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Existence of Exact Walrasian Equilibria in Non Convex Economies

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Abstract The existence of an exact Walrasian equilibrium in non convex economies is still a largely unexplored issue. In this paper an existence result for exact equilibrium in non convex economies is provided by following the "almost-near" approach introduced by Postlewaite and Schmeidler for convex economies. More precisely, we show that for any non convex economy there is a set of "perturbed" economies with the same number of agents exhibiting an exact Walrasian equilibrium; moreover as the number of agents tends to infinity the perturbed economies can be chosen as much close as we like to the original one.

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Keywords Exact Walrasian equilibrium; non convex economies; perturbed economies

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1. Introduction

The existence of an *exact* walrasian equilibrium in non-convex economies is still a largely unexplored issue. Mas-Colell (1977) shows that that in the space of differentiable economies there exists an open (in an appropriate topology) and dense set of economies such that if one considers a sequence of finite economies with an increasing number of consumers and with limit in this set then, eventually, an exact walrasian equilibrium exists. Smale (1974) shows the existence of an *extended* equilibrium in a nonconvex differentiable economy. In addition to the differentiability of the economies, Mas Colell's work is constrained by the use of sequences of purely competitive economies, while Smale's work relies upon the use of a nonconventional concept of equilibrium.

Postlewaite and Schmeidler (1981) introduce an "almost-near" approach to deal with existence issues in convex economies: they show that if an allocation of any *convex* economy is "almost" walrasian at price p, then it is possible to construct an economy "near" (in terms of an "average" metric) the original where that allocation is walrasian at the same price p. The motivation of the approach is that "If we don't know the characteristics [of the agents in an economy], but rather, we must estimate them, it is clearly too much to hope that the allocation would be Walrasian with respect to the estimated characteristics even if it were Walrasian with respect to the true characteristics. …… [Thus,] one could not easily pronounce that the procedure generating the allocation was not Walrasian by examining the allocations unless one is certain that there have been no errors in determining the agents' characteristics" (Postlewaite and Schmeidler (1981, pp. 105-106).[1] More recent economic applications of the "almost-near" approach along Postlewaite and Schmeidler's interpretation have been provided by Kubler and Schmedders (2005) and Kubler (2007).[2]

Postlewaite and Schmeidler's result is obtained constructively by perturbing the preferences of agents in the original convex economy in such a way that the indifference surface passing through the bundle of the approximate walrasian equilibrium coincides with the original indifference curves outside the budget set while inside the budget set it is

[1] Anderson (1986) develops the "almost-near" argument within a very general framework and, relying on nonstandard analysis and an appropriate formal language, provides an abstract theorem showing that objects "almost" satisfying a property are "near" an object exactly satisfying that property. He emphasizes also that this approach can be used to obtain existence results and applies his abstract result to show the existence of exact decentralization of core allocations (Anderson (1986, p. 231)).

[2] Kubler and Schmedders also quote Blum, Cucker, Shub, and Smale (1997, Chapter 8) as an example of application of this approach to computation theory.

flattened onto the budget surface, with continuous extensions also to neighboring surfaces. This method, in principle could be extended to show that close to nonconvex economies with near walrasian equilibria there exists an economy with an exact equilibrium. However, their perturbation rule requires that at the exact equilibrium price of the nearby economy the demand set of agents is convex, which is a quite disturbing feature.

In this paper we introduce a rule for perturbing the original nonconvex economy which allows to retain nonconvexity of preferences of the perturbed economy also at the equilibrium price, and we show that for any nonconvex economy there is a *set* of perturbed economies with the same number of agents as the original which exhibit an exact walrasian equilibrium. Moreover, as the number of agents tends to infinity the perturbed economies can be chosen as much close (in terms of an appropriate metric) as we like. The intuition behind our result is very simple: consider a n consumer, k good pure exchange economy satisfying all standard assumptions except convexity of preferences. Since, under our hypotheses, there exists a strictly positive price vector ensuring that the aggregate supply vector belongs the convex hull of the aggregate demand set at that price (see, e.g., Hildenbrand (1975, p. 150)). Hence at this price p it is possible to perturb the economy by shrinking and translating the indifference curves and/or changing the initial endowments perpendicularly to the price vector in such a way that the aggregate supply vector of the perturbed economy belongs to the aggregate demand set, that is, the perturbed economy has an exact equilibrium at price p. In addition, by Shapley and Folkman Theorem, the number of consumers whose endowments and/or preferences have to be perturbed is independent upon the number of consumers (to be precise, is not greater than k+1.) Therefore, as the number of consumers increases the distance between the original economy and the perturbed economy tends to zero in terms of Postlewaite and Schmeidler's metric.

