any positive integer greater than 1 (so that inequality values depend only on the relative, not the absolute, frequency distribution of incomes); and

Differentiability (D), which is the requirement that for all $\mathbf{x} \in \mathbf{X}$, I should have continuous first and second partial derivatives $(\partial I / \partial x_i \forall i \in N(\mathbf{x})$ and $\partial^2 I / \partial x_i^2 \forall i \in N(\mathbf{x})$ respectively) in each income in the vector.

Some basic definitions relating to relative and absolute inequality measures, and 'compromise' versions of these, are now provided. (Note that relative inequality measures are also referred to as 'right-wing' measures, and absolute inequality measures as 'left-wing' measures.)

Definition 1 (Relative Inequality Measure). An inequality measure $I : \mathbf{X} \rightarrow \mathcal{R}$ is *relative* if and only if it is *scale-invariant*, that is, if and only if, for all $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x}) = I(\lambda \mathbf{x})$ for any $\lambda \in \mathcal{R}_{++}$.

Definition 2 (Absolute Inequality Measure). An inequality measure $I : \mathbf{X} \rightarrow \mathcal{R}$ is *absolute* if and only if it is *translation-invariant*, that is, if and only if, for all $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x}) = I(\mathbf{x} + \mathbf{t})$ where $\mathbf{t} = (t,...,t)$ for any $t \in \mathcal{R}$ and $n(\mathbf{t}) = n(\mathbf{x})$.

Definition 3 (Unit-Consistent Inequality Measure). An inequality measure $I : \mathbf{X} \rightarrow \mathcal{R}$ is *unit-consistent* if and only if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $I(\mathbf{x}) < I(\mathbf{y})$ implies $I(\lambda \mathbf{x}) < I(\lambda \mathbf{y})$ for any $\lambda \in \mathcal{R}_{++}$ (see Zheng 2007).

Definition 4 (Centrist Inequality Measure). An inequality measure $I:\mathbf{X}\rightarrow\mathcal{R}$ is *centrist* if and only if, for all $\mathbf{x}\in\mathbf{X}$, (i) $I(\mathbf{x})<I(\lambda\mathbf{x})$ for any $\lambda>1$ and (ii) $I(\mathbf{x})>I(\mathbf{x}+\mathbf{t})$ where $\mathbf{t}=(t,...,t)$ for any $t\in\mathcal{R}_{++}$ and $n(\mathbf{t})=n(\mathbf{x})$ (see Zheng 2007).

Definition 5 (Bossert-Pfingsten Restriction). A centrist inequality measure $I:\mathbf{X}\rightarrow\mathcal{R}$ will be said to obey the *Bossert-Pfingsten restriction* (see Bossert and Pfingsten 1990) if and only if, for all $\mathbf{x}\in\mathbf{X}$, $I(\mathbf{x})=I(\mathbf{x}+a[\pi\mathbf{x}+(1-\pi)\mathbf{t}])$, where $a\in\mathcal{R}$, $\pi\in[0,1]$, and $\mathbf{t}=(t,...,t)$ for any $t\in\mathcal{R}_{++}$ and $n(\mathbf{t})=n(\mathbf{x})$.
[The restriction stated above provides a particular operationalization of the notion of a centrist inequality measure by specifying a plausible condition under which the measure should remain unchanged for some combination of a uniform scale increase and a uniform incremental increase in all incomes of a distribution.]

Next, the notion of ‘level-sensitivity’ is defined. Level-sensitivity essentially demands that, when a population is partitioned into equi-dimensional non-overlapping income groups, then, other things equal, a given increase in subgroup inequality should cause aggregate inequality to rise by more the poorer (in terms of mean income) the subgroup is. More formally:

Level-Sensitivity (Axiom LS). An inequality measure $I:\mathbf{X}\rightarrow\mathcal{R}$ is *level-sensitive* if and only if, for all $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}$, if $\mathbf{x}=(\mathbf{x}_1,...,\mathbf{x}_K)$, $\mathbf{y}=(\mathbf{y}_1,...,\mathbf{y}_K)$, $\mathbf{z}=(\mathbf{z}_1,...,\mathbf{z}_K)$,

$n(\mathbf{x}_j)=n(\mathbf{x}_{j+1})(=n(\mathbf{x})/K)\forall j=1,...,K-1$,

$n(\mathbf{x}_j)=n(\mathbf{y}_j)=n(\mathbf{z}_j)\forall j\in\{1,...,K\}$,

$\mu(\mathbf{x}_j)=\mu(\mathbf{y}_j)=\mu(\mathbf{z}_j)\ \forall j\in\{1,...,K\}$,

