Economics
The Open-Access, Open-Assessment E-Journal

Discussion Paper
No. 2008-36 | December 10, 2008 | *http://www.economics-ejournal.org/economics/discussionpapers/2008-36*

Modeling Maximum Entropy and Mean-Field Interaction in Macroeconomics

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Abstract

The representation of the economic system, from a complexity perspective, focuses on interactions among heterogeneous agents in conditions of uncertainty. Heterogeneity entails asymmetric reactions to shocks and, through interaction mechanisms and feedback loops at micro, macro and meso level, these diverse reactions influence behaviours of other agents. Such a system cannot be modelled with mainstream economics' tools. In this work we propose a stochastic dynamic model with heterogeneous firms. Their responses to stochastic shocks, in order to maximize profit, modifies their financial ratios, determining in this way the evolution of the system. The model is analytically solved by means of maximum entropy maximization and master equation's solution techniques (Aoki and Yoshikawa, 2006).

Paper submitted to the special issue "Reconstructing Macroeconomics" (http://www.economics-ejournal.org/special-areas/special-issues)

JEL: E1, E6

Keywords: Business cycles; heterogeneity; financial fragility; stochastic aggregation

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1 Introduction

The use of master equation and maximum entropy is definitely not new in economics, but it has been basically limited so far to financial analysis. Recent works of Masanao Aoki (Aoki, 1996, 2002) and Aoki and Yoshikawa (2006) open the possibility of an useful application of these tools in macroeconomic modeling, in particular the analytical solution of heterogeneous agents models, for which the problem of the aggregation appears particularly tricky. In this way, they open the possibility of developing new analytical frameworks that overcome the representative agent hypothesis and its unsound foundations (Keen, 2001; Kirman, 1992). Following the approach introduced in these works and in Di Guilmi (2008), this paper presents a financial fragility model with heterogeneous agents, originally conceived by Greenwald and Stiglitz (1993) and Delli Gatti et al. (2005), and solves it in a dynamic stochastic framework.

Greenwald and Stiglitz (1993) presented a financial fragility model in which the representative firm, facing uncertain market conditions, adjusts its production level in order to avoid bankruptcy, origining business fluctuations. Delli Gatti et al. (2005) modify the original model allowing firms to be heterogeneous, as regards size and financial conditions. The relevant performances of the latter model in replicating empirical facts demonstrated once again that economy can be better modeled as a complex dynamical system rather than a single and perfect informed agent that produces and consumes.

Structures with non identical and interacting agents are usually solved by means of computer simulations and their analytical solution cannot be reached with conventional economic tools. Aoki, introducing in economics the concept of mean-field interaction, made feasible the analytical modeling of indirect interaction among heterogeneous agents. Mean-field can be defined as the average interaction model that approximates interactions among agents that, otherwise, could not be analytically treated (Opper and Saad, 2001). Agents are grouped according to their micro-state; the joint configuration of their micro-states individuates and determines the aggregate behavior of the system, i. e. the macro-state. Since economy is populated by a large number of agents, researchers cannot predict all their possible configurations, but they can infer the stochastic laws that describes units' behavior. These laws can be estimated through master equation's solution techniques. As Aoki and Yoshikawa (2006) stress: "Precise behavior of each agent is *irrelevant*. Rather we need to recognize that microeconomic behavior is fundamentally stochastic and we need to resort to proper statistical methods to study the macroeconomy consisting of a large number of such agents."

The structure of the work is the following: in section 2, we specify the hypothesis for firms and define the stochastic structure of the system; then, in section 3, we develop and solve the MaxEnt analysis for inference about statical probability; analysis about dynamic behavior of the system is performed in section 4, where we introduce and solve the master equation that fully describe our economy; section 5 concludes.