2. Existence of an exact walrasian equilibrium in nonconvex economies

Consider the space $\mathscr{E}_n$ of pure exchange economies $\mathcal{E}_n((u_h), (\omega_h))_{h\in N}$ with n consumers and k goods satisfying the assumptions of strict positivity of the initial endowment vector ω_h and of continuity and strict monotonicity of utility function u_h for each consumer $h \in N = \{1,2,\ldots, n\}$. The consumption set of consumers will always be assumed to be the non-negative orthant of the k-dimensional Euclidean space. Denote by $A_h(\cdot,\omega_h)$, $A^{(n)}(\cdot,(\omega_h))$ and $\omega^{(n)}$, respectively, the demand correspondence of agent h, the aggregate demand correspondence and the aggregate endowment of economy $\mathcal{E}_n$. Symbols co, d and d_H

indicate, respectively, the convex hull operator, the Euclidean distance and the Hausdorff distance. Symbol co$\mathcal{E}_n$ denotes the convexified version of economy $\mathcal{E}_n$; i.e. the economy whose demand correspondence of consumer h is co$A_h(\cdot,\omega_h)$. For any utility function u_h, set $P_{u_h} = \left\{(x,y) \in \Re_+^{2k} \middle| u_h(x) \geq u_h(y)\right\}$. Given a couple of utility functions u_h and $\hat{u}_h$, the distance δ between the preferences underlying these functions is defined as follows (see Debreu (1969)): $\delta(u_h, \hat{u}_h) = d_H(P_{u_h}, P_{\hat{u}_h}) = \inf\left\{\varepsilon \in (0,\infty) \middle| P_{u_h} \subseteq N_\varepsilon(P_{\hat{u}_h}) \text{ and } P_{\hat{u}_h} \subseteq N_\varepsilon(P_{u_h}) \right\}$ where $N_\varepsilon(\cdot)$ is the closed ε-ball around a set. We shall use the same metric m_n used by Postlewaite and Schmeidler (1981): $m_n(\mathcal{E}_n, \hat{\mathcal{E}}_n) = \frac{1}{n}\sum_{h\in N}\left(\delta(u_h,\hat{u}_h) + \frac{\|\omega_h - \hat{\omega}_h\|}{\omega^{(n)} + \hat{\omega}_h^{(n)}}\right)$, where $\mathcal{E}_n\left((u_h),\ (\omega_h)\right)_{h\in N}$ and $\hat{\mathcal{E}}_n\left((\hat{u}_h),\ (\hat{\omega}_h)\right)_{h\in N}$ are economies in $\mathscr{E}_n$. A *walrasian equilibrium* of economy $\mathcal{E}_n$ is a non-negative price vector $p_n{}^*$ and an allocation $(x_{nh}{}^*)_{h\in N}$ such that: $\omega^{(n)} \in A^{(n)}(p_n{}^*,\ (\omega_h))$ and $x_{nh}{}^* \in A_h(p_n{}^*,\omega_h)$ for every $h\in N$. The set of walrasian equilibria in economy $\mathcal{E}_n$ is indicated by $\mathcal{W}(\mathcal{E}_n)$. A walrasian equilibrium of the *convexified* economy co$\mathcal{E}_n$ is defined in an obvious way and the set of these equilibria is indicated by $\mathcal{W}(\text{co}\mathcal{E}_n)$. In the following result it is worth keeping in mind that under our assumptions, $\mathcal{W}(\text{co}\mathcal{E}_n)$ is non-empty for every $n \in \mathbb{N}$ (see Lemma 2 in Section 3.)

Theorem. *Let $\mathcal{E}_n((u_h),\ (\omega_h))_{h\in N} \in \mathscr{E}_n$ be a pure exchange economy satisfying the stated assumptions and let $(p_n{}^*,(x_{nh}{}^*)_{h\in N}) \in \mathcal{W}(\text{co}\mathcal{E}_n)$. Then, there exists a set $\mathfrak{X}_n(p_n{}^*) \subset \mathscr{E}_n$ such* *that if* $\hat{\mathcal{E}}_n((\hat{u}_h), (\hat{\boldsymbol{\omega}}_h))_{h\in N}) \in \mathfrak{X}_n(p_n^*)$, *then* $\mathcal{W}(\hat{\mathcal{E}}_n) \neq \varnothing$; *moreover,* $m_n(\hat{\mathcal{E}}_n, \mathcal{E}_n) \to 0$ as $n \to \infty$. *Furthermore, for every* $n \in \mathbb{N}$ *there exists in* $\mathfrak{X}_n(p_n^*)$ *economies* $\tilde{\mathcal{E}}_n((\tilde{u}_h), (\tilde{\boldsymbol{\omega}}_h))_{h\in N})$ *and* $\mathcal{E}_n^*((u_h^*), (\boldsymbol{\omega}_h^*))_{h\in N}$ *with* $\tilde{u}_h = u_h$ *and* $\boldsymbol{\omega}_h^* = \boldsymbol{\omega}_h$ *for every* $h \in N$ *and* $(p_n^*, (x_{nh}^*)_{h\in N}) \in \mathcal{W}(\mathcal{E}_n^*)$.