$\mu(\mathbf{x}_j)<\mu(\mathbf{x}_{j+1})\forall j=1,...,K-1$,

$I(\mathbf{x}_j)=I(\mathbf{x}_{j+1})=I$ (say) $\forall j=1,...,K-1$, and

[$I(\mathbf{x}_j)=I(\mathbf{y}_j)\ \forall j\in\{1,...,K\}\setminus\{s\}$ and

$I(\mathbf{x}_j)=I(\mathbf{z}_j)\ \forall j\in\{1,...,K\}\setminus\{t\}$]

for some subgroups s and t such that $I(\mathbf{y}_s)=I(\mathbf{z}_t)=I+\Delta,\ \Delta\in\mathcal{R}_{++}$,

and $s<t$ (so that $\mu(\mathbf{x}_s)<\mu(\mathbf{x}_t)$), then:

$I(\mathbf{y}) > I(\mathbf{z}) > I(\mathbf{x})$.

Finally, we state the axiom of sub-group decomposability:

Subgroup Decomposability (Axiom SD). For all $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x})$ is a *subgroup decomposable inequality measure* if and only if $I(\mathbf{x}) = I_W(\mathbf{x}) + I_B(\mathbf{x})$, where $I_W(\mathbf{x})$ is the *within-group component* of inequality, defined as:

$$I_W(\mathbf{x}) = \sum_{j=1}^{K} w(\mathbf{x}_j) I(\mathbf{x}_j) \ ,$$

where $w(\mathbf{x}_j)$ is a weight attached to subgroup *j*'s inequality level, with $w(\mathbf{x}_j)$ depending only on subgroup *j*'s population share $\pi(\mathbf{x}_j)$ or income share $\sigma(\mathbf{x}_j)$ or both; and

$I_B(\mathbf{x})$ is the *between-group component* of inequality, defined as:

$$I_B(\mathbf{x}) = I(\mathbf{x}_1^0, ..., \mathbf{x}_j^0, ..., \mathbf{x}_K^0),$$

where, for all $\mathbf{x} \in \mathbf{X}$, $\mathbf{x}^0$ is the vector obtained by setting $x_i^0 = \mu(\mathbf{x}) \forall i = 1, ..., n(\mathbf{x})$.

Notice that if a population is partitioned into non-overlapping income groups, then as long as the group-specific inequality levels and population shares remain unchanged, it is reasonable – even if the group-specific income shares should change - to expect the within-group component of a decomposable inequality index to also remain unchanged. Decomposability subjected to this reasonable restriction can be called '*proper decomposability*', and it is easy to see that proper decomposability implies the requirement that the group-specific weights $w(\mathbf{x}_j)$ appearing in the definition of subgroup decomposability should depend only on the subgroup population-shares (and, in particular, not at all on the subgroup income-shares):

Proper Subgroup Decomposability (Axiom PSD). Axiom PSD is derived from Axiom SD by replacing the phrase 'where $w(\mathbf{x}_j)$ is a weight attached to subgroup *j*'s inequality level, with $w(\mathbf{x}_j)$ depending only on subgroup *j*'s population share $\pi(\mathbf{x}_j)$ or income share $\sigma(\mathbf{x}_j)$ or both' with the phrase 'where $w(\mathbf{x}_j)$ is a weight attached to subgroup *j*'s inequality level, with $w(\mathbf{x}_j)$ depending only on subgroup *j*'s population share $\pi(\mathbf{x}_j)$'.

A number of important results relating to the characterization of subgroup decomposable inequality indices have been established in the literature. Some of these results are summarized in what follows:

Result 1 (Shorrocks 1980). For all $\mathbf{x} \in \mathbf{X}$, a relative inequality measure I belongs to the set $\boldsymbol{I}^*$ and satisfies subgroup decomposability if and only if it is a positive multiple of a member of the following class $I_c(\mathbf{x})$ of Generalized Entropy measures:

$$I_c(\mathbf{x}) = \frac{1}{n(\mathbf{x})c(c-1)} \sum_{i=1}^{n(\mathbf{x})} [(x_i / \mu(\mathbf{x}))^c - 1], c \in \mathcal{R}, c \neq 0,1;$$

$$= \frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \left(\frac{x_i}{\mu(\mathbf{x})} \right) \left(\ln \frac{x_i}{\mu(\mathbf{x})} \right), c = 1;$$

$$= \frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \left(\ln \frac{\mu(\mathbf{x})}{x_i} \right), c = 0.$$

