Efficiency and Stability in Complex Financial Markets

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Abstract

The authors study a simple model of an asset market with informed and non-informed agents. In the absence of non-informed agents, the market becomes information efficient when the number of traders with different private information is large enough. Upon introducing non-informed agents, the authors find that the latter contribute significantly to the trading activity if and only if the market is (nearly) information efficient. This suggests that information efficiency might be a necessary condition for bubble phenomena—induced by the behavior of non-informed traders—or conversely that throwing some sands in the gears of financial markets may curb the occurrence of bubbles.

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1 Introduction

Financial markets have increased tremendously in size and complexity in the last decades, with the proliferation of hedge funds and the expansion of derivative markets. Within the neo-classical paradigm, the expansion in *i)* the diversity of traders and *ii)* in the repertoire of financial instruments is, generally, enhancing the efficiency of the market[1]. Indeed, *i)* unfettered access to trading in financial markets makes more liquidity available and it eliminates arbitrages, thus pushing the market closer to the theoretical limit of *perfectly competitive, informationally efficient markets.* Likewise, the expansion in the repertoire of trading instruments provides a wider range of possibilities to hedge risks and it drives the system closer to the theoretical limit of *dynamically complete markets* [Merton and Bodie 2005]. Both conclusions rely on non-trivial assumptions, notably the absence of information asymmetries. Indeed, financial stability is related to the effects of asymmetric information[2] and most of the responsibility for market failures is, in one way or another, usually put on market imperfections[3]. It is hard to deny the evidence of the perverse effects of asymmetric information. Still market imperfections are inevitable even in stable periods, which suggests that the problem lies in understanding when such deviations from ideal conditions are amplified by the internal dynamics of the market, leading to a full blown crisis.

This paper suggests that the more markets are close to ideal conditions, the more they are prone to the proliferation and amplification of market imperfections. This point has already been made in the literature [Brock et al 2008, Caccioli et al 2009, Marsili 2009] concerning the expansion in the repertoire of financial instruments. Specifically, [Brock et al 2008] show that adding more and more Arrow's securities in a market with heterogeneous adaptive traders, brings the system to a dynamic instability. A similar conclusion was drawn in [Caccioli et al 2009], though based on different models. Ref. [Marsili 2009] discusses instead an equilibrium model and it shows that as the number of possible trading instruments increases the market approaches the theoretical limit of complete markets but allocations develop a marked sensitivity to price indeterminacy, and the volume of trading implied by hedging in the interbank market diverges.

Here we address the issue of stability, in relation to information efficiency. Information efficiency refers to the ability of the market to allocate investment to activities which provide profitable return opportunities. In brief, traders who have a private information on the performance of an asset will buy or sell shares of the corresponding stock in order to make a profit. As a result, prices will move in order to incorporate this information, thus reducing the profitability of that piece of information. In equilibrium, when all informed traders are allowed to invest, prices must be such that no profit can be extracted from the market on the basis of their

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[1]See however [Hart 1975], [Cass and Citanna 1998].

[2]According to F.S. Mishkin "Financial instability occurs when there is a disruption to financial markets in which asymmetric information and hence adverse selection and moral hazard problems become much worse, so that financial markets are unable to channel funds efficiently to those with the most productive investment opportunities." [Mishkin and Herbertsson 2006, Mishkin 1996].

[3]See, for example, [Turnbull 2008] for a detailed account of perverse effect which led to the 2007 – 2008 crisis in credit markets.

information.

In this respect, markets behave as information processing and aggregating devices, and in the ideal limit, market prices are expected to reflect all possible information, which is the content of the celebrated *Efficient Market Hypothesis* [Fama 1970]. Paradoxically, however, when markets are really informationally efficient, traders have no incentive to gather private information, because prices already convey all possible information. Hence, as realized long ago [Grossman and Stiglitz 1980], traders' behavior does not transfer any information into prices, which implies that efficient markets cannot be realized.