The last part of the previous result means that the walrasian equilibrium of $\mathrm{co}\mathcal{E}_n$ (which always exists under our assumptions) is the walrasian equilibrium of an appropriately non-convex economy $\mathcal{E}_n^*$ obtained by perturbing only the preferences of the original economy $\mathcal{E}_n$, while the perturbed non convex economy $\tilde{\mathcal{E}}_n$ exhibiting an exact equilibrium differs from the original only by the endowments. In addition, economies $\mathcal{E}_n^*$ and $\tilde{\mathcal{E}}_n$ become as close to $\mathcal{E}_n$ as we like when the number of consumers is "big enough". Following Postlewaite and Schmeidler's interpretation, a consequence of the previous result is that a walrasian equilibrium of the convexified version of any large nonconvex economy should be interpreted as an exact walrasian equilibrium of a nonconvex economy "close" to the original one obtained by perturbing the preferences.

3. Proofs

The next two results are well-known.

Lemma 1. (see, e.g., Balasko (1988, p. 77)) *Let* $p \in \Re^k_+$ *be a price vector. Then,* $A_h(p, \omega_h) = A_h(p, \hat{\omega}_h)$ *for every* $\hat{\omega}_h \in B_h(p, \omega_h) = \left\{x \in \Re^k_+ \middle| p \cdot x = p \cdot \omega_h\right\}$.

Lemma 2. (see, e.g., Hildenbrand (1974, p. 150)) *For every n,* $\mathcal{W}\left(\mathrm{co}\mathcal{E}_n\right) \neq \emptyset$. *Moreover, if* $(p_n^*, (x_{nh}^*)_{h\in N}) \in \mathcal{W}\left(\mathrm{co}\mathcal{E}_n\right)$, *then* $p^* \in \Re^k_{++}$.

From now on p_n^* is an equilibrium price vector associated to economy $\mathrm{co}\mathcal{E}_n((u_h), (\omega_h))_{h\in N}$. By Lemma 2 the budget surface $B_h(p_n^*,\omega_h)$ of consumer h is compact. By Urysohn's Lemma (see, e.g. Willard (1970, p. 102)), given a real number $\varepsilon > 0$ and for every $h \in N$ there exists a continuous function $\gamma_{h\varepsilon} : \Re^k_+ \to \left[0,1\right]$ (which depends also on p_n^* and ω_h) such that $\gamma_{h\varepsilon}(x) = 1$ if $x \in B_h(p_n^*,\omega_n)$ and $\gamma_{h\varepsilon}(x) = 0$ if $x \in \Re^k_+ \setminus S_{h\varepsilon}(B_h(p_n^*, \omega_h))$ where $S_{h\varepsilon}(B_h(p_n^*, \omega_h)) = \left\{y \in \Re^k_+ \middle| p_n^* \cdot \omega_h - \varepsilon < p_n^* \cdot y < p_n^* \cdot \omega_h + \varepsilon)\right\}$ is the open ε-"slice" containing set $B_h(p_n^*,\omega_h)$ (see Figure 1, where segment B is the budget line and the shaded area is the ε-slice containing it.)

Given two vectors $x_h, y_h \in \Re^k_+ \setminus \left\{0\right\}$ such that $p_n^* \cdot (x_h - y_h) = 0$, let $t_h(\cdot\,; \varepsilon, p_n^*, x_h, y_h)$ be a mapping defined as follows: $t_h(x; \varepsilon, p_n^*, x_h, y_h) = x + \min_{j=1,2,\ldots,k}\left(\frac{x_j}{y_{hj}}\right)\gamma_{h\varepsilon}(x)(x_h - y_h)$, where function $\gamma_{h\varepsilon}$ has been defined previously. Intuitively, transformation $t_h(\cdot\,; \varepsilon, p_n^*, x_h, y_h)$

translates any point x in $\Re^l_+$ by the vector $\min_{j=1,2,\ldots,k}\left(\frac{x_j}{y_{hj}}\right)\gamma_{h\varepsilon}(x)(x_h - y_h)$ perpendicular to $p_n{}^*$.

In Figure 1 the curved arrows describe the effects of transformation t_h on points on the budget line: for example, point y_h is mapped into point x_h.