Result 2 (Chakravarty and Tyagarupananda 1998, Bosmans and Cowell 2010). For all $\mathbf{x} \in \mathbf{X}$, an absolute inequality measure I belongs to the set $\boldsymbol{I}^*$ and satisfies subgroup decomposability if and only if it is a continuous and strictly increasing function of the following class $I_b(\mathbf{x})$ of measures:

$$I_b(\mathbf{x}) = \frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} [e^{b(x_i - \mu(\mathbf{x}))} - 1], b \in \mathcal{R}, b \neq 0;$$

$$= \frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} [x_i - \mu(\mathbf{x})]^2, b = 0.$$

Result 3 (Chakravarty 2000). For all $\mathbf{x} \in \mathbf{X}$, an absolute inequality measure I belongs to the set $\boldsymbol{I}^*$ and satisfies proper subgroup decomposability if and only if it is a positive multiple of the variance, given by:

$$V(\mathbf{x}) = \frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} (x_i - \mu(\mathbf{x}))^2.$$

Result 4 (Chakravarty and Tyagarupananda 2009). For all $\mathbf{x} \in \mathbf{X}$, a centrist inequality measure I belongs to the set ***I****, obeys the Bossert-Pfingsten restriction [namely the requirement that $I(\mathbf{x}) = I(\mathbf{x} + a[\pi\mathbf{x} + (1-\pi)\mathbf{t}])$, where $a \in \mathcal{R}$, $\pi \in [0,1]$, and $\mathbf{t} = (t,...,t)$ for any $t \in \mathcal{R}_{++}$ and $n(\mathbf{t}) = n(\mathbf{x})$], and is subgroup decomposable, if and only if it is a member of the following class $\hat{I}_c(\mathbf{x})$ of transformed Generalized Entropy measures:

$$\hat{I}_c(\mathbf{x}) = \frac{1}{n(\mathbf{x})c(c-1)} \sum_{i=1}^{n(\mathbf{x})} [\{(x_i + v) / (\mu(\mathbf{x}) + v)\}^c - 1], c \in \mathcal{R}, c \neq 0,1;$$

$$= \frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \left(\frac{x_i + v}{\mu(\mathbf{x}) + v} \right) \left(\ln \frac{x_i + v}{\mu(\mathbf{x}) + v} \right), c = 1;$$

$$= \frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \left(\ln \frac{\mu(\mathbf{x}) + v}{x_i + v} \right), c = 0,$$

where $v = (1-\pi)/\pi$, and c depends on both a and π.

Result 5 (Zheng 2007). For all $\mathbf{x} \in \mathbf{X}$, a unit-consistent inequality measure I belongs to the set ***I**** and satisfies subgroup decomposability if and only if it is a positive multiple of a member of the following class $\tilde{I}_c(\mathbf{x})$ of measures:

$$\tilde{I}_c(\mathbf{x}) = \frac{1}{c(c-1)n(\mathbf{x})\mu^d(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} [x_i^c - \mu^c(\mathbf{x})], \; c, d \in \mathcal{R}, c \neq 0,1;$$

$$= \frac{1}{n(\mathbf{x})\mu^{d-1}(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \left(\frac{x_i}{\mu(\mathbf{x})} \right) \left(\ln \frac{x_i}{\mu(\mathbf{x})} \right), \; c = 1, d \in \mathcal{R};$$

$$= \frac{1}{n(\mathbf{x})\mu^d(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \left(\ln \frac{\mu(\mathbf{x})}{x_i} \right), \; c = 0, d \in \mathcal{R}.$$

[In the interests of formal accuracy, it should be pointed out that in the Bosmans-Cowell (2010) version of Result 2, the axioms of normalization and differentiability are dispensed with, and Result 3 (Chakravarty, 2000) does not really invoke the replication invariance property.] Result 3 relates to the characterization of a properly subgroup decomposable inequality measure which is absolute, while Results 1, 2, 4 and 5 relate to the characterization of subgroup decomposable measures which are, respectively, relative, absolute, centrist, and unit-consistent. How do these measures fare in relation to level-sensitivity? This issue is examined in the following section.