The interplay between informed and non-informed traders is one of the key elements in explaining market dynamics. Informed traders – so-called fundamentalists – typically have a stabilizing effect whereas non-informed traders – e.g. trend-followers or chartists in general – can destabilize the market, and induce bubble phenomena[4]. Research in Heterogeneous Agents Models [Hommes 2006] have provided solid support to the thesis that when trading activity is dominated by non-informed traders, bubbles and instabilities develop.

Our goal, here, is to establish a relation between market efficiency and the interplay between informed and non-informed traders. Specifically, we provide support to the idea that *non-informed traders dominate if and only if the market is sufficiently close to information efficiency.* In addition, as markets become informationally efficient, they develop a marked susceptibility to perturbations and instabilities.

Our discussion steps from the simple asset market model studied in [Berg et al 2001], which describes a population of heterogeneous individuals, who receive a private signal on the return – or dividend – of a given asset. Given their private signal, agents invest in the asset and their demand determines the price, via market clearing. Agents learn how to optimally exploit their private information in their trading activity, which has the effect that their information is incorporated into prices. As the number N of agents with a different private information increases, prices gradually converge to the returns. Beyond a critical number of agents, prices converge exactly to returns. Therefore, the model provides a stylized picture of how markets aggregate information into prices and become informationally efficient.

In this setting, we introduce non-informed agents who adopt the same learning dynamics, but which base their decision on public information, rather than on a private signal. In particular, we take the sign of the last return as public signal, which mimics a chartist behavior, in its simplest form. Our main result is that chartists take over a sizable share of market activity only when the market becomes informationally efficient.

The rest of the paper is organized as follows. The next section introduces the model and the notation. In the following section we first recall the results with only informed traders and then discuss the effect of introducing non-informed traders. The paper concludes with a discussion of the extension and relevance of the results.

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[4]A similar intuition seems to underlie the origin of financial instability in Minsky's theory [Minsky 1992]: The gradual shift of financial institutions from hedge funds to Ponzi's scheme, where the working of the "veil of money" acquires a dynamics which is unrelated to the underlying economy, is similar in nature to the prevalence of non-informed traders.

2 The model: Information efficiency and chartists

Let us consider a market where a single asset is being traded an infinite number of periods. Let there be $N+1$ traders, N informed (fundamentalists) and one chartist (uninformed), which we shall refer to as agent $i = 0$. There are N units of asset available at each time and at the end of each period the asset pays a return.

The return depends only on the state of nature in that period, $\omega = 1, \ldots, \Omega$, and is denoted by R^ω. The state of nature is determined in each period, independently, according to the uniform distribution on the integers $1, \ldots, \Omega$.

Traders do not observe the state directly, but informed traders ($i = 1, \ldots, N$) receive a signal on the state according to some fixed private information structure, which is determined at the initial time and remains fixed. More precisely, a signal is a function from the state ω to a signal space, which for simplicity we assume to be $M \equiv \{-, +\}$. The signal observed by trader i if state is ω is k_i^ω. The information structure available to i is the vector $(k_i^\omega)_{\omega \in \Omega}$. Trader $i = 0$, instead, does not receive any signal on the state ω, but he observes a public variable $k_0 \in \{-1, +1\}$, such as the sign of the last price change.

We focus on a random realization of this setup, where the value of the return R^ω in state ω is drawn at random before the first period, and does not change afterwards. Returns thus only change because the state of nature changes. As in Ref. [Berg et al 2001] we take R^ω Gaussian with mean $\bar{R}$ and variance s^2/Ω. Likewise, the information structure is determined, by setting $k_i^\omega = +1$ or -1 with equal probability, independently across traders i *and* states ω.