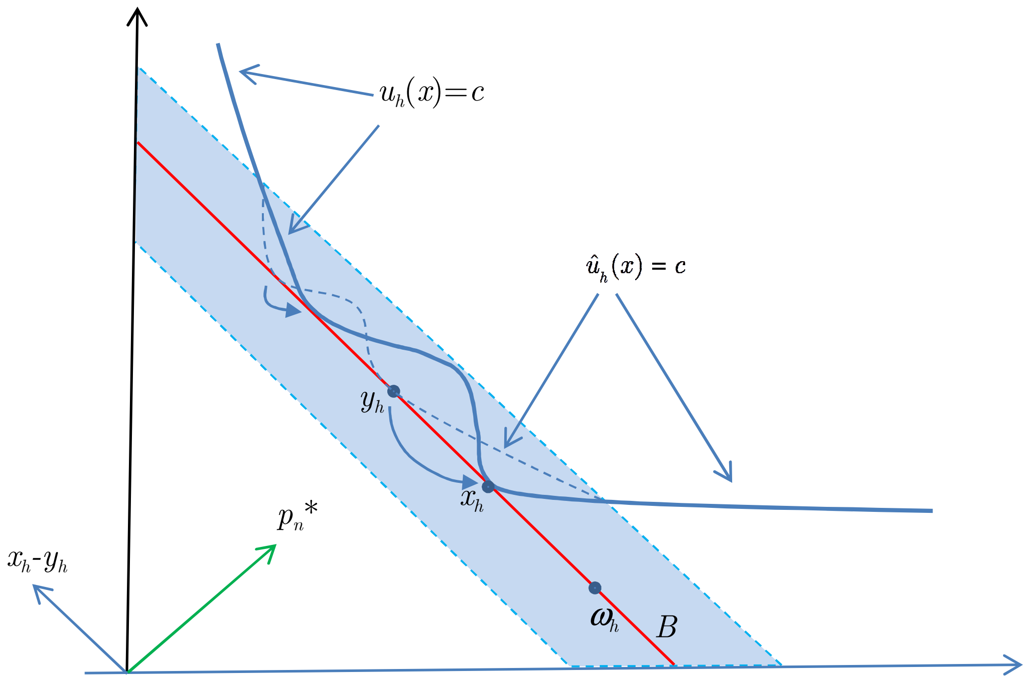

Figure 1. The continuous segment is the budget line $B_h(p_n{}^*, \omega_h)$. The shaded area is the ε-slice $S_{h\varepsilon}(B_h(p_n{}^*, \omega_h))$ cotaining the budget line. Under transformation t_h point y_h is mapped into point x_h.

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In the following result, notice that if $x_h, y_h \in B_h(p_n{}^*, \omega_h)$, then $p_n{}^* \cdot (x_h - y_h) = 0$:

Lemma 3. *Given* $\varepsilon > 0$ *and* $p_n{}^* \in \Re^k_{++}$, *for every* $x_h, y_h \in B_h(p_n{}^*, \omega_h)$, *map* $t_h(\cdot\,; \varepsilon, p_n{}^*, x_h, y_h)$ *satisfies the following properties*:

(i) $t_h(\cdot;\, \varepsilon,\, p_n{}^*,\, x_h,\, y_h)$ *maps* $\Re^k_+$ *into itself and is continuous*;

(ii) $p_n * \cdot t_h(x;\varepsilon, p_n *, x_h, y_h) = p_n * \cdot x$ *for every* $x \in \Re^k_+$;

(iii) $t_h(y_h;\ \varepsilon,\ p_n*,\ x_h,\ y_h) = x_h$;

(iv) *for every* $x \in S_{h\varepsilon}(B_h(p_n*,\omega_h))$ *there exists* $\lambda > 1$ *such that* $\lambda x \in S_{h\varepsilon}(B_h(p_n*,\omega_h))$ *and* $t_h(\lambda x;\ \varepsilon,\ p_n*,\ x_h,\ y_h) > x$;

(v) $t_h(x;\ \varepsilon,\ p_n*,\ x_h,\ y_h) = x$ for every $x \in \Re^k_+ \setminus S_{h\varepsilon}(B_h(p_n*,\omega_h))$.

Proof. (i) Continuity is obvious. Take any $x \in \Re^k_+$, then, for $i = 1,2,\ldots, k$,

$$t_{hi}(x;\varepsilon, p_n *, x_h, y_h) = x_i + \min_{j=1,\ldots,k}\left(\frac{x_j}{y_{hj}}\right)\gamma_{h\varepsilon}(x)(x_{hi} - y_{hi}) \geq x_i + \min_{j=1,\ldots,k}\left(\frac{x_j}{y_{hj}}\right)\gamma_{h\varepsilon}(x)x_{hi} - \left(\frac{x_i}{y_{hi}}\right)y_{hi} = \min_{j=1,\ldots,k}\left(\frac{x_j}{y_{hj}}\right)\gamma_{h\varepsilon}(x)x_{hi} \geq 0.$$

Assertions (ii) and (iii) can immediately be verified by substitution. As for (iv), set: $\lambda^* = \sup\left\{\lambda \in \left[1,\infty\right) \middle| \lambda x \in S_{h\varepsilon}(B_h(p_n *, \omega_h))\right\}$. Clearly, $\lambda^* > 1$. We have: $t_{hi}(\lambda * x;\varepsilon, p_n *, x_h, y_h) = \lambda * x_i > x_i$, $i = 1,2,\ldots,k$. By continuity, the assertion follows. Fact (v) follows from the stated properties of function $\gamma_{h\varepsilon}$.