2 Hypothesis

2.1 Firms' behaviour

We set up a model in continuous time for a system of heterogeneous and interacting agents, partitioned into groups or states. Firms behavior is modelled following Greenwald and Stiglitz (1993) and Delli Gatti et al. (2005). The economy is populated by a fixed number of firms $N = N(t)$, each indexed by i for any given time. Firms are classified in two groups, according to their equity ratio, i. e. the ratio among net worth and total assets. Firms with an equity ratio $a_i(t)$ lower than the threshold $\bar{a}(t)$ have a positive probability of bankruptcy. The system's vector of states $\boldsymbol{\omega}$ is identified by firm's equity ratio in the following way:

$$\boldsymbol{\omega} = \{\omega_i(t) = H\left(a_i(t)|\, \bar{a}\right)\ \forall i\ \leq N\} \quad : \quad H(a_i(t)|\bar{a}) = \begin{cases} 1 \iff a_i(t) < \bar{a} \\ 0 \iff a_i(t) \geq \bar{a} \end{cases}$$

For analytical reasons we set $a^1(t)$ for firms which equity ratio is under the threshold and $a^2(t)$ for firms which equity ratio is over the threshold. In this way it is feasible to obtain the mean-field approximation of interactions among agents. More precisely, $a^j : j = 0, 1$ can be regarded as a statistic V of all the equity ratios for each state:

$$a^j = V\{a_1, ..., a_i, ..., a_N\}\ :\ H(a_i|\bar{a}) = j$$

The production function:

$$q_i(t) = 2(k_i(t))^{1/2} \tag{1}$$

where k is the physical capital, not subject to deterioration, and q is the physical output, determines a demand of capital function equal to:

$$k_i(t) = \frac{1}{2}(q_i(t))^2 \tag{2}$$

Firms sell all the output they optimally decide to produce and there are no stocks. The uncertainty of the market is represented by an iid stochastic multiplicative shock on price. In particular, each firm's selling price is equal to:

$$p_i(t) = P(t)\tilde{u}_i(t) \tag{3}$$

where $P(t)$ is the average price on the market and $\tilde{u_i}(t)$ has uniform distribution with a support that, without loss of generality, can be fixed in the interval $[0.75; 1.25]$, with $\mathbb{E}[\tilde{u}] = 1$. Once a firm get failed, it faces bankruptcy costs growing with the size of firm and quantified by:

$$C_i(t) = c(P_i(t)q_i(t))^2 = c(P(t)u_i(t)q_i(t))^2 \quad : \quad 0 < c < 1 \tag{4}$$

2.2 Stochastic structure

2.2.1 Macro and micro states

The system is articulated in two micro states, corresponding to the two possible types of firms. The cardinality of the j-th state, i. e. the number of firms in state $j = 0, 1$ is given by

$$\left.\begin{array}{l} card\, N^1(t) = \#\{\omega_i(t) = 1 \quad \forall\, i \in I\} = N^1(t) \\ \\ N^0(t) = N - N^1(t) \end{array}\right\} \Rightarrow \mathbf{N}(t) = \big(N^0(t), N^1(t)\big) \tag{5}$$

By assumption, the dynamics of the occupation number N^j follows a continuous time jump Markov process, defined over a state space $\Omega = (x, y)$ equipped with the counting measure $N_{(.)}(.) : \Omega \times \mathbb{T} \to \mathbb{N}$, so that $(\Omega, N_{(.)}(.))$ is a countable sample space. In the following x indicates the case of firms with equity ratio below the threshold $\bar{a}$ and y the alternative case:

$$\omega = x \Leftrightarrow \omega_i(t) = 1 \;\vee\; \omega = y \Leftrightarrow \omega_i(t) = 0 \tag{6}$$

such that $N_{(.)}(.)$ evaluates the cardinality of microstates: $N_{(\omega)}(t) = N^1(t) \iff \omega = x$ and $N_{(\omega)} = N^0(t) \iff \omega = y$. The relative frequency of firms is indicated in small letters: $n_k = N_k/N$. A-priori probability of $\omega = 1$ is indicated by η:

$$p(\omega = x) = \eta \Leftrightarrow p(\omega = y) = 1 - \eta$$

A firm entries the system in state x and fails (exiting from the system) only if it is in state x; in order to maintain constant the number of firms N we assume that each bankrupted firm is immediately substituted by a new one. Therefore failures of firms do not modify the value of $N^1 := N(x)$.

