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Measuring Group Disadvantage with Inter-distributional Inequality Indices: A Critical Review and Some Amendments to Existing Indices

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Abstract A long literature on inter-distributional inequality (IDI) has developed statistical tools for measuring the extent of inequality between two groups (e.g. men versus women). Firstly, I introduce the property of group-specific disadvantage focus (GDF). Indices satisfying this property are only sensitive to inequalities that are disadvantageous to one specific group. Then the paper reviews some of the most prominents IDI indices proposed in the last four decades. The assessment focuses on whether these indices satisfy GDF and, if not, how they react to inequalities that are disadvantageous to different groups. I also discuss whether these indices are informative, or not, regarding other interesting features related to IDI comparisons, e.g. distributional equality, absence of distributional overlap and presence of first-order stochastic dominance. Finally, I propose amendments to several of these indices in order to render them in fulfillment of GDF and more informative on the mentioned distributional features.

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Introduction

The concern for differences in the distribution of wellbeing characteristics among groups within societies has earned a long-standing interest in the Social Sciences and Political Philosophy. This concern has often emphasized the potential presence of socio-economic discrimination of different natures (e.g. Becker, 1971; Phelps, 1972; Arrow, 1973). In general, it has been associated with concepts of inequality of opportunities.[1] The normative view for between-groups differences related to ethnicity or gender states that they are intrinsically unfair (particularly when the groups are defined over characteristics beyond their members' control), and instrumentally detrimental to individuals and societies (e.g. Arneson, 1989; Cohen, 1989; Nussbaum and Glover, 1995; Roemer, 1998; Fleurbaey, 2001; Sen, 2001).

From a quantitative perspective, one way of measuring the extent of differences in wellbeing between groups is to use indices that capture between-group inequalities and that declare the total absence of between-group inequality if and only if the conditional distributions of wellbeing are identical across groups.[2] There is also an interest in quantifying between-group inequalities with a focus on capturing inequality if and when it is (more) detrimental to one specific group as opposed to other(s), i.e. a concept of relative economic disadvantage. Even though several authors have focused on inequalities detrimental to one group,[3] only recently formal definitions of the concept have been put forward, with a concern for censoring inequalities when they are not detrimental to the group of concern. The most recent and neat definition by del Rio et al. (2011), based on the work of Jenkins (1994), applies to comparisons of actual distributions against counterfactuals. This approach effectively deals with distributions of the same population size.

In this paper I first propose a property of a(n index's) sensitivity to inequality that is detrimental exclusively to one specific group and that is applicable to indices of inter-distributional inequality that deal with populations of *different* size.

[1] For a good review of the literature on inequality of opportunity see Fleurbaey (2008). Also Roemer (1998).

[2] This condition is consistent with a literalist definition of inequality of opportunity by Roemer (1998, p. 15-6) as well as with Van De Gaer's rule (Ooghe et al., 2007). It is also consistent with Fleurbaey's concept of circumstance neutralization (Fleurbaey, 2008, p. 25). There are alternative ways of measuring between-group inequality. For instance, it could be measured as the residual inequality after within-group inequality has been suppressed (e.g. by replacing individual's wellbeing values with those of their group mean). Such approach has been followed, among others, by Roemer (2006); Elbers et al. (2008); **?**); Lanjouw and Rao (2008).

[3] This literature is abundant. Some important examples are Gastwirth (1975), Butler and McDonald (1987), Dagum (1987), Jenkins (1994), van Krem (2009), Gradin et al. (2010) and del Rio et al. (2011).

I call this property Group-Specific Disadvantage Focus (GDF). An advantage of this definition is that it can be related to indices that measure inequality on quantile space, or on probability space.

Secondly, I explore how we can measure inequalities with metrics satisfying (GDF), i.e. with an exclusive focus on one specific group's disadvantage. Since there are several indices of inter-distributional inequality (IDI) already available, I propose some ways of measuring this focused inequality by suggesting some amendments to existing indices which do not measure IDI with a focus on specific disadvantages in their current forms.

Thirdly, I take the opportunity to extend this review of existing indices in order to evalute whether these indices are informative, or not, regarding other interesting features related to IDI comparisons. For instance, I assess whether these indices are able to pinpoint situations in which two distributions are identical. Remarkably most of them are not. I also assess whether they are informative as to the absence of distributional overlap and/or presence of first-order stochastic dominance. I propose some further amendments that improve the indices' informative content on these features.