Lemma 4. *Given* $\varepsilon > 0$ *and for every* $p_n * \in \Re^k_{++}$ *and* $x_h, y_h \in B_h(p_n *, \omega_h)$ *there exists a positive number* K_h *such that if the utility functions* u_h *and* $\hat{u}_h$ *satisfy* $\hat{u}_h(x) = u_h(t_h(x;\varepsilon, p_n *, x_h, y_h))$, *then* $\delta(u_h, \hat{u}_h) \leq K_h$.

Figure 1 illustrates the effects of transformation t_h on the indifference curve $u_h(x) = c$ in case $k = 2$: The dotted curve is the part of the indifference curve of $\hat{u}_h$ in the ε- slice containing the budget line. For any other point outside this set, the indifference curve of $\hat{u}_h$ coincides with the indifference curve of u_h.

Proof of Lemma 4. By Lemma 3(v), $\hat{u}_h(x) = u_h(x)$ for $x \in \Re^k_+ \setminus S_{h\varepsilon}(B_h(p_n{}^*, \omega_h))$. So, preferences of consumer h differ only inside set $S_{h\varepsilon}(B_h(p_n{}^*, \omega_h))$. Given the strict positivity of $p_n{}^*$, this set is compact. Hence, also the Cartesian product $S^2_{h\varepsilon} = S_{h\varepsilon}(B_h(p_n{}^*, \omega_h)) \times S_{h\varepsilon}(B_h(p_n{}^*, \omega_h))$ is compact. It follows that there exists a positive real number K_h such that its diameter $r(S^2) = \sup\left\{\alpha \in \Re_+ \middle| \alpha = d((x,y),(x',y')),\, (x,y),(x',y') \in S^2_{h\varepsilon}\right\}$ is less than K_h.

Take $(x,y) \in P_{u_h}$ with $x, y \in S^2_{h\varepsilon}$ and suppose that $(x,y) \notin P_{\hat{u}_h}$ (otherwise there is nothing to prove), i.e. $\hat{u}_h(x) \le \hat{u}_h(y)$ or $u_h(t_h(x)) \le u_h(t_h(y))$. Since $x \in S_{h\varepsilon}(B_h(p_n{}^*, \omega_h))$, then by Lemma 3(ii), $t_h(x) \in S_{h\varepsilon}(B_h(p_n{}^*, \omega_h))$. By Lemma 3(iv) there exists $\lambda > 1$ such that $\lambda t_h(y) \in S_{h\varepsilon}(B_h(p_n{}^*, \omega_h))$ and $t_h(\lambda t_h(y)) > t_h(y)$. Set $x' = \lambda t_h(y)$ and $y' = y$. By monotonicity, it follows that $u_h(t_h(\lambda t_h(y)) = \hat{u}_h(x') > u_h(t_h(y)) = \hat{u}_h(y')$, that is $(x',y') \in P_{\hat{u}_h}$ with $(x',y') \in S^2_{h\varepsilon}$. Therefore, $d((x,y),(x',y')) \le K_h$.

Suppose now that $(x,y) \in P_{\hat{u}_h}$ with $(x,y) \in S^2_{h\varepsilon}$, and, again, $(x,y) \notin P_{u_h}$. Then, $u_h(x) \le u_h(y)$. Take $x' = \lambda y$ and $y' = y$ where $\lambda > 1$ is such that $\lambda y \in S_{h\varepsilon}(B_h(p_n{}^*, \omega_h))$. Hence, $u_h(x') \ge$

$u_h(y')$, that is, there exists a point $(x',y') \in P_{u_h}$ with $(x',y') \in S^2_{h\varepsilon}$. Therefore, $d((x,y),(x',y')) \le K_h$. It follows that $\delta(u_h, \hat{u}_h) \le K_h$.