2.2.2 Transition rates

Firms move from x to y or vice versa according to the following transition rates:

$$\begin{array}{l} b(N^j) = r(N^j + 1|N^j) = \zeta \frac{N - N^j}{N}(1 - \eta) \\ d(N^j) = l(N^j - 1|N^j) = \iota \frac{N^j}{N}\eta \end{array} \tag{7}$$

being ζ the transition probability from state y to x (firms whose financial position is deteriorated from a period to another, with equity ratio that becomes lower than $\bar{a}$) and ι the probability of the inverse transition (firms whose equity ratio improved becoming bigger than $\bar{a}$). b and d recall "births" and "deaths" of stochastic processes while r ("right") and l ("left") stand for increment or decrement in the cardinality. Having already indicated with N^j the occupation number of firms in state j, transition rates can be evaluated according to:

$$\begin{array}{l} b(N^j) = r(N^j + 1|N^j) = \lambda(N - N^j) : \lambda = \zeta(1 - \eta) \\ d(N^j) = l(N^j - 1|N^j) = \gamma(N^j) : \gamma = \iota\eta \end{array} \tag{8}$$

In statistical mechanics terms, this kind of system can be defined as a statistical ensemble with conservative cardinality, described by a continuous time Markov process over a discrete state space with the structure of a birth-death process.

A firm goes bankrupted if its own capital A_i becomes null. Given that the only variable that firms cannot know is the price shock, bankruptcy probability can be conveniently expressed as a function of it. In particular, it is possible to identify a failure condition in terms of a shock price threshold $\bar{u}_i(t)$ below which $A_i(t)$ becomes ≤ 0. It can be specified as:

$$\tilde{u}(t') \leq \left(\frac{P(t)}{P(t')}\right)(rk_i(t)/q_i(t) - a^1(t)\frac{k_i(t)}{P(t')q_i(t)}) \equiv \bar{u}_i(t')$$

where $t' - t = \delta t \to 0^+$. Substituting equation (2) into the above expression and normalizing reference price $P(t) = P(t')$ to 1, the r.h.s of above equation becomes:

$$\bar{u}(t') \equiv \frac{q^1(t)}{2}(r - a^1(t)) \tag{9}$$

Since the random variable $\tilde{u}$ has a delimited support, in order to have consistent values of the probability, the critical thresholds of shock prices should be normalized according to:

$$\begin{cases} \bar{u} = 0.75 \iff \tilde{u}_i(t) < 0.75 \\ \bar{u} \in (0.75; 1.25) \iff 0.75 < \tilde{u}_i(t) < 1.25 \\ \bar{u} = 1.25 \iff \tilde{u}_i(t) > 1.25 \end{cases} \tag{10}$$

Then, the probability of failure for a firm $\mu(t)$ can be expressed explicitly as a function of $\tilde{u}(t)$:

$$\mu(t) = F(\tilde{u}(t)) = p(\tilde{u}(t) \leq \bar{u}(t)) = \frac{\bar{u}(t) - 0.75}{0.5} = 2\bar{u}(t) - 1.5 \tag{11}$$