At the beginning of each period, a state ω and a public information k_0 are drawn, and private information $m = k_i^\omega$ is revealed to informed agents ($i > 0$). All traders decide to invest a monetary amount z_i^m in the asset, depending on the signals ($m = k_i^\omega$ for $i > 0$ or k_0 for $i = 0$) they receive. The price of the asset p^{ω, k_0} is then extracted from the market clearing condition

$$Np^{\omega,k_0} = \sum_{i=1}^{N} \sum_{m=\pm 1} z_i^m \delta_{k_i^\omega, m} + \sum_{m=\pm 1} z_0^m \delta_{k_0, m}. \tag{1}$$

Agents do not know the price at which they will buy the asset when they decide their investment z_i^m. The price depends on the state ω and on k_0 because the amount invested by each agent depends on the signal they receive, which depends on ω [Shapley and Shubik 1977, Pliska 1997]. At the end of the period, each unit of asset pays a monetary amount R^ω. If agent i has invested z_i^m units of money, he will hold $z_i^m/p^{\omega,k_0}$ units of asset, so the expected payoff of agents is

$$u_i(z_i) = \frac{1}{\Omega} \sum_\omega \sum_m \delta_{k_i^\omega, m} \left(\frac{R^\omega}{p^{\omega, k_0}} - 1 \right). \tag{2}$$

How will agents choose their investments? One can consider either competitive equilibria or take a dynamical approach where agents are assumed to learn over time how to invest optimally. As in Ref. [Berg et al 2001] these two different choices are going to bring to the same equilibria, so we focus on the latter. In particular,

each informed agent $i > 0$ has a propensity to invest $U_i^m(t)$ for each of the signals $m = \pm 1$. His investment $z_i^m = \Phi(U_i^m)$ at time t is an increasing function of $U_i^m(t)$ ($\Phi : R \to R^+$) with $\Phi(x) \to 0$ if $x \to -\infty$ and $\Phi(x) \to \infty$ if $x \to \infty$. After each period agents update $U_i^m(t)$ according to the marginal success of the investment:

$$U_i^m(t+1) = U_i^m(t) + \left(R^{\omega_t} - p_t^{\omega_t,k_0}\right)\delta_{m,k_i^{\omega_t}} - \Gamma\epsilon, \qquad i = 1,\ldots,N \qquad (3)$$

where ω_t is the state at time t and $p_t^{\omega_t,k_0}$ is the realized price. The idea in Eq. (3) is that if for a given signal m agent i observes returns R^ω which are higher that prices, she will increase her propensity U_i^m to invest, under that signal. At odds with Ref. [Berg et al 2001], the learning dynamics for informed agents ($i > 0$) also takes into account the cost of information, through the term ϵ. More precisely, investment is considered attractive ($U_i^m > 0$) only if the returns under signal m exceed prices by more than ϵ. Similarly, the non-informed agent updates her propensity to trade according to

$$U_0^m(t+1) = U_0^m(t) + \left(R^{\omega_t} - p_t^{\omega_t,k_0}\right)\delta_{m,k_0} \qquad (4)$$

and invest an amount $z_0^m = \Phi(U_0^m)$, depending on the value $m = k_0$ of public information at time t.

3   Results

Let us briefly recall the behavior of the market in the absence ($z_0 = 0$) of non-informed traders and $\epsilon = 0$. Ref. [Berg et al 2001] shows that the learning dynamics converges to the allocations $\{z_i^m\}$ which correspond to the solution of the minimization of the function

$$H = \frac{1}{2}\sum_{\omega=1}^{\Omega}(R^\omega - p^\omega)^2, \qquad p^\omega = \frac{1}{N}\sum_{i=1}^{N}\sum_{m=\pm 1}z_i^m\delta_{k_i^\omega,m} \qquad (5)$$

The function H is the squared distance of prices from returns. As more and more different types of informed agents enter the market prices approach returns. There is a critical value n_c of informed traders beyond which $H = 0$, which implies that prices equal returns ($p^\omega = R^\omega$) for each state $\omega = 1,\ldots,\Omega$. This corresponds to the strong form of market information efficiency, when all private information is incorporated into prices[5] [Malkiel 1992]. The region $H = 0$ is also characterized