Since there are several indices of distributional change, or IDI, in this paper I focus on indices that are characterized by: i) being useful especifically for two-group comparisons, ii) being more informative than just comparing two means, and iii) being useful when the two distributions have different sample sizes.[4] I first review the PROB index by Gastwirth (1975), followed by the closely related indices of relative distributions by Le Breton et al. (2008). I show that these indices do not fulfill GDF because in some cases they compensate inequalities detrimental to one group with inequalities detrimental to the other group, while in other cases they just add up the two forms of inequalities together. In relation to that, most of these indices do not distinguish situations of equal distributions from other situations wherein there is inequality among distributions. Finally, while several of these indices are helpful to pinpoint situations of lack of distributional overlap, they are not informative to the presence of first-order stochastic dominance. I propose some simple amendments to these indices that render them more informative about the abovementioned features; chiefly, the extent of group-specific disadvantage.

I then review the family of percentile-based indices of Ebert (1984) and Vinod (1985). I show that, again, these indices do not fulfill GDF either because they

[4] When sample sizes are identical the literature on counterfactual comparisons, e.g. del Rio et al. (2011), provides the relevant indices. However, even without the explicit purpose, mobility indices may also be amendable to render them suitable for the analysis of between-group inequalities with GDF and identical populations. Good examples of such indices are provided by Cowell's measures of distributional change (Cowell, 1985), by Fields and Ok (1996, 1999) and by Schluter and van de Gaer (2011).

compensate group-specific detrimental inequalities or because they add them up indiscriminately. As for other features, while Ebert's index does differentiate between distributional equality and other situations, Vinod's does not. Neither index is helpful to detect absolute lack of distributional overlap. I propose simple amendments to these indices that render them both in fulfillment of GDF and more informative in terms of lack of distributional overlap.

The I turn to an assessment of the $P(1,1)$ measure of Butler and McDonald (1987) and the measures of Dagum (1980, 1987). Notwithstanding these indices' merits and usefulness in other situations, I show that they do not fulfill GDF and that they do not distinguish a situation of distributional equality from other cases of inequality. Finally, I complete the review with an appraisal of the family of ethical distance functions proposed by Shorrocks (1982) and Chakravarty and Dutta (1987). Ethical distance indices are different from the previous ones in that they compare equally-distributed-equivalent (EDE) standards from the distributions.[5] This requires a first aggregation step in which each distribution is mapped into its respective EDE standard. Then two such standards are compared. Despite this difference, I include these indices in the review because they have been proposed as alternatives to, and contrasted with, some IDI indices (see Shorrocks (1982)). I show that, notwithstanding their merit and appeal, this family of indices does not fulfill GDF. The indices are also of little help for pinpointing situations of distributional equality, first-order stochastic dominance and/or absolute absence of distributional overlap.

The next section defines the property of group-specific disadvantage focus. Then the review and proposal of new amendments is done in subsequent sections: one for the PROB measure and indices based on relative distributions; followed by a section on the percentile indices of Ebert and Vinod; then followed by a section on the $P(1,1)$ measure of Butler and McDonald, a section on the REA measures of Dagum and a section on ethical distance indices. Finally the paper ends with some concluding remarks.

A focus on group-specific disadvantage

Two distributions may be different in many ways. For instance, they may have different means. Or even if they have equal means, they may differ in their variance, skewness or kurtosis. More importantly, from a wellbeing perspective, these inter-distributional differences may render one distribution more desirable than the other one as a "lottery". The stochastic dominance literature discusses this type of partial-ordering comparisons. But even when stochastic dominance rela-

[5] EDE standards were introduced by Atkinson (1970).

tionships do not hold over the whole admissible range of a wellbeing variable, one may be able to make statements about whether certain parts of a distribution are more advantageous for one group vis-a-vis another one. For instance, consider income distributions A and B. Both are symmetric and have equal means, but people in A are closely clustered around the mean, whereas people in B exhibits significantly higher variance. In that case, one may find that the poorest people in B are poorer than the poorest people in A whereas the richest people in B are richer than the richest people in A. In such situations, one may be interested in measuring only the amount of inequality that is detrimental to, say, A. If that is the purpose then one may want to have an index that is sensitive to the fact that the richest people in A are poorer than the richest people in B, while being *insensitive* to the fact that the poorest people in A are better-off than the poorest people in B.

The purpose of such a focused approach is served by comparing the percentiles of the two groups, i.e. people who are in the same relative wellbeing position within their own group (i.e. the same position in the Pen's Parade). [6] Let $y(p)$, $p \in [0,1]$, be the p percentile of distribution Y. Then I propose the following definition of Group-specific Disadvantage Focus for an index that is meant to capture only inequalities that are detrimental to a distribution X when compared to a distribution Y:

Definition 1 *An index measuring inter-distributional inequality between Y and X satisfies the property of group-specific disadvantage focus (GDF) if and only if it is sensitive to the gap* $y(p) - x(p) \forall p \in [0,1] \mid y(p) \geq x(p)$ *and it is insensitive to the gap* $y(p) - x(p) \forall p \in [0,1] \mid y(p) \leq x(p)$*. In particular, the index does not decrease (increase) if the gap* $y(p) - x(p)$ *increases (decreases) given that initially* $y(p) \geq x(p)$ *and the index does not react to changes in* $y(p) - x(p)$ *as long as* $y(p) \leq x(p)$ *before and after the changes.*