As the threshold $\bar{a}$ identifies the minimum value of equity ratio which ensures the surviving of the firm, i. e. the value of $a_i(t)$ for which $\mu_i(t) = 0$, it can be then quantified by:

$$\bar{a}(t') = r - \frac{1.5}{q^1(t)} \tag{12}$$

With an analogue procedure, the transition probabilities ζ and ι can be specified as dependent on the price shock. Indicating the critical values respectively with $\bar{u}_\zeta(t)$ and $\bar{u}_\iota(t)$ we thus obtain:

$$\begin{array}{l} \tilde{u}_i(t) \leq \frac{q^0(t)}{2}\left(r + \bar{a}(t) - a^0(t)\right) \equiv \bar{u}_\zeta(t) \\ \tilde{u}_i(t) > \frac{q^1(t)}{2}\left(r + \bar{a}(t) - a^1(t)\right) \equiv \bar{u}_\iota(t) \end{array}$$

with ranges of variation of the two thresholds truncated as in (10). It is straightforward now to get the transition probability for each state:

$$\zeta(t) = p(\tilde{u}(t) \leq \bar{u}_\zeta(t)) = 2\bar{u}_\zeta(t) - 1.5 \tag{13}$$

$$\iota(t) = 1 - p(\tilde{u}(t) \leq \bar{u}_\iota(t)) = -2\bar{u}_\iota(t) + 2.5 \tag{14}$$

2.2.3 Firms profit maximization

A firm decides the optimal quantity to produce in order to maximize its profit, using all the available information. Under the stated hypothesis the object function of a generic firm i can be then expressed as:

$$\max_{q_i(t)} F(q_i(t)) := \{\mathbb{E}\left[P(t)u_i(t')q_i(t)\right] - rk_i(t) - C_i(t)\mu(t')\} \tag{15}$$

By assumption, firms take into consideration the present level of failure probability, therefore $\mathbb{E}\left[\mu(t')\right] = \mu(t)$ and prices are normalized such that $\mathbb{E}\left[P(t')\right] = P(t') = P(t) = 1$ without loss of generality. Using equation (2) and considering that $\mathbb{E}[\tilde{u}] = 1$, the argument of the (15) can be rewritten as:

$$q_i(t) - rk_i(t) - C_i(t)\mu(t) = q_i(t) - r\frac{1}{2}(q_i(t))^2 - c(q_i(t))^2\mu(t)$$

Consequently, two different optimal levels of production can be identified, for firms in state x and for firms in state y, respectively:

$$\begin{aligned} q^{1*} &= (r + 2c\mu(t))^{-1} \\ q^{0*} &= r^{-1} \end{aligned} \tag{16}$$

since $\mu = 0$ for firms in state y. Therefore the aggregate production:

$$Y(t) = N^0(t)q^{0*} + N^1(t)q^{1*} = \frac{N^1}{r + 2c\mu(t)} + \frac{N^0}{r} \tag{17}$$

comes out to be dependent on the occupation numbers.

3 The MaxEnt problem

The first step of model's solution consists in estimating the statical probability for a firm to be in one of the two states in a given instant. In particular, the aim is to obtain an estimation without imposing any hypothesis that may drive or restrict firms behavior but just imposing a macro-economic constraint to make probability to be consistent with system's structure. The inference here performed is based on maximum entropy method (MaxEnt). Indeed, the use of statistical entropy emerges as a particularly useful instrument for inference in a context where so few information are available[1]. Maximization of the entropy functional returns the maximum likelihood estimation of probability distribution function in Gibbs form. The statistical entropy of the system is measured, according to Aoki (1996), by the Shannon entropy.

Here arguments of maximization are the occupation numbers and the problem has the following formulation:

$$\max_{N^1, N^0} \; H(N^1, N^0) = -N^1 \log(N^1) - N^0 \log(N^0) \tag{18}$$

[1]MaxEnt problems have been widely addressed in the works of Jaynes (see e. g. Jaynes (1957)). For economic applications see Liossatos (2004) and Landini et al. (2008).

s.t.

$$\begin{cases} N^1(t) + N^0(t) = N \\ N^1(t)y^1(t) + N^0(t)y^0(t) = Y(t) \end{cases} \tag{19}$$