3. Some Observations on Subgroup Decomposability and Level-Sensitivity

While both subgroup decomposability and level-sensitivity appear to be attractive properties of an inequality index, it may not always be possible for an inequality measure to satisfy both properties. We illustrate this proposition by considering the Gini coefficient G of inequality which, though it is not a subgroup decomposable (nor even subgroup consistent) measure, does lend itself to decomposability in the special case in which the population is partitioned into non-overlapping income groups (see Anand 1983). Specifically, it can be shown that if a population is divided into, say, K non-overlapping income groups of the same size, so that $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_j, ..., \mathbf{x}_K)$ with $n(\mathbf{x}_j) = n(\mathbf{x}_k) \forall j, k \in \{1, ..., K\}$ [and therefore $\pi(\mathbf{x}_j) = \pi(\mathbf{x}_k) = 1/K \forall j, k \in \{1, ..., K\}$], then one can write:

$G(\mathbf{x}) = G_B(\mathbf{x}) + G_W(\mathbf{x})$, where

$G_B(\mathbf{x}) = 1 + \frac{1}{K} - \frac{2}{K}\sum_{j=1}^{K}(K+1-j)\sigma_j$; and

$G_W(\mathbf{x}) = \frac{1}{K}\sum_{j=1}^{K}\sigma_j G_j$.

Of interest is the fact that in the expression for the within-group component of aggregate inequality, the weight on the *j*th subgroup's inequality level is σ_j / K: if the groups are indexed in ascending order of mean-income, then it is clear that when $G_j = G_k \forall j, k \in \{1, ..., K\}$, a given increase in inequality will raise aggregate inequality by more the *richer* (in terms of mean income) the subgroup is, since the weight on G_j, σ_j / K, is an increasing function of j: this precisely reverses what the axiom of level-sensitivity demands.

What can be said at a more general level about subgroup decomposability and level-sensitivity? A first and immediately obvious conclusion that emerges from a consideration of the concepts and definitions discussed in the preceding section is that there is a mutual incompatibility between the properties of proper decomposability and level-sensitivity of an inequality measure. This follows from noting that when a population is partitioned into non-overlapping income groups of equal size, any properly decomposable inequality measure I belonging to the set ***I**** will (by definition) have a within-group inequality component which is a weighted sum of subgroup inequality levels where the weights depend only on the subgroup population shares – which must all be equal since the subgroups are of equal size: a given increase in subgroup inequality will therefore cause overall inequality to rise by the same extent, irrespective of the average level of prosperity of the subgroup. The outcome is that level-sensitivity is a casualty. This leads to our first observation:

Observation 1. There exists no properly decomposable inequality measure $I \in \boldsymbol{I}^*$ which is level-sensitive.

Observation 1 suggests that if level-sensitivity is a desired normative property of an inequality index, then insistence on *proper* decomposability may have to be sacrificed. Indeed, the following observation, it can be shown, is true:

Observation 2. There exists a relative inequality measure $I \in \boldsymbol{I}^*$ which satisfies both subgroup decomposability and level-sensitivity.