The sensitivity part of Definition 1 is similar to the monotonicity axiom of del Rio et al. (2011) for counterfactual comparisons, while the insensitivity part is similar to their focus axiom. Now Definition 1 can be expressed also in terms of cumulative probabilities. This dual expression is useful for applications based on ordinal variables. It stems from the fact that, if it is true that $y(p) \geq x(p)$ over the

[6] A similar approach was advocated in the inequality-of-opportunity literature by Roemer (1998). He proposed that in order to measure inequality of opportunity between different groups of people (defined in terms of their specific sets of life circumstances), people in a given percentile within their own group should be compared against people from the same percentile in a different group. The percentile is used as a measure of relative effort within the group, under certain assumptions.

interval $p \in [\underline{p}, \overline{p}]$, and is also the case that $y(\underline{p}) = x(\underline{p})$ and $y(\overline{p}) = x(\overline{p})$, then the following equation holds:

$$\int_{x(\underline{p})}^{x(\overline{p})} [F_X(z) - F_Y(z)]_+ \, dz = \int_{\underline{p}}^{\overline{p}} [y(p) - x(p)]_+ \, dp, \tag{1}$$

where $F_X(z)$ is the cumulative density function (cdf) of X and $[m]_+ \equiv \max\{m, 0\}$. In words, (1) says that the sum of positive gaps, $y(p) - x(p)$, over the interval $[\underline{p}, \overline{p}]$, is equal to the sum of positive gaps of cdfs, $F_X(z) - F_Y(z)$, in the interval $[x(\underline{p}), x(\overline{p})]$ (or $[y(\underline{p}), y(\overline{p})]$) defined by $[\underline{p}, \overline{p}]$. Hence a dual for definition 1 can be proposed:

Definition 2 *An index measuring inter-distributional inequality between Y and X satisfies the property of group-specific disadvantage focus (GDF) if and only if it is sensitive to the gap $F_X(z) - F_Y(z) \mid F_X(z) \geq F_Y(z)$ and it is insensitive to the gap $F_X(z) - F_Y(z) \mid F_X(z) \leq F_Y(z)$. In particular, the index does not decrease (increase) if the gap $F_X(z) - F_Y(z)$ increases (decreases) given that initially $F_X(z) \geq F_Y(z)$ and the index does not react to changes in $F_X(z) - F_Y(z)$ as long as $F_X(z) \leq F_Y(z)$ before and after the changes.*

Definition 2 is useful for indices that map from probability space like the PROB index and those based on relative distributions. It can also be considered for applications with ordinal variables. Notice also the connection between the two definitions and first-order stochastic dominance. The following three statements are identical:

(i) Distribution Y (weakly) first-order dominantes X.

(ii) $y(p) \geq x(p) \, \forall p \in [0, 1]$.

(iii) $F_X(z) \geq F_Y(z) \, \forall z$.

Hence indices that satisfy GDF are expected to be informative about the presence of first-order stochastic dominance, especially in its weak form, as is shown below.

The PROB measure and relative distributions: review and amendments

The PROB measure of Gastwirth (1975) is defined as: $PROB \equiv \int_{-\infty}^{\infty} [1 - F_X(z)] f_Y(z) \, dz$. It measures the probability of finding an individual in X having at least as much of z as a random individual in Y (hence Y is the *reference* distribution and X is the *compared* distribution). *PROB* does not fulfill GDF because it pits inequalities that are detrimental to X against inequalities that are detrimental to Y. To see this notice the following simple decomposition stemming from adding and subtracting $\int_{-\infty}^{\infty} F_Y(z) f_Y(z)\, dz$ and considering that $\int_{0}^{\infty} F_Y(z) f_Y(z)\, dz = 0.5$:

$$PROB = \int_{-\infty}^{\infty} [F_Y(z) - F_X(z)]_+ f_Y(z)\, dz - \int_{-\infty}^{\infty} [F_X(z) - F_Y(z)]_+ f_Y(z)\, dz + 0.5 \tag{2}$$

Hence it is clear from (2) that inequalities detrimental to Y $\left([F_Y(z) - F_X(z)]_+\right)$ are compensated with inequalities detrimental to X $\left([F_X(z) - F_Y(z)]_+\right)$. For this reason $PROB$ cannot distinguish a situation of distributional equality from others of distributional inequality. Whenever $f_X = f_Y$, $PROB = 0.5$.[7] However the reverse is not true, as is clear from (2). As it stands, $PROB$ does not take any specific value that signals first-order stochastic dominance. By contrast, $PROB$ is useful to pinpoint absences of distributional overlap. For instance: $PROB = 0 \leftrightarrow F_X\left(z^Y_{\min}\right) = 1$, where $z^Y_{\min}$ is the minimum value for which Y has support. When $PROB = 0$ the richest person in X is not better off than the poorest person in Y (whose value of z is $z^Y_{\min}$). On the other extreme: $PROB = 1 \leftrightarrow F_X\left(z^Y_{\max}\right) = 0$. When $PROB = 1$ the poorest person in X is richer than the richest person in Y.