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[5]A market is efficient with respect to an information set if the public revelation of that information would not change the prices of the securities. *Strong efficiency* refers to the case where the information set includes the information available to any of the participants in the market, including private information. Note indeed that an agent who knew simultaneously the partial information of all agents would be able to know the state ω, with probability one, for $N \to \infty$. Indeed the probability that there are two states ω and ω' with different returns and that no agent can distinguish them is well approximated by $\Omega(\Omega-1)2^{-(N+1)}$, which vanishes for $N \to \infty$, in the case $\Omega \propto N$ we consider here. If ω were public knowledge, we would expect that $p^\omega = R^\omega$ for all
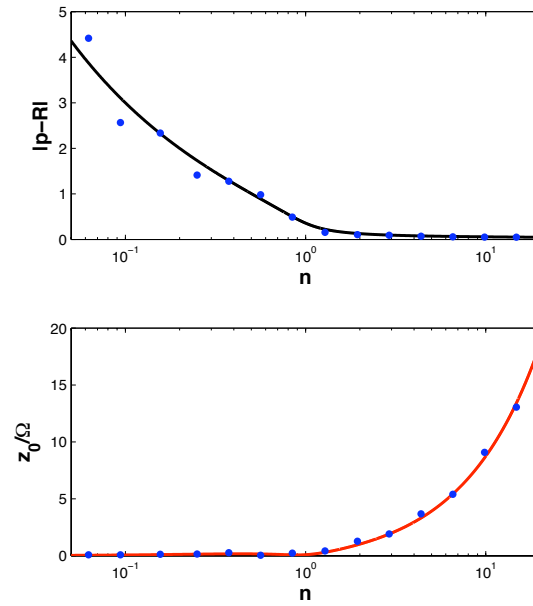
Figure 1: *Top panel: distance $|p - R| = \sqrt{\sum_{\omega=1}^{\Omega} E_{k_0} \left[R^\omega - p^{\omega,k_0} \right]^2}$ of prices from returns in competitive equilibrium. The full line represents the analytical solution for the case $s = \overline{R} = 1$ and $\epsilon = 0.1$, points refer to numerical simulations of systems with $\Omega = 32$, $s = \overline{R} = 1$ and $\epsilon = 0.1$. Bottom panel: monetary amount invested by the trend follower z_0 for the same values of the parameters.*
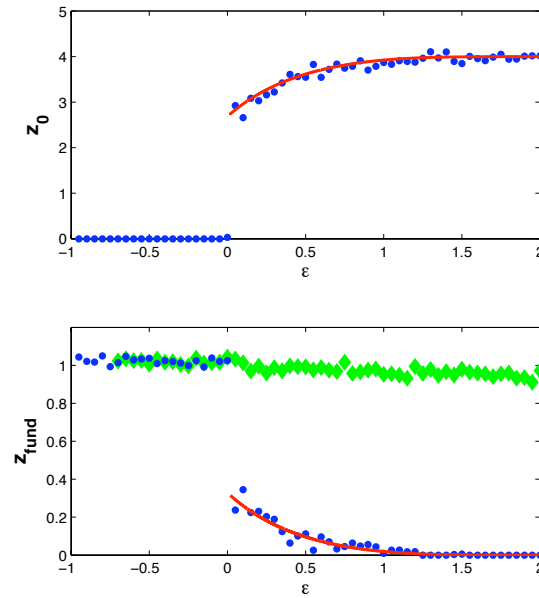

Figure 2: *Top panel: monetary amount invested by the trend follower. Bottom panel: monetary amount invested by a fundamentalist in presence (blue points) or absence (green diamonds) of the trend follower. Points refer to simulations of systems with $\Omega = 32$, $n = 4$ and $s = \overline{R} = 1$. Full lines represent the corresponding analytical solution.*

by a divergent *susceptibility*, which means that allocations $\{z_i^m\}$ have a marked dependence on structural parameters[6].