Proof of Theorem. By definition, $p_n{*} \in \Re^k_{++}$ satisfies the condition: $\omega^{(n)} \in \mathrm{co}A^{(n)}(p_n{*},(\omega_h))$. That is, there are t $(1 \le t \le k+1)$ real numbers $\alpha_{ni} > 0$ such that $\sum_{i=1}^{t} \alpha_{ni} = 1$ and $\omega^{(n)} = \sum_{i=1}^{t} \alpha_{ni} x_i^{(n)} = \sum_{i=1}^{t} \alpha_{ni} \sum_{h=1}^{n} x_{nhi}$ where $x_i^{(n)} \in A^{(n)}(p_n{*},(\omega_h))$ and $x_{nhi} \in A_h(p_n{*},\omega_h)$ for every i = 1,2., …, t and every h = 1, …, n. Therefore, $\omega^{(n)} = \sum_{h=1}^{n} y_{nh}$, where $y_{nh} = \sum_{i=1}^{t} \alpha_{ni} x_{nhi} \in \mathrm{co}A_h(p_n{*},\omega_h)$.

Denote by $N(p_n{*})$ the set of allocations which are feasible in terms of vectors in set $\mathrm{co}A^{(n)}(p_n{*},(\omega_h))$ and which maintains constant consumers' income with respect to price $p_n{*}$ and to the initial allocation $(\omega_h)_{h\in N}$ i.e. $N\left(p_n{*}\right) = \left\{\left(x_h\right)_{h\in N} \in \Re^{k\times n}_{+} \middle| \sum_{h\in N} x_h \in \mathrm{co}A^{(n)}(p_n{*},(\omega_h)), p_n{*} \cdot x_h = p_n{*} \cdot \omega_h, h \in N\right\}$. By Lemma 1, any allocation in $N(p_n{*})$ maintains unchanged the demand set of agents at price $p_n{*}$, that is $A_h(p_n{*},\hat{\omega}_h) = A_h(p_n{*},\omega_h)$ for every $h\in H$ - and, therefore, $A^{(n)}(p_n{*},(\omega_h)) = A^{(n)}(p_n{*},(\hat{\omega}_h))$ - whenever $(\hat{\omega}_h)_{h\in N} \in N(p_n{*})$.

Take any $(\hat{\omega}_h)_{h\in N} \in N\left(p_n{}^*\right)$. Since $\hat{\omega}^{(n)} \in \mathrm{co}A^{(n)}(p_n{}^*,(\omega_h))$, hence, $\hat{\omega}^{(n)} \in \mathrm{co}A^{(n)}(p_n{}^*,(\hat{\omega}_h))$.

Then $\hat{\omega}^{(n)} = \sum_{i=1}^{\hat{t}} \hat{\alpha}_i^n \hat{x}_i^{(n)} = \sum_{i=1}^{\hat{t}} \hat{\alpha}_{ni} \sum_{h=1}^{n} \hat{x}_{nhi}$ with $0 \le \hat{\alpha}_i^n \le 1$, $\sum_{i=1}^{\hat{t}} \hat{\alpha}_i^n = 1$ and $1 \le \hat{t} \le k+1$, $\hat{x}_i^{(n)} = \sum_{h\in N} \hat{x}_{nhi} \in A^{(n)}(p_n{}^*,\ (\hat{\omega}_h))$, and $\hat{x}_{nhi} \in A_h(p_n{}^*,\ \hat{\omega}_h)$ for every $i = 1,2,\ldots,\ \hat{t}$ and every $h = 1,2,\ldots,\ n$. Therefore, $\hat{\omega}^{(n)} = \sum_{h\in N} \hat{y}_{nh}$ where $\hat{y}_{nh} = \sum_{i=1}^{\hat{t}} \hat{\alpha}_i^n \hat{x}_{nhi} \in \mathrm{co}A_h(p_n{}^*,\omega_h) = \mathrm{co}A_h(p_n{}^*,\ \hat{\omega}_h)$. By Shapley-Folkman Theorem there exists a subset $\hat{J}_n \subset N$ with $\#\hat{J}_n \le k+1$, such that $\hat{\omega}^{(n)} = \sum_{h'\in N\setminus\hat{J}_n} \hat{y}_{nh'} + \sum_{h''\in\hat{J}_n} \hat{y}_{nh''}$ where $\hat{y}_{nh'} \in A_{h'}(p_n{}^*,\hat{\omega}_{h'})$ and $\hat{y}_{nh''} \in \mathrm{co}A_{h''}(p_n{}^*,\hat{\omega}_{h''})$. Let $\left\{\hat{x}_{nh}\right\}_{h\in N}$ be a family of vectors defined as follows: $\hat{x}_{nh'} = \hat{y}_{nh'}$ for $h' \in N\setminus\hat{J}_n$ and $\hat{x}_{nh''} \in \left\{x \in A_{h''}(p_n{}^*,\hat{\omega}_{h''}) \middle| d(x,\hat{y}_{nh''}) \le d(z,\hat{y}_{nh''}),\ z \in A_{h''}(p_n{}^*,\hat{\omega}_{h''})\right\}$ for $h'' \in \hat{J}_n$.