In summary: $PROB$ does not satisfy GDF, does not exclusively identify distributional equality or first-order stochastic dominance, but it does identify lack of distributional overlap. However, some simple measures based on $PROB$ can be used in conjunction with it in order to provide more information on the above-mentioned distributional features. I propose the following:

$$PROB^{\alpha}_{Y}(Y - X) \equiv (\alpha + 1) \int_{-\infty}^{\infty} [F_Y(z) - F_X(z)]^{\alpha}_{+} f_Y(z)\, dz, \tag{3}$$

$$PROB^{\alpha}_{Y}(X - Y) \equiv (\alpha + 1) \int_{-\infty}^{\infty} [F_X(z) - F_Y(z)]^{\alpha}_{+} f_Y(z)\, dz, \tag{4}$$

where α is a parameter and the subindex Y in $PROB^{\alpha}_{Y}(Y - X)$ denotes that the reference distribution is Y.[8] It is straightforward to notice that both (3) and (4) fulfill GDF. It is also the case that: $f_X = f_Y \leftrightarrow \left(PROB^{\alpha}_{Y}(Y - X) = 0 \wedge PROB^{\alpha}_{Y}(X - Y) = 0\right)$. Hence, used together, both indices (for any positive value of α) identify distributional equality. Two interesting

[7] When $PROB < 0.5$ the distribution of Y has some advantage over X's such that the probability of finding someone in X having at least as much of z as a randomly chosen person from Y is lower than the probability that would ensue from identical distributions. A similar interpretation, favouring X's distribution over Y's, ensues when $PROB > 0.5$.

[8] Analogue indices can be defined using X as the reference distribution.

sets of indices are related to the cases when $\alpha = 0$ and $\alpha = 1$. When $\alpha = 0$ the indices help to pinpoint situations of first-order stochastic dominance since: $PROB_Y^0(Y - X) = 1 \leftrightarrow X \succeq_{FD} Y$, where $\succeq_{FD}$ reads "weakly first-order dominates".[9] When $\alpha = 1$, both $PROB_Y^1(Y - X)$ and $PROB_Y^1(X - Y)$ are sensitive to changes in the percentile gaps, and they are helpful to detect absence of distributional overlap because: $PROB_Y^1(X - Y) = 1 \leftrightarrow F_X\left(z_{\max}^Y\right) = 0$. When $\alpha = 1$ the following relationship holds:

$$2PROB = PROB_Y^1(Y - X) - PROB_Y^1(X - Y) + 1 \tag{5}$$

Since these indices map from probability space, it is easy to show that they fulfill properties of population replication invariance and ratio scale invariance.

Relative distributions

The $PROB$ index and this paper's amendments are closely related to indices stemming from discrimination curves based on cumulative relative distributions. A cumulative relative distribution function maps the cumulative distribution of a reference distribution, $F_Y(z)$, into the interval $[0,1]$. Specifically, the cumulative distribution function is: $G_{X/Y}(F_Y) \equiv F_X[y(F_Y)]$ and the discrimination curve is the drawing of $G_{X/Y}(F_Y)$ on an horizontal axis of F_Y .[10] Le Breton et al. (2008) studied dominance conditions for the discrimination curve and proposed some indices based on the area between the discrimination curve and the 45 degree line. Two of their measures are relevant for this paper:

$$AAD = \int_0^1 \left|G_{X/Y}(F_Y) - F_Y\right| dF_Y \tag{6}$$

$$C = \int_0^1 \left[G_{X/Y}(F_Y) - F_Y\right] dF_Y, \tag{7}$$

where AAD is the average absolute deviation between the discrimination curve and the distributional equality line (45 degree). [11] Now notice that: $2AAD = PROB_Y^1(Y - X) + PROB_Y^1(X - Y)$ and $2C = PROB_Y^1(X - Y) - PROB_Y^1(Y - X)$ .Hence it is easy to see that both AAD and C do not fulfill GDF. In the first case both types of inequalities, i.e those detrimental to X and those detrimental to Y, are added up; while in the second case they compensate each other. As for other distributional features, C does not distinguish between distributional

[9] Likewise: $PROB_Y^0(X - Y) = 0 \leftrightarrow X \succeq_{FD} Y$.
[10] Hence when: $f_X = f_Y$ the discrimination curve is a 45 degree line.
[11] Le Breton et al. (2008) use different names for these indices.

equality and other situations. By contrast, $AAD = 0 \leftrightarrow f_X = f_Y$. Neither AAD nor C are informative about first-order stochastic dominance, but both are informative about the absence of distributional overlap since: $AAD, C = 0.5 \leftrightarrow F_X\left(z^Y_{\min}\right) = 1$.