What happens when we introduce chartists ($z_0 > 0$) and information costs ($\epsilon > 0$)? First we find that allocations $\{z_i^m \geq 0\}_{i=0,\dots,N}$ are again given by the solution of the minimization of a function, which takes the form

$$H_\epsilon = \frac{1}{2} \sum_{\omega=1}^{\Omega} E_{k_0} \left[R^\omega - p^{\omega,k_0} \right]^2 + \frac{\epsilon}{2N} \sum_{i=1}^{N} \sum_{m=\pm 1} z_i^m, \tag{6}$$

where p^{ω,k_0} is given in Eq. (1) in terms of z_i^m, $i = 0,\dots,N$, $m = \pm 1$. The proof proceeds, on one side, by taking the partial derivatives of H_ϵ with respect to z_i^m and analyzing the Kuhn-Tucker conditions for the minimization of H_ϵ. These tell us that if the partial derivative of H_ϵ vanishes in the minimum, then $z_i^m > 0$. Otherwise, if the derivative is positive, then $z_i^m = 0$. On the other side, one easily finds that

$$\frac{\partial H_\epsilon}{\partial z_i^m} = -E_{k_0,\omega} \left[U_i^m(t+1) - U_i^m(t) \right] \tag{7}$$

which implies that the Kuhn-Tucker conditions correspond exactly to the conditions for the stationary state (with $U_i^m \to -\infty$ when $z_i^m = 0$).

This result paves the way for the extension of the statistical mechanics approach to this case. Some simple heuristic arguments can be useful in order to understand the basic behaviour of the system. Let us consider the case of small ϵ and small n. Then the first term in Eq. (6) dominates the second and the minimum is expected to be close to that without chartists. When n increases, however, the value of H decreases making the two terms comparable. When this happens, i.e. when $n \approx n_c$ and $H \approx 0$, then it starts to become possible to achieve a small value of H_ϵ by decreasing the size of the second term, increasing, at the same time, z_0^m in order to keep average prices of the same order of average returns. Hence we expect z_0^m to be large and of order N when the market becomes close to information efficient. The results of numerical simulations as well as the analytical solution for competitive market equilibrium (see appendix for more details), shown in Fig. 1, confirm this picture. Upon increasing the number of informed agents, the system undergoes a transition from inefficient to efficient market. Correspondingly, the share of trades due to uninformed agent starts raising only once information has been aggregated by informed traders. It has to be noticed that the introduction of the information cost ϵ makes sure that a perfect efficiency of the market is recovered only at $\epsilon = 0$. It is then instructive to look at the behaviour of the chartists as a function of ϵ. Figure 2 shows signatures of a phase transition occurring at $\epsilon = 0$. Indeed, for $\epsilon < 0$ the chartists barely operates in the market, while they start trading as soon as $\epsilon > 0$
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states ω. hence $H = 0$ is equivalent to the strong form of information efficiency.

[6]The susceptibility Φ relates a small uncertainty in a structural parameter, such as e.g. R^ω, to the uncertainty in allocations $\delta z_i^m \simeq \Phi \delta R^\omega$. A divergent susceptibility $\Phi \to \infty$ signals the fact that equilibria with different allocations are possible even for the same structural parameters, i.e. that the minimum of Eq. (5) is not unique.

4   Conclusions

We have shown that, in a simple asset market model, non-informed traders contribute a non-negligible fraction of the trading activity only when the market becomes informationally efficient. In the simple setting studied here, non-informed traders do not have a destabilizing effect on the market as in the models of Ref. [Hommes 2006]. At the same time, when non-informed traders dominate, their activity does not spoil information efficiency.