Consider now the set $\mathfrak{X}_n(p_n{}^*)$ of perturbed economies $\hat{\mathcal{E}}_n((\hat{u}_h),(\hat{\omega}_h))_{h\in N}$ defined as follows: $(\hat{\omega}_h)_{h\in N} \in N(p_n{}^*)$ and $\hat{u}_h(x) = u_h(t_h(x))$ where $t_h(x) = x + \min\left(\dfrac{x_j}{\hat{y}_{nhj}}\right)\gamma_{h\varepsilon}(x)(\hat{x}_{nh} - \hat{y}_{nh})$ for $h \in N$ (in what follows, for the sake of simplicity, we drop parameters ε, $\hat{x}_{nh}$ and $\hat{y}_{nh}$ in t_h.) Clearly, $p_n{}^* \cdot (\hat{x}_{nh} - \hat{y}_{nh}) = 0$ for $h \in N$ and, moreover, $\hat{u}_{h'} = u_{h'}$ for $h' \in N\setminus\hat{J}_n$.

We show that $(p_n{}^*,(\hat{y}_{nh})_{h\in N}) \in \mathcal{W}\left(\hat{\mathcal{E}}_n\right)$ for every $\hat{\mathcal{E}}_n((\hat{u}_h),(\hat{\omega}_h))_{h\in N}$ in $\mathfrak{X}_n(p_n{}^*)$. First, by construction $\hat{\omega}^{(n)} = \sum_{h\in N} \hat{\omega}_h = \sum_{h\in N} \hat{y}_{nh}$, so allocation $(\hat{y}_{nh})_{h\in N}$ is feasible. That

$\hat{y}_{nh} \in B_h(p_n*, \hat{\omega}_h)$ for every h = 1,2, …, n, follows by construction. Vector $\hat{y}_{nh'}$ is optimal for agent $h' \in N \setminus \hat{J}_n$ because $\hat{y}_{nh'} \in A_{h'}(p_n*, \hat{\omega}_{h'})$. We now show that for every $h'' \in \hat{J}_n$ $\hat{u}_{h''}(\hat{y}_{nh''}) \geq \hat{u}_{h''}(x)$ for every $x \in B_{h''}(p_n*, \hat{\omega}_{h''})$. To this end, notice that, by Lemma 3(ii), transformation t_h at price p_n* maps the budget hyperplane into itself. So, by monotonicity, we can focus only on points on the latter. Thus, suppose that there exists $\tilde{x} \in B_{h''}(p_n*, \hat{\omega}_{h''})$ such that $\hat{u}_{h''}(\tilde{x}) > \hat{u}_{h''}(\hat{y}_{nh''})$. Hence $u_{h''}(t_{h''}(\tilde{x})) > u_{h''}(t_{h''}(\hat{y}_{nh''}))$. By Lemma 3(iii), this implies that $u_{h''}(t_{h''}(\tilde{x})) > u_{h''}(\hat{x}_{nh''})$. Since $t_{h''}(\tilde{x}) \in B_{h''}(p_n*, \hat{\omega}_{h''})$, this contradicts the fact that $\hat{x}_{nh''} \in A_{h''}(p_n*, \hat{\omega}_{h''})$ for $h'' \in \hat{J}_n$.

We show that $m_n(\mathcal{E}_n, \hat{\mathcal{E}}_n) \to 0$ as $n \to \infty$. First, as already noticed, in the perturbed economy $\hat{\mathcal{E}}_n$, $\hat{u}_{h'}(x) = u_{h'}(x)$ for $h' \in N \setminus \hat{J}_n$. Hence, $m_n(\mathcal{E}_n, \hat{\mathcal{E}}_n) =$ $\frac{1}{n}\sum_{h''\in\hat{J}_n} \delta(u_{h''}, \hat{u}_{h''}) + \frac{1}{n}\sum_{h\in N}\left(\frac{\|\omega_h - \hat{\omega}_h\|}{\omega^{(n)} + \hat{\omega}^{(n)}}\right)$. Therefore, by Lemma 4,

$$m_n(\mathcal{E}_n, \hat{\mathcal{E}}_n) \leq \frac{k+1}{n}K + \frac{1}{n}\sum_{h\in N}\frac{\|\omega_h\| + \|\hat{\omega}_h\|}{\omega^{(n)} + \hat{\omega}^{(n)}} = \frac{1}{n}((k+1)K + 1).$$ Thus, $m_n\left(\hat{\mathcal{E}}_n, \mathcal{E}_n\right) \to 0$ as $n \to \infty$.