The first constraint ensures the normalization of the probability function, maintaining the number of firms in each group below the total number of firms. In the second, y^1 and y^0 represent the value of production of a single firm for both states. This condition ensures the consistency of the estimation with the underlying economic model. Moreover, it represents a factor of indirect interaction among agents, linking the estimation to aggregate production and, then, to the situation of the other firms in the system. The lagrangean is:

$$\begin{aligned} \ell = \quad & -N^1(t)\log\left(N^1(t)\right) - N^0(t)\log\left(N^0(t)\right) + \delta_1(t)N^1(t) + \delta_1(t)N^0(t) - \delta_1(t)N + \\ & +\delta_2(t)N^1(t)y^1(t) + \delta_2(t)N^0(t)y^0(t) - \delta_2(t)Y(t) \end{aligned}$$

with first order conditions[2]:

$$\begin{cases} \frac{\partial \ell}{\partial N^1(t)} = -\log(N^1(t)) - 1 + \delta_1(t) + \delta_2(t)y^1(t) \\ \frac{\partial \ell}{\partial N^0(t)} = -\log(N^0(t)) - 1 + \delta_1(t) + \delta_2(t)y^0(t) \\ \frac{\partial \ell}{\partial \delta_1(t)} = N - N^1(t) - N^0(t) \\ \frac{\partial \ell}{\partial \delta_2(t)} = Y(t) - N^1(t)y^1(t) - N^0(t)y^0(t) \end{cases} \tag{20}$$

Equating the (20) to 0, and substituting $\delta_1(t) = 1 - \alpha(t)$ and $\delta_2(t) = -\beta(t)$, we obtain:

$$\begin{cases} N^1(t) = e^{-(\alpha(t)+\beta(t)y^1(t))} \\ N^0(t) = e^{-(\alpha(t)+\beta(t)y^0(t))} \\ N^1(t) + N^0(t) = N \\ N^1(t)y^1(t) + N^0(t)y^0(t) = Y(t) \end{cases}$$

Substituting the first two equations in the third and rearranging, it becomes:

$$e^{-\alpha(t)} = \frac{N}{e^{-\beta(t)y^1(t)} + e^{-\beta(t)y^0(t)}}$$

which, substituted in the last of the (20), generates:

$$e^{-\beta(t)y^1(t)}y^1(t) + e^{-\beta(t)y^0(t)}y^0(t) = Y(t)\frac{e^{-\beta(t)y^1(t)} + e^{-\beta(t)y^0(t)}}{N}$$

Indicating $\bar{y}(t) = Y(t)/N$, we obtain therefore:

$$(y^1(t) - \bar{y}(t))e^{-\beta(t)y^1(t)} + (y^0(t) - \bar{y}(t))e^{-\beta(t)y^0(t)} = 0 \tag{21}$$

that, in statistical mechanics terms, is defined as *quantum anomalies* equation. Its l.h.s. measures the distance of the actual production from an ideal situation

[2]As demonstrated in (Landini, 2005, pag. 146) the first order condition are also sufficient.

in which all firms, being safe from failure, are in the state with the higher level of production. $\beta(t)$ then emerges as a useful synthetic indicator for the situation of the economy. Its estimation as a solution for equation (21) permits to explicit the theoretical probability of an agent to be in state x or y, conditioned on the present value of N^1 and N^0. Since:

$$N^j(t) = e^{-\alpha(t)} e^{-\beta(t) y^j(t)}$$

for $j = 0, 1$, then:

$$p^j(t) = \frac{N^j(t)}{N} = \frac{e^{-\beta(t) y^j(t)}}{Z} \tag{22}$$

where Z represents the partition function. Solution of equation (21) is the estimation for $\beta(t)$:

$$\beta(t) = \ln\left(-\frac{y^1(t) - \bar{y}(t)}{y^0(t) - \bar{y(t)}}\right)\left(y^1(t) - y^0(t)\right)^{-1} \tag{23}$$