When combined with the insights of the literature on Heterogeneous Agent Models [Hommes 2006], the very fact that non-informed traders start trading massively when market efficiency is approached in fact suggests that information efficiency can lead to bubbles and instabilities. Substantiating this claim, would require first to extend the framework of Ref. [Hommes 2006] to the case of fundamentalists with different types of private information, recovering a picture for information efficiency similar to that provided by Ref. [Berg et al 2001]. Then one should investigate the effect of introducing non-informed traders – i.e. genuine trend-followers. Besides understanding whether information efficiency is also in that case a necessary condition for non-informed traders to dominate, one could also address the interesting question of the effect of chartists on information efficiency.

Ultimately, our results suggest that excessive insistence on information efficiency in market regulation policies – as e.g. in the debate on the Tobin tax [Haq et al 1996] – could have the unintended consequence of propelling financial bubbles, such as those which have plagued international financial markets in the recent decades.

Appendix: The statistical mechanics analysis

The competitive equilibrium solution of our problem can be obtained through the minimisation of the following Hamiltonian function

$$H = \frac{N^2}{4\Omega} \sum_{\omega,k_0} (R^\omega - p^{\omega,k_0})^2 + \frac{\epsilon}{2} \sum_{i,m} z_i^m, \tag{8}$$

with $p^{\omega,k_0} = \frac{1}{N} \sum_{i,m} z_i^m \delta_{k_i^\omega,m} + \sum_{k_0} \frac{z_0^{k_0}}{N}$. In order to compute the minima of H we introduce the partition function

$$Z(\beta) = \int_0^\infty dz_0^+ \int_0^\infty dz_0^- \cdots \int_0^\infty dz_N^+ \int_0^\infty dz_N^- e^{-\beta H\{z_i^m\}}. \tag{9}$$

In the limit $\beta \to \infty$ integrals are dominated by those configurations $\{z_i^m\}$ that minimise the Hamiltonian. The central quantity to compute is the free energy $f_\beta = -\beta^{-1} \log Z(\beta)$, which has to be averaged over the realisations of the disorder, namely $\{k_i^\omega, R^\omega\}$. In the following we are going to consider $k_i^\omega = \pm 1$ with equal probability $\forall i, \omega$, and we take $R^\omega = R + \frac{\tilde{R}}{\sqrt{N}}$, where $\tilde{R}$ are Gaussian variables with zero mean and variance equal to s^2. In order to compute the average over the disorder $\langle f_\beta \rangle$ we can resort to the so called replica trick through the identity $\log Z = \lim_{M \to 0} (Z^M - 1)/M$. The problem reduces then to that of computing the

average over the disorder of the partition function of M non interacting replicas of the system:

$$
\begin{aligned}
\left\langle Z^M \right\rangle &= \left\langle \mathrm{Tr}_{\{z\}} \prod_a \delta \left(N\overline{R} - \sum_i \overline{z}_{i,a} - \overline{z}_{0,a} \right) \right.\\
&\quad \times \left. e^{-\beta \left[\sum_{a,\omega,k0}(NR^\omega - \sum_{i,m} z_i^m \delta_{k_i^\omega,m} - z_0^{k0})^2 + \epsilon \sum_{i,a} \frac{z_{i,a}^+ + z_{i,a}^-}{2} \right]} \right\rangle,
\end{aligned}
$$

with $a \in \{1,\ldots,M\}$, $i \in \{1,\ldots,N\}$, $\omega \in \{1,\ldots,\Omega\}$, $m \in \{-1,1\}$ and $k0 \in \{-1,1\}$ and $\overline{z}_{i.a} \equiv (z_{i,a}^+ + z_{i,a}^-)/2$. We verified through numerical simulations that, for the specific public signal k_0 that we considered in this paper, $\left\langle z_0^+ \right\rangle = \left\langle z_0^- \right\rangle$ so, in order to simplify the calculation, we make the assumption $z_0^+ = z_0^- \equiv z_0$. After performing a Hubbard-Stratonovich transformation in order to linearize the quadratic term of the Hamiltonian, taking the average over the quenched variables introduces an effective interaction between replicas:

$$
\begin{aligned}
\left\langle Z^n \right\rangle &= \left\langle \int \{dQ_{a,b}\}\{d\hat{Q}_{a,b}\}\{d\hat{R}\}\mathrm{Tr}_{\{z\}} \right.\\
&\quad e^{-\sum_{a,b} \hat{Q}_{a,b}\left(NQ_{a,b} - \sum_i \Delta_i^a \Delta_i^b \right) - \sum_a \hat{R}_a\left(N\overline{R} - \sum_i \overline{z}_{i,a} - z_{0,a} \right)} \\
&\quad \times e^{-\beta N/\Omega \sum_{a,b,\omega}(\tilde{R}^\omega)^2 \left(\frac{\beta Q_{a,b}}{\alpha} + \delta_{a,b} \right)^{-1} - \beta\epsilon \sum_{i,a} \overline{z}_{i,a}} \\
&\quad \times \left. e^{-\frac{\Omega}{2}\mathrm{Tr}\log\left(\frac{\beta Q_{a,b}}{\alpha} + \delta_{a,b} \right)} \right\rangle,
\end{aligned}
$$

where we have introduced the overlap matrix $Q_{a,b}$ and the variables $\Delta_{i.a} \equiv (z_{i,a}^+ - z_{i,a}^-)/2$, while $\hat{Q}_{a,b}$ and $\hat{R}_a$ are conjugated variables that come from integral representations of δ functions:

$$
\delta(X - X_0) \propto \int d\hat{X} e^{-\hat{X}(X-X_0)}. \tag{10}
$$

In order to make further progress we consider the replica symmetric ansatz, namely we take

$$
Q_{a,b} = q_0 + \alpha\frac{\Phi}{\beta}\delta_{a,b} \tag{11}
$$

$$
\hat{Q}_{a,b} = -\frac{\beta^2 \hat{q}_0}{\alpha^2} + \frac{\beta^2 \hat{q}_0/\alpha^2 + \beta w/\alpha}{2}\delta_{a,b} \tag{12}
$$

The resulting expression is handled in such a way to be able to use saddle point methods in the limit $N, \beta \to \infty$ (see [Caccioli et al 2009] for more details on a similar calculation). The final result is given in terms of the free energy

$$
f(q_0, \Phi, \hat{q}_0, w, \hat{R}, z_0) = \frac{s^2 + q_0}{1 + \Phi} + 2\frac{\hat{R}\overline{R}}{\alpha} - 2\frac{\hat{R}z_0}{\alpha} + \frac{\Phi\hat{q}_0}{\alpha} - \frac{wq_0}{\alpha} + \frac{2}{\alpha}\left\langle \min_{z \geq 0}\{V(z)\} \right\rangle_t, \tag{13}
$$

with the potential $V(z)$ given by

$$V(z) = \frac{w}{2}\Delta^2 - \sqrt{\hat{q}_0}t\Delta - \hat{R}\overline{z} + \epsilon\overline{z} \tag{14}$$

and where we used $\langle \cdots \rangle_t$ to denote averages over the normal variable t. The corresponding saddle point equations are

$$w = \frac{\alpha}{1 + \Phi} \tag{15}$$

$$\hat{q}_0 = \frac{\alpha(s^2 + q_0)}{(1 + \Phi)^2} \tag{16}$$

$$\overline{R} = z_0 + \langle \Delta^* \rangle_t \tag{17}$$

$$q_0 = \langle \Delta^{*2} \rangle_t \tag{18}$$

$$\Phi = \frac{\langle t\Delta^* \rangle_t}{\sqrt{\hat{q}_0}} \tag{19}$$

$$\hat{R} = 0, \tag{20}$$

$$\tag{21}$$

where

$$\Delta^*(t) = \theta(t - \tau)\frac{\sqrt{\hat{q}_0}}{w}(t - \tau) + \theta(-t - \tau)\frac{\sqrt{\hat{q}_0}}{w}(-t - \tau) \tag{22}$$