As far as the last part of the assertion is concerned, choose $(\omega_h*)_{h\in N} = (\omega_h)_{h\in N} \in N\left(p_n*\right)$. By the same argument at the beginning of this proof, there are t $(1 \leq t \leq k+1)$ numbers $\alpha_{ni} > 0$ satisfying the condition $\sum_{i=1}^{t}\alpha_{ni} = 1$ such that $\omega^{(n)} = \sum_{i=1}^{t}\alpha_{ni}x_i^{(n)} = \sum_{i=1}^{t}\alpha_{ni}\sum_{h=1}^{n}x_{nhi}$ where $x_i^{(n)} \in A^{(n)}(p_n*, (\omega_h))$ and $x_{nhi} \in A_h(p_n*, \omega_h)$ for every i = 1,2., …, t and every h = 1,

…, n. Therefore, $\omega^{(n)} = \sum_{h=1}^{n} y_{nh}$, where $y_{nh} = \sum_{i=1}^{t} \alpha_{ni} x_{nhi} \in \mathrm{co}A_h(p_n{}^*, \omega_h)$. Thus, $(p_n{}^*, (y_{nh})_{h\in N}) \in \mathcal{W}\left(\mathrm{co}\mathcal{E}_n\right)$. By Shapley-Folkman Theorem there exists a subset $J_n \subset N$ with $\# J_n \leq k+1$, such that $\omega^{(n)} = \sum_{h' \in N\setminus J_n} y_{nh'} + \sum_{h'' \in J_n} y_{nh''}$ where $y_{nh'} \in A_{h'}(p_n{}^*, \omega_{h'})$ and $y_{nh''} \in \mathrm{co}A_{h''}(p_n{}^*, \omega_{h''})$. Let $\left\{x_{nh}\right\}_{h\in N}$ be a family of vectors defined as follows: $x_{nh'} = y_{nh'}$ for $h' \in N \setminus J_n$ and $x_{nh''} \in \left\{x \in A_{h''}(p_n{}^*, \omega_{h''}) \middle| d(x, y_{nh''}) \leq d(z, y_{nh''}), z \in A_{h''}(p_n{}^*, \omega_{h''})\right\}$ for $h'' \in J_n$. Consider now the perturbed economy $\mathcal{E}_n^*((u_h^*), (\omega_h))_{h\in N}$ obtained from the original by changing only utility functions as follows: $u_h^*(x) = u_h(t_h(x))$ where $t_h(x) = x + \min\left(\dfrac{x_j}{y_{nhj}}\right)\gamma_{h\varepsilon}(x)(x_{nh} - y_{nh})$ for $h \in N$.

By a similar argument used before, it is possible to show that $(p_n{}^*, (y_{nh})_{h\in N}) \in \mathcal{W}\left(\mathcal{E}_n^*((u_n^*), (\omega_h))_{h\in N}\right)$. However, we already noticed that $(p_n{}^*, (y_h)_{h\in N}) \in \mathcal{W}\left(\mathrm{co}\mathcal{E}_n\right)$. Finally, since $\omega^{(n)} \in \mathrm{co}A^{(n)}(p_n{}^*, (\omega_h))$, choose $\tilde{\omega}^{(n)} \in \mathrm{Exco}A^{(n)}(p_n{}^*, (\omega_h))$ and $(\tilde{\omega}_h)_{h\in N} \in N(p_n{}^*)$ such that $\sum_{h\in N} \tilde{\omega}_h = \tilde{\omega}^{(n)}$, where Ex indicates the set of extreme points. Since, by standard results on convex hulls (see, for example Lay (1992, Chapter 2)) and by Lemma 1, $\mathrm{Exco}A^{(n)}(p_n{}^*, (\omega_h)) = \mathrm{Ex}A^{(n)}(p_n{}^*, (\omega_h)) = \mathrm{Ex}A^{(n)}(p_n{}^*, (\tilde{\omega}_h))$, one obtains that

$\tilde{\boldsymbol{\omega}}^{(n)} \in \mathrm{Ex}A^{(n)}(p_n{}^*,(\tilde{\boldsymbol{\omega}}_h))$. Therefore, $\tilde{\boldsymbol{\omega}}^{(n)} = \sum_{h\in N} \tilde{y}_{nh}$ where $\tilde{y}_{nh} \in \mathrm{Ex}A_h(p_n{}^*,\tilde{\boldsymbol{\omega}}_h)$ (see Price (1940)). Thus, $\tilde{y}_{nh} \in A_h(p_n{}^*,\tilde{\boldsymbol{\omega}}_h)$ for every $h \in N$. So, $(p_n{}^*,(\tilde{y}_{nh})_{h\in N}) \in \mathcal{W}(\tilde{\mathcal{E}}_n((u_n),(\tilde{\boldsymbol{\omega}}_h))_{h\in N})$.

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