Given the connection between $PROB$, on one hand, and AAD and C, on the other, the amendments proposed for the former are also relevant for the latter. An additional proposal can be made by combining $PROB^1_Y(Y-X)$ and $PROB^1_Y(X-Y)$ with AAD :

$$R_Y(Y-X) = \frac{PROB^1_Y(Y-X)}{2AAD} \tag{8}$$

$$R_Y(X-Y) = \frac{PROB^1_Y(X-Y)}{2AAD} \tag{9}$$

$R_Y(Y-X)$ provides a measure of the proportion of the inter-distributional inequality that is detrimental to Y, when the distribution of Y is taken as reference. An example of its usefulness is provided by the two cases in Figure 1. $PROB^1_Y(X-Y)$ yields the same value for both cases. By contrast, $R_Y(X-Y) = 1$ for the case of the left panel, whereas $R_Y(X-Y) < 1$ for the case of the right panel.

The indices by Ebert and Vinod: review and amendments

In a seminal contribution Ebert (1984) characterized axiomatically a family of indices based on Minkowski distances that is useful for IDI measurement. These indices are direct functions of the percentile gaps. The family for two groups with different population sizes is:

$$d^r(X,Y) \equiv \left[\int_0^1 |y(p) - x(p)|^r \, dp\right]^{\frac{1}{r}} \quad \forall r \geq 1 \tag{10}$$

This proposal is similar to that of Vinod (1985) in that both are direct functions of the percentile gaps. Vinod's measure of "overall economic advantage" is:

$$V(X,Y) \equiv \int_0^1 [y(p) - x(p)] \, dp = \mu_Y - \mu_X, \tag{11}$$

where μ_Y is the mean of distribution Y. Again it is easy to check that d^r and V do not fulfill GDF. As in the case with AAD, d^r is sensitive to both types of inequalities, which are added up by the index. By contrast, V compensates them.

For that reason V does not distinguish between distributional equality and other situations; whereas, like AAD: $d^r(X,Y) = 0 \leftrightarrow f_X = f_Y$. Neither d^r nor V are informative regarding situations of first-order stochastic dominance or absence of overlap.

However simple amendments related to both d^r and V fulfill GDF and, combined, provide more information about distributional equality and first-order dominance. The two amended indices are:

$$d^r_{Y-X} \equiv \int_0^1 |y(p) - x(p)|^r_+ \, dp \;\; \forall r \geq 1 \tag{12}$$

$$d^r_{X-Y} \equiv \int_0^1 |x(p) - y(p)|^r_+ \, dp \;\; \forall r \geq 1 \tag{13}$$

Clearly, both (12) and (13) fulfill GDF. Together these indices also pinpoint distributional equality because: $d^r_{Y-X} = d^r_{X-Y} = 0 \leftrightarrow f_X = f_Y$. They also detect first-order dominance since: $\left[d^r_{Y-X} > 0 \wedge d^r_{X-Y} = 0\right] \leftrightarrow X \succeq_{FD} Y$. However neither the amendments nor the original indices take specific values if and only if there is absence of overlap; except, in the case of $d^r(X,Y)$, when either $y(p)$ or $x(p)$ are equal to zero for all p. Finally the amendments are related to d^r are V according to the following expressions:

$$[d^r(X,Y)]^r = d^r_{Y-X} + d^r_{X-Y} \;\; \forall r \geq 1 \tag{14}$$

$$V(X,Y) = d^1_{Y-X} - d^1_{X-Y} \tag{15}$$

Unlike the indices in the previous section, d^r and V are not bounded from above and do not fulfill ratio scale invariance. However both of these properties can be met easily by dividing the indices by their maxima:

$$dd^r(X,Y) \equiv \frac{d^r(X,Y)}{\left[\int_0^1 y(p)^r \, dp\right]^{\frac{1}{r}} + \left[\int_0^1 x(p)^r \, dp\right]^{\frac{1}{r}}} \tag{16}$$

$$VV(X,Y) \equiv \frac{V(X,Y)}{\mu_Y + \mu_X} \tag{17}$$

The amended indices, (12) and (13), can be normalized the same way as in (16).