To see this, recall from Result 1 that the only relative inequality measures in ***I**** which satisfy subgroup decomposability are positive multiples of members of the class of generalized entropy indices I_c. As a matter of convention, the only members of I_c in common circulation are restricted to the case in which the parameter c assumes non-negative values: specifically, $c = 1$ and $c = 0$ correspond to the two well-known Theil indices T_1 and T_2 respectively, while $c = 2$ yields one-half the squared coefficient of variation C. None of these three indices is level-sensitive: it is well-known that the weight on the inequality level of subgroup j in the within-group component of inequality is σ_j for T_1, π_j for T_2, and $\pi_j \sigma_j^2$ for C. The implications for level-sensitivity are plain: T_1 and C are level-*insensitive*, while T_2 is level-*neutral*. The picture, however, becomes promising when we consider negative values for the parameter c. Specifically, if we set $c = -1$, then we obtain an inequality measure – call it I_{-1} - given by: $I_{-1}(\mathbf{x}) = (1/n(\mathbf{x}))\sum_{i=1}^{n(\mathbf{x})}\left(\frac{\mu(\mathbf{x}) - x_i}{x_i + x_i}\right)$. The index I_{-1} is, as it happens, closely related to a member of the Atkinson (1970) family of measures, given by: $A_\lambda(\mathbf{x}) = 1 - [(1/n(\mathbf{x}))\sum_{i=1}^{n(\mathbf{x})} x_i^\lambda]^{1/\lambda} / \mu(\mathbf{x}), \lambda < 1, \lambda \neq 0$. When $\lambda = -1$, it is easily verified that I_{-1} is a strictly increasing transform of A_{-1}: specifically, $I_{-1} = [A_{-1} / 2(1 - A_{-1})]$. (It may also be noted, in passing, that the inequality measure I_{-1} is quite similar in formulation to one advanced by Jayaraj and Subramanian 2006, which can be derived as a normalized Canberra distance function, and is given by the expression $I_{Canberra}(\mathbf{x}) = (1/n(\mathbf{x}))\sum_{i=1}^{n(\mathbf{x})}\left(\frac{\mu(\mathbf{x}) - x_i}{\mu(\mathbf{x}) + x_i}\right)$: this latter index, however, is not decomposable.) What is relevant to note is that the decomposition of $I_{-1}(\mathbf{x})$ is defined by: $I_{-1}(\mathbf{x}) = I_{-1B}(\mathbf{x}) + I_{-1W}(\mathbf{x})$, where $I_{-1B}(\mathbf{x}) = (1/2)\left(\sum_{j=1}^{K}\frac{n_j \mu}{n \mu_j} - 1\right)$ and $I_{-1W}(\mathbf{x}) = \sum_{j=1}^{K} w_j I_{-1j}$, with $w_j = \pi_j / \sigma_j, j = 1,...,K$. Since the weight on subgroup inequality is a *declining* function of the subgroup income-share, I_{-1} will satisfy the level-sensitivity requirement.

But what if our distributional values were left-wing rather than right-wing? Observation 3 below addresses this question.

Observation 3. There exists an absolute inequality measure $I \in \boldsymbol{I}^*$ which satisfies both subgroup decomposability and level-sensitivity.

Result 2 enables us to see the truth of Observation 3. The class of indices $I_b(\mathbf{x}) = \frac{1}{n(\mathbf{x})}\sum_{i=1}^{n(\mathbf{x})}[e^{b(x_i - \mu(\mathbf{x}))} - 1], b \in \mathcal{R}, b \neq 0$ is the class of exponential inequality indices, and is ordinally equivalent to the Kolm (1976) class of measures. Chakravarty (2000) has established that a subgroup decomposition of I_b yields a within group component in which the weight on the inequality value for the *j*th subgroup is given by $w_j = \pi_j e^{b(\mu_j - \mu)}$; if the population is partitioned into K non-overlapping income-groups of the same size, then $w_j = (1/K)e^{bh_j}$, where $h_j \equiv \mu_j - \mu, j = 1,...,K$, so that

$\frac{dw_j}{dh_j} = (b / K)e^{bh_j} < 0$ for $b < 0$. That is to say, I_b is level-sensitive whenever b is negative. Thus, the exponential inequality measures, for negative values of the parameter b, are both subgroup decomposable and level-sensitive.

Notice now that since all relative inequality indices are also unit-consistent, we are assured by Observation 2 that there exists a unit-consistent relative inequality measure belonging to the set ***I**** which is also level-sensitive. Unfortunately, we have no such assurance regarding absolute inequality measures from Observation 3, since absolute measures may or may not be unit-consistent. Result 2 confines our attention to those absolute indices which are either exponential indices or the variance. Zheng (2007) points out that the family of exponential indices is *not* unit-consistent. The variance, however, *is* a unit-consistent measure, but Result 4 (Chakravarty 2000) asserts that the only absolute inequality measure in the set ***I**** which is properly decomposable is the variance; and from Observation 1 we know that no properly decomposable index belonging to the set ***I**** is level-sensitive. This leads to the following negative observation:

Observation 4. There exists no absolute unit-consistent inequality measure $I \in \boldsymbol{I}^*$ which is level-sensitive.

Observation 4 is a harsh verdict for those who would value both subgroup decomposability and level-sensitivity but whose distributional judgments favour only left-wing inequality indices. For those who are happy to settle for centrist measures, the present state of knowledge may be inadequate to arrive at a definitive conclusion on the prospects of meeting the requirements of both subgroup decomposability and level-sensitivity, as reflected in the following observation.

Observation 5. Since (to the best of this author's awareness) there is no characterization available of unit-consistent, centrist inequality measures which are subgroup decomposable, it is not known if there exists a unit-consistent and centrist measure which is both decomposable and level-sensitive.