Using equations (16), we can express equation (23) in the following way:

$$\beta(t) = \ln\left(\frac{N^0(t)}{N^1(t)}\right)\left(\frac{r + 2c\mu(t)}{2c\mu(t)}\right) = \ln\left(\frac{N - N^j(t)}{N^j(t)}\right)\left(\frac{r + 2c\mu(t)}{2c\mu(t)}\right) \tag{24}$$

If the proportion of firms in state x increases, β drops, until it becomes less than 0, revealing an under-performance of the economy due to the risk of bankruptcy of heavily indebted firms. Indeed, the sign of β is determined by the relative proportion of firms in the two states. The distance from 0 basically due to the disproportion among the two occupation numbers (enforced by the interest rate effect), given that:

$$\begin{array}{ll} (N^1(t) \to N) & \Rightarrow \beta \to -\infty \\ (N^1(t) \to 0) & \Rightarrow \beta \to +\infty \end{array}$$

In this view β may be regarded also as an index of the uncertainty of the system, given that in the case of an approximately equal proportion of firms in the two states the parameter will tend to 0.

Now we can explicit equations (22) for $j = 0, 1$:

$$p^0(t) = Z^{-1}\left(\frac{N - N^1(t)}{N^1(t)}\right)^{\frac{1}{2c\mu(t)}} \tag{25}$$

$$p^1(t) = Z^{-1}\left(\frac{N - N^1(t)}{N^1(t)}\right)^{\frac{r+2c\mu(t)}{r2c\mu(t)}} \tag{26}$$

These equations represent the maximum likelihood estimations for the probability density function for a firm to enter in state x or y, where Z is the partition function.

4 Dynamic analysis and master equation

Since the evolution of the occupation numbers is assumed to be a jump Markov chain, the dynamics of the joint probabilities, and, by this way, the stochastic evolution of the system can be conveniently described by a master equation[3]. In particular, the dynamics of the probability of having N_k firms in state x at time t can be described in the following way:

$$\begin{aligned} \tfrac{dP(N_k,t)}{dt} &= b(N_{k-1}(t))P(N_{k-1}(t)) + d(N_{k+1}(t))P(N_{k+1}(t)) + \\ &- \{[(b(N_k(t)) + d(N_k(t)))P(N_k(t))]\} \end{aligned} \tag{27}$$

with boundary conditions:

$$\begin{cases} P(N,t) = b(N^1(t))P(N^1-1,t) + d(N(t))P(N,t) \\ P(0,t) = b(1)P(1,t) + d(0)P(0,t) \end{cases} \tag{28}$$

These conditions ensure that the distributions functions consider only consistent values, that is to say $N^1 \in [0;N]$. Equation (28) is a balance flow equation between probability fluxes in and out from state x.

An analytical solution for master equations can be obtained only under very specific conditions (Risken, 1989), thus we solve it by means of an approximation method based on led and lag operators[4]. The state variable is modified, assuming that the fraction of firms in state x at a given moment is determined by its expected mean (m), the drift, and by an additive fluctuations component of order $N^{-1/2}$ around this value, the spread:

$$N^1(t) = Nm(t) + \sqrt{N}s \tag{29}$$

In Appendix A it is demonstrated that, starting from (29) it is possible to obtain an asymptotically approximated solution for each of the two components of the dynamics:

$$\frac{dm}{d\tau} = \lambda m(t) - (\lambda+\gamma)m^2(t) \tag{30}$$

$$\frac{\partial Q}{\partial \tau} = [2(\lambda+\gamma)m(t) - \lambda]\frac{\partial}{\partial s}(sQ(s)) + \frac{[\lambda m(t)(1-m(t)) + \gamma m^2(t)]}{2}\left(\frac{\partial}{\partial s}\right)^2 Q(s) \tag{31}$$

The ODE (30) describes the dynamics of the trend and can be defined as macroeconomic equation. The probability flow of the spreading component is quantified by equation (31), that is termed as *Fokker-Planck equation*. The asymptotic solution of this PSDE identifies the stationary distribution for fluctuations. In the same way, setting the l.h.s. of macroscopic equation (30) to 0, we identify the steady state value for m:

$$m^* = \frac{\lambda}{\lambda+\gamma} \tag{32}$$

[3]See Aoki (2002, chap. 3), Di Guilmi (2008, chap. 3) and Landini (2005, page 252).