Other approaches: the indices by Butler and McDonald

Butler and McDonald (1987) proposed a group of so-called Pietra indices for the measurement of IDI. I review their index $P(1,1)$ because it is the one that uses more information from the cumulative distributions of the two compared groups:

$$P(1,1) = \frac{\int_0^{F_Y(\mu_X)} y(p)\,dp}{\mu_Y} - \frac{\int_0^{F_X(\mu_Y)} x(p)\,dp}{\mu_X} \tag{18}$$

$P(1,1)$ measures the difference between the proportion of total income in Y held by people who have income not higher than the average income in X minus the proportion of total income in X held by people who have income not higher than the average income in Y. Following the same assessment criteria as above, first, it can be shown that $P(1,1)$ does not fulfill GDF. In order to prove this, imagine that Y and X are both symmetric with equal mean, μ, hence: $F_Y(\mu_X) = F_X(\mu_Y)$. In that situation: $P(1,1) = \frac{\int_0^{0.5}[y(p)-x(p)]dp}{\mu}$. Clearly, there is no reason why the gaps $y(p) - x(p)$ should have the same sign in the integration interval (e.g. imagine the two distributions differ in their kurtosis). Hence $P(1,1)$ may compensate percentile gaps that are detrimental to different groups. For that same reason it is possible that $P(1,1) = 0$ in several situations, besides distributional equality. Hence the measure is not helpful in identifying distributional equality. Likewise it is not difficult to find examples showing that the measure is not helpful in identifying first-order stochastic dominance either. In the absence of overlap $P(1,1) = 1$ if the poorest person in X is richer than the richest person in Y, and $P(1,1) = -1$ if the poorest person in Y is richer than the richest person in X. However the reverse relationships are not true. For instance, it suffices for $P(1,1) = 1$ that the richest person in Y has less than the mean income of X and the poorest person in X has more than the mean income of Y.

Amendments to $P(1,1)$ that may render it in fulfillment of GDF, or more informative about the distributional features under discussion, do not seem to be straightforward. Moreover, as shown above, $d^r(X,Y)$ and $V(X,Y)$ provide a better starting point for proposing IDI indices that fulfill GDF and are informative about other interesting distributional features.

Other approaches: the indices by Dagum

Dagum (1987) proposed a measure of relative economic affluence (REA) which, in this paper's notation, is defined as: $D_{X/Y} = 1 - \frac{d_Y}{d_X}$, where:

$$
\begin{aligned}
d_X &= \int_0^{\infty} dF_Y(y) \int_0^{y} (y-x)\, dF_X(x), &(19)\\
d_Y &= \int_0^{\infty} dF_X(x) \int_0^{x} (x-y)\, dF_Y(y). &(20)
\end{aligned}
$$

It is easy to show that $D_{X/Y}$ does not fulfill GDF. First, notice that $d_X - d_Y = \mu_Y - \mu_X$.[12] Hence: $D_{X/Y} = \frac{\mu_Y - \mu_X}{d_X} = \frac{V(X,Y)}{d_X}$. This means that D compensates percentile gaps that are detrimental to different groups in the same way that V does. So neither can fulfill GDF. Like V, D does not identify distributional equality because, even though $D_{X/Y} = 0$ if $f_X = f_Y$, the reverse is not true. The necessary and sufficient requirement for $D_{X/Y} = 0$ is: $\mu_X = \mu_Y$. Likewise D does not identify situations of first-order dominance because, even though $X \succeq_{FD} Y$ implies $d_Y \geq d_X$, the reverse is not true. By contrast, D is useful for pinpointing absence of overlap. When the richest person in Y is poorer than the poorest person in X then $d_X = 0$ (and $d_Y = \mu_X - \mu_Y$) and the reverse is also true. When the richest person in X is poorer than the poorest person in Y then $d_Y = 0$ (and $d_X = \mu_Y - \mu_X$) and the reverse is also true. d_X compares every member of Y against all the people in X who have less income and quantifies the respective gaps. That is why the measures are well suited to detect absence of overlaps without resorting to percentiles or probabilities: if everybody in X is richer than everybody in Y then $d_X = 0$.The reverse is true because, unless the two distributions are degenerate and equal to each other, $d_X = 0$ requires that $f_X(y) = 0$ for every value of y on the support of Y (and then $F_X(y) = 0$ over the same support). Amendments to $D_{X/Y}$ that may render it, or its constituent statistics, in fulfillment of GDF, or more informative about the distributional features under discussion, do not seem to be straightforward either.

In an earlier contribution, Dagum (1980) proposed using d_X, or d_Y, as the basic statisics for measures of economic distance normalized by their respective minima and maxima.[13]

[12] This result stems from equations (4) and (5) of Dagum (1987, p. 6).