$$\tau = \frac{\epsilon}{\sqrt{\hat{q}_0}}. \tag{23}$$

Using these equations it is possible to compute $\langle H_\epsilon \rangle = \frac{q_0 + s^2}{(1 + \Phi)^2}$. It is useful to define the three functions

$$\psi_r(\tau) = 2\int_\tau^\infty dt\, e^{-t^2/2}(t - \tau) = \sqrt{\frac{2}{\pi}}e^{-\tau^2/2} - \tau\,\mathrm{erfc}\left(\frac{\tau}{\sqrt{2}}\right) \tag{24}$$

$$\psi_q(\tau) = 2\int_\tau^\infty dt\, e^{-t^2/2}(t - \tau)^2 = (1 + \tau^2)\mathrm{erfc}\left(\frac{\tau}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}\tau e^{-\tau^2/2} \tag{25}$$

$$\psi_\Phi(\tau) = 2\int_\tau^\infty dt\, e^{-t^2/2}t(t - \tau) = \mathrm{erfc}\left(\frac{\tau}{\sqrt{2}}\right) \tag{26}$$

It is now possible to express equations (17), (18) and (19) in terms of these non-linear functions. We can now look for a parametric solution in terms of τ, and consider α as an independent variable. From the definition of τ we have $\hat{q}_0 = \epsilon^2/\tau^2$. Inserting (15) into (19) we find

$$\alpha = \frac{1 + \Phi}{\Phi}\psi_\Phi(\tau), \tag{27}$$

while from (18) we get

$$q_0 = \frac{\epsilon^2}{\tau^2}\frac{\psi_q(\tau)}{\psi_\Phi^2(\tau)}\Phi^2. \tag{28}$$

Inserting these expressions into (16) we obtain

$$\frac{\epsilon^2}{\tau^2} = \frac{s^2\psi_\Phi(\tau)}{\Phi(1+\Phi)} + \frac{\epsilon^2}{\tau^2}\frac{\psi_q(\tau)\Phi}{\psi_\Phi(\tau)(1+\Phi)}, \tag{29}$$

from which

$$\Phi_\pm = \frac{-1 \pm \sqrt{1 + 4\psi_\Phi(\tau)s^2\frac{\tau^2}{\epsilon^2}\left(1 - \frac{\psi_q(\tau)}{\psi_\Phi(\tau)}\right)}}{2(1 - \psi_q(\tau)/\psi_\Phi(\tau))}. \tag{30}$$

Since Φ has the meaning of a distance between replicas the only physical solution is $\Phi = \Phi_+$. Inserting this expression for Φ in the previous equations makes possible to express all order parameters and α in terms of the functions ψ_r, ψ_q, ψ_Φ and of the free parameters ϵ and τ.

A parametric solution can be found also for the case of α fixed and ϵ variable. From (19) we find

$$\Phi = \frac{\psi_\Phi(\tau)}{\alpha - \psi_\Phi(\tau)}. \tag{31}$$

From (18)

$$q_0 = \frac{\epsilon^2}{\tau^2}\frac{(1+\Phi)^2}{\alpha^2}\psi_q(\tau). \tag{32}$$

Finally, inserting this expression in (16) we can now express ϵ as:

$$\frac{\epsilon^2}{\tau^2} = \frac{\alpha s^2}{(1+\Phi)^2}\frac{1}{1 - \frac{\psi_q(\tau)}{\alpha}}. \tag{33}$$

As before, using this expression, is now possible to write the order parameters in terms of ψ_r, ψ_q, ψ_Φ and of the free parameters α and τ.

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