[4]For a detailed exposition see Aoki (2002) or Landini (2005).

The direct integration of equation (30) returns:

$$m(\tau) = \frac{\lambda}{(\lambda+\gamma) - ke^{-\psi\tau}} \quad : \quad \begin{cases} k = 1 - \frac{m^*}{m(0)} \\ \psi = \frac{(\lambda+\gamma)^2}{\lambda} \end{cases} \tag{33}$$

Solution of the equation for the spread component, presented in Appendix B, is represented by the distribution function, here indicated with θ, for the spread s, that identifies the probability distribution of fluctuations:

$$\theta(s) = C \exp\left(-\frac{s^2}{2\sigma^2}\right) \quad : \quad \sigma^2 = \frac{\lambda\gamma}{(\lambda+\gamma)^2} = m^* \frac{\gamma}{\lambda+\gamma} \tag{34}$$

Both the dynamics of fraction of firms occupying state 1 and the distribution of fluctuations appear as fully dependent on transition rates.

The temporal evolution of aggregate production can be then quantified accordingly. Rearranging equation (17), using the (32), the long run level of aggregate production can be expressed as:

$$Y^e = N\left[\frac{1}{r} - \frac{\lambda}{\lambda+\gamma}\frac{2c\mu}{r(r+2c\mu)}\right] = N\left[\frac{1}{r} - \frac{\lambda}{\lambda+\gamma}\left(y^1 - y^0\right)\right] \tag{35}$$

Production dynamics appears then to be dependent on the transition rates λ and γ and on the differences in firms level of production between microstates. The dynamics of these factors are studied in the following section.

4.1 Equilibrium distribution and critical points

The analytical condition for the equilibrium probability is obtained by equating the master equation to 0 for all the possible macro-states, condition known as *detailed balance*. Considering the applicability in markovian spaces of the Brook's lemma (Brook, 1964) and making use of the Hammersley and Clifford theorem (Clifford, 1990), the stationary probability of the process for N^j, provided that detailed balance holds, can be expressed by:

$$P^e(N^j) \propto Z^{-1} e^{-\beta N U(N^j)} \tag{36}$$

where $U(N^j)$ is the Gibbs potential (Picardello and Woess, 1999). This leads to the identification of a functional form for the probability η:

$$\eta(N^j) = N^{-1} e^{\beta g(N^j)} \tag{37}$$

where $g(N^j)$ is a function that evaluates the relative difference in the outcome as a function of N^j. Large values of β associated with positive values of $g(N^j)$ cause $\eta(N^j)$ to be larger than $1-\eta(N^j)$, making transition from state y to state x more likely to occur than the opposite one. On the other hand, values of β close to 0, make $\eta(N^j)$ close to 0.5. In models with two states, for a great N, the equation of the potential is:

$$U(N^j) = -2\int_0^{N^j} g(N^j)dy - \frac{1}{\beta}S(\mathbf{N})$$

where $S(\mathbf{N})$ is the Shannon entropy for the vector of occupation numbers. In order to individuate the stationary points of probability dynamics, we have to identify its peak. β is an inverse multiplicative factor for entropy: this implies then that, for very large values of β, the entropy component has a negligible impact and the stationary points can be unambiguously identified. With reference to our economic system, a relative high value of β translates in a low number of firms exposed at bankruptcy risk and a limited impact of the bankruptcy's probability μ (Aoki, 1996, pp. 55 and followings). On the contrary, as β approaches 0, the weight of the entropy component grows. Aoki (2002) shows that the points in which the potential is minimized are also the critical point of the aggregate dynamics of p_j. Therefore, in order to individuate the peak of probability dynamic we need to find the local minimum of the potential. The first order derivative of the potential with respect to N^j:

$$g(N^j) = -\frac{1}{2\beta}\frac{\partial S(N^j)}{\partial N^j} = -\frac{1}{2\beta}\ln\left(\frac{N^j}{N - N^j}\right) \tag{38}$$