[13] More generally, Dagum also suggested considering the following family of statistics based on generalized means, even though he focused on d_X^1 :

$$
\begin{aligned}
d_X^r &= \left[\int_0^{\infty} dF_Y(y) \int_0^{y} (y-x)^r\, dF_X(x)\right]^{\frac{1}{r}}, \; r \neq 0 &(21)\\
d_X^0 &= e^{\int_0^{\infty} dF_Y(y) \int_0^{y} \ln(y-x) dF_X(x)}. &(22)
\end{aligned}
$$

Likewise one can show that d_X does not fulfill GDF. For instance, following Shorrocks (1982), d_X can be decomposed in the following way:

$$d_X = \frac{V(X,Y)}{2} + \frac{1}{2}\int_0^1\int_0^1 |\, y(p_y) - x(p_x)\,|\, dp_x dp_y, \tag{23}$$

where p_x and p_y are percentiles of X and Y, respectively. Hence d_X compensates *and* adds up gaps that are detrimental to diffferent groups. So it does not fulfill GDF. d_X and d_Y are also unable to identify distributional equality. As shown by Shorrocks (1982), when there is distributional equality $d_X = d_Y = \mu_X G(X) = \mu_Y G(Y)$, where $G(X)$ is the Gini coefficient of X. However the reverse is not true because two different distributions can have the same mean and Gini coefficient. For instance if Y is obtained from X by performing two transfers of the same amount, but one regressive and one progressive, involving two pairs of individuals in different parts of the distribution, then both distributions, despite being unequal, have the same mean and the same value for the Gini coefficient. [14]

Like D, d_X and d_Y do not identify situations of first-order stochatic dominance, but they are good for detecting absence of distributional overlap due to the abovementioned reasons.

An alternative framework: Indices based on welfare comparisons

Thus far, the indices reviewed are characterized by: i) being useful especifically for two-group comparisons, ii) being more informative than just comparing two means, and iii) being useful when the two distributions have different sample sizes (as mentioned in the introduction). All these indices stem from aggregations of several comparisons of different parts of the two distributions (e.g. in the case of $d^r(X,Y)$ and $V(X,Y)$) pairwise comparisons of quantiles are performed, and then these are aggregated). An alternative to this approach to IDI measurement has been proposed by Shorrocks (1982) and Chakravarty and Dutta (1987). Their proposed indices are characterized by a *different order of aggregation*: first, an equally distributed equivalent (EDE) standard is computed for each distribution separately, and then the two EDE statistics are compared. These indices are also known as "ethical distance functions" and they measure the differences in the welfare provided by two distributions through the metric of the EDE standard introduced by Atkinson (1970).

[14] For instance if $X = (1,2,3,4)$ and $Y = (0.5, 2.5, 3.5, 3.5)$.

Let x_{EDE} be the EDE standard of X. Chakravarty and Dutta (1987) show that IDI measures like $d^r(X,Y)$ are not coherent with a welfarist comparison approach. By coherent, they mean that the index should be related monotonically to the absolute value of the difference between the two EDE standards. Instead they propose the following family of EDE-standard-based measures:

$$S(X,Y) = K \mid x_{EDE} - y_{EDE} \mid, K > 0. \tag{24}$$

When $K = 1$, S is the distance index suggested by Shorrocks (1982). Notwithstanding the merits of these indices in terms of their coherence with an approach based on a social evaluation function, they are unlikely to fulfill GDF, because they do not explicitly document relative advantages at different parts of the distributions, which is due to the order of aggregation. For instance, consider the following EDE standard: $x_2 = \left[\int_0^1 \sqrt{x(p)}dp\right]^2$.[15] With that standard: $S = K \mid \left(\int_0^1 \left[\sqrt{x(p)} + \sqrt{y(p)}\right] dp\right) \left(\int_0^1 \left[\sqrt{x(p)} - \sqrt{y(p)}\right] dp\right) \mid$. Hence gaps that are detrimental to different groups are compensated. It is not difficult to show the occurrence of this same feature with other choices for the EDE standard. As for the other distributional aspects emphasized in this review, measures like $S(X,Y)$ are not useful to identify distributional equality because two different distributions can have the same EDE standard. Likewise, they are not informative as to the presence of first-order stochastic dominance because first-order dominance is sufficient but not necessary in order to have differences between x_{EDE} and y_{EDE}. Finally, they are not helpful either when it comes to detecting absence of overlap.

Concluding remarks

This paper's review focused on indices that measure IDI involving distributions with different population sizes. The main question of the review was whether these indices fulfill a property of group-specific disadvantage focus (GDF). This property is necessary for indices that quantify inequalities that are detrimental to one specific group. Such concern for group-specific disadvantages has been articulated recently in the literature that measures labour market discrimination using counterfactual distribution techniques. The latter compare actual versus counterfactual situations individual-by-individual. By contrast, traditional IDI indices compare actual distributions of populations with different sizes. Then, it is natural

[15] x_2 is a member of the family of EDE standards based on generalized means, considered by Atkinson (1970).

that most of the reviewed IDI indices map from either percentiles or probabilities. Hence in this paper, I first set out to define GDF in the context of IDI measurement with different population sizes.