It may be added that the available evidence on this question is not encouraging. Result 4 (Chakravarty and Tyagarupananda 2009) presents a class $\hat{I}_c$ of centrist inequality measures belonging to the set ***I****** which are decomposable, but, as pointed out by Zheng (2007), none of these indices is unit-consistent. Result 5 (Zheng, 2007) presents a class $\tilde{I}_c$ of unit-consistent inequality measures belonging to the set ***I****** which are decomposable. Two classes of centrist measures which are subsets of the $\tilde{I}_c$ class are the following ones (see Zheng 2007):

$$I^*(\mathbf{x}) = \frac{1}{n(\mathbf{x})\mu^d(\mathbf{x})}\sum_{i=1}^{n(\mathbf{x})}[x_i^2 - \mu^2(\mathbf{x})],\ \ 0 < d < 2.$$

$$I^{**}(\mathbf{x}) = \frac{1}{n(\mathbf{x})c(c-1)}\sum_{i=1}^{n(\mathbf{x})}[x_i^c - \mu^c(\mathbf{x})], 0 < c < 2, c \neq 1.$$

I^* is what Zheng (2007) refers to as a generalization of the Krtscha (1994) measure. It can be verified that the subgroup decompositions of the families of indices I^* and I^{**} yield the following outcomes:

$I^*(\mathbf{x}) = I_B^*(\mathbf{x}) + I_W^*(\mathbf{x})$, where $I_B^* = \frac{\sum_{j=1}^{K} n_j \mu_j^2}{n\mu^d} - \mu^{2-d}$ and $I^*_W = \sum_{j=1}^{K} w_j I_j^*$, with $w_j = \pi_j^{1-d}\sigma_j^d$, $j = 1,...,K$; and

$I^{**}(\mathbf{x}) = I_B^{**}(\mathbf{x}) + I_W^{**}(\mathbf{x})$, where $I_B^{**} = \frac{1}{c(c-1)}\left[\frac{1}{n}\sum_{j=1}^{K} n_j \mu_j^c - \mu^c\right]$ and $I_W^{**} = \sum_{j=1}^{K} w_j I_j^{**}$, with $w_j = \pi_j$, $j = 1,...,K$.

An examination of the weights on subgroup inequality levels in the within-group component of inequality suggests that I^* is level-*insensitive*, while I^{**} is level-*neutral*. Briefly, the decomposable and centrist measures proposed by Chakravarty and Tyagarupananda are not unit-consistent, while the decomposable and centrist measures proposed by Zheng are unit-consistent but not level-sensitive. Whether there exist decomposable, centrist, unit-consistent and level-sensitive inequality measures is an open question.

4. Summary and Conclusion

This article has been mainly a quick review of a set of important results on the characterization of decomposable inequality measures, classified into relative, absolute, centrist, and unit-consistent indices, and an examination of the mutual compatibility of the properties of subgroup decomposability and level-sensitivity. For inequality measurement to be coherent, it appears that inequality measures must be unit-consistent. For inequality measurement to be informed by non-extreme distributional values, it also seems to be desirable that inequality measures be centrist. Thus, in the interests of both coherence and normative appeal, there would appear to be a strong case to confine attention to the set of unit-consistent and centrist measures. Decomposability is an extremely convenient property for an inequality index to possess, though it is not clear that this property is imbued with any particularly striking normative values (except in so far as what the philosopher Derek Parfit [1997] has called 'prioritarianism' is compatible with the strong separability underlying additively decomposable inequality indices). Level-sensitivity is a fairly compelling property of an inequality measure, requiring as it does that inequality be regarded as a more severe problem the poorer the population experiencing it is. Level-neutrality is a weaker requirement, demanding only that inequality should be regarded as a problem whose severity does not diminish as a population becomes poorer. In an 'ideal' situation, one may wish to have inequality measures which are centrist, unit-consistent, subgroup decomposable and level-sensitive. Whether such measures exist is still (as far as the present author is aware) an open question. What can, however, be asserted is that there does exist a symmetric, normalized, continuous, differentiable, Schur-concave and replication-invariant measure which is unit-consistent, centrist, subgroup decomposable and level-neutral. This is the index, or rather family of indices (see Zheng 2007), given by $I^{**}(\mathbf{x}) = \frac{1}{n(\mathbf{x})c(c-1)}\sum_{i=1}^{n(\mathbf{x})}[x_i^c - \mu^c(\mathbf{x})]$, $0 < c < 2, c \neq 1$.

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