Neither of the indices reviewed satisfies GDF. Several are also limited in the information they provide on some interesting distributional features including distributional equality, presence of first-order stochastic dominance and absence of distributional overlap. However, as the paper shows, in most cases it is straightforward to amend these indices in order to render them in fulfillment of GDF and, often, more informative in terms of the additional distributional features mentioned above.

The examination of several indices suggests that fulfillment of GDF by IDI indices, for two distributions with different population sizes, may require that the indices map explicitly from either percentiles or probabilities.

References

Arneson, R. (1989). Equality and equal opportunity for welfare. *Philosophical studies*, 56: 77–93.

Arrow, K. (1973). *Discrimination in the labor market*, chapter The theory of discrimination. Princeton University Press.

Atkinson, A. (1970). On the measurement of economic inequality. *Journal of Economic Theory*, 2(3): 244–63.

Becker, G. (1971). *The economics of discrimination*. The University of Chicago Press.

Butler, R., and McDonald, J. (1987). Intedistributinal income inequality. *Journal of Business and Economic Statistics*, 5(1): 13–7.

Chakravarty, S., and Dutta, B. (1987). A note on measures of distance between income distributions. *Journal of Economic Theory*, 41: 185–8.

Cohen, G. (1989). On the currency of egalitarian justice. *Ethics*, 99(4): 906–44.

Cowell, F. (1985). Measures of distributional change: an axiomatic approach. *Review of Economic Studies*, LII: 135–51.

Dagum, C. (1980). Inequality measures between income distributions with applications. *Econometrica*, 48(7): 1791–803.

Dagum, C. (1987). Measuring the economic affluence between populations of income receivers. *Journal of Business and Economic Statistics*, 5(1): 5–8.

del Rio, C., Gradin, C., and Canto, O. (2011). The measurement of gender wage discrimination: the distributional approach revisited. *Journal of Economic Inequality*, 9(1): 57–86.

Ebert, U. (1984). Measures of distance between income distributions. *Journal of Economic Theory*, 32: 266–74.

Elbers, C., Lanjouw, P., Mistiaen, J., and Ozler, B. (2008). Reinterpreting between-group inequality. *Journal of Economic Inequality*, 6: 231–45.

Fields, G., and Ok, E. (1996). The meaning and measurement of income mobility. *Journal of Economic Theory*, 71: 349–77.

Fields, G., and Ok, E. (1999). Measuring movement of incomes. *Economica*, 66(264): 455–71.

Fleurbaey, M. (2001). Egalitarian opportunities. *Law and Philosophy*, 20(5): 499–530.

Fleurbaey, M. (2008). *Fairness, Responsibility and Welfare*. Oxford University Press.

Gastwirth, J. (1975). Statistical measures of earnings differentials. *The American Statistician*, 29(1): 32–5.

Gradin, C., Del Rio, C., and Canto, O. (2010). Gender wage discrimination and poverty in the EU. *Feminist Economics*, 16(2): 73–110.

Jenkins, S. (1994). Earnings discrimination measurement: a distributional approach. *Journal of Econometrics*, 61: 81–102.

Lanjouw, P., and Rao, V. (2008). Revisiting between-group inequality measurement: an application to the dynamics of caste inequality in two Indian villages. Manuscript.

Le Breton, M., Michelangeli, A., and Peluso, E. (2008). Wage discrimination measurement: in defense of a simple but informative statistical tool. Econpubblica Working Paper 128.

Nussbaum, M. C., and Glover, J. (1995). *Women, culture and development: a study of human capabilities*. WIDER studies in Development Economics. Clarendon Press.

Ooghe, E., Schokkaert, E., and Van de Gaer, D. (2007). Equality of opportunity versus equality of opportunity sets. *Social Choice and Welfare*, 28: 209–30.

Phelps, E. (1972). The statistical theory of racism and sexism. *The American Economic Review*, 62(4): 659–61.

Roemer, J. (1998). *Equality of Opportunity*. Harvard University Press.

Roemer, J. (2006). Economic development as opportunity equalization. Cowles Foundation Discussion Paper 1583.

Schluter, C., and van de Gaer, D. (2011). Upward structural mobility, exchange mobility, and subgroup consistent mobility measurement: U.S.-German mobility rankings revisited. *Review of Income and Wealth*, 57(1): 1–22.

Sen, A. (2001). *Development as Freedom*. Oxford University Press.

Shorrocks, A. (1982). On the distance between income distributions. *Econometrica*, 50(5): 1337–9.

van Krem, P. (2009). Generalized measures of wage differentials. IRISS Working Papers 2009-08.

Vinod, H. (1985). Measurement of economic distance between blacks and whites. *Journal of Business and Economic Statistics*, 3(1): 78–88.

Figure 1: Left panel: the two CDFs overlap in part of their common support. Right panel: the two CDFs cross once