Setting $U' = 0$ and using equation (23), we get an explicit formulation for $g(N^1)$ in stationary conditions:

$$g(N^1) = \frac{y_0 - y_1}{2} \tag{39}$$

that quantifies the mean difference (for states) of the outcome.
From equation (38), it follows that the point of local minimal of the potential is given by:

$$U' = 0 \Rightarrow e^{2\beta g(N^j)} = \frac{N^j}{N - N^j} \tag{40}$$

that, if the rates of entries and exits are equated, (i. e. if $\iota = \zeta$), reproduces exactly the (29). Therefore, making use of the (37) we can write:

$$P(N^j) = \frac{e^{2\beta g(N^j)}}{e^{2\beta g(N^j)} + e^{-2\beta g(N^j)}} \tag{41}$$

that is the maximum likelihood estimation of the Gibbs distribution of the number of firms in state j. Let us analyze the different behavior of the stationary distribution for different values of β.

For $\beta \to \infty$, equation (38) shows that the critical points in which the potential is minimized are also the zeros of the function $g(N^j)$:

$$U'(m^*) = -2g(m^*) = 0 \tag{42}$$

This suggests that β may be interpreted as an inverse index of uncertainty. From equation (39) it follows that:

$$g(N^j) = 0 \Leftrightarrow y_0 - y_1 = 0$$

Under these conditions, there is no uncertainty in the system, since no firm can go bankrupted. Indeed β can go to infinity if $N^0 \to N$ or if $\mu \to 0$, since both situation imply a convergence among the different targets of production at micro level and, then, a minimum degree of uncertainty in the system.

For $\beta \to 0$, in order to individuate the critical points, a further deepening is needed. β can go to 0 if and only if $\frac{N^j}{N-N^j} \to 1$, that is to say, if the system has the same proportion of firms in the two states. But this is not informative about the behavior of the $g(N)$ since the first factor of the equation (38) goes to infinity (given that $\beta \to 0$) while the second goes to 0. We can employ Cox and Miller hazard function (Cox and Miller, 1996), setting an underlying density function analogous to a standard Brownian motion's first-passage(Grimmett and Stirzaker, 1992), and express it as a function of m:

$$F(m) = \left[1 + e^{-2\beta m}\right]^{-1} \Rightarrow h(m) = \frac{2\beta}{1 + e^{-2\beta m}} \tag{43}$$

The aim is to calculate the probability that a firm passes from a state to another in response to a small variation in the difference of relative production, conditional on the current difference among y_0 and y_1, quantified by $g(m)$ in equation (39). We can then rewrite the conditional hazard function in the following way:

$$P(v \leq y_1 - y_0 | m^*) = \left[1 + e^{-2\beta g(m^*)}\right]^{-1} \tag{44}$$

and then:

$$h(y_1 - y_0 | m^*) = \frac{2\beta\eta(m^*)}{1 + e^{-2\beta g(m^*)}} \tag{45}$$

Supposing that $\eta(m^*) = m^*$, we finally obtain:

$$h(y_1 - y_0 | m^*) = 2\beta m^* \tag{46}$$

Therefore, we may conclude that for values of β close to 0, the critical point of probability dynamics, here found by minimizing the potential, is a value of m^* approximately equal to β itself. In other words, β may be considered the as the conditional hazard rate in the range where β is small. The potential then is minimized for a fraction m^* of firms in state x when the value of the conditional hazard function is approximately equal to β.

5 Concluding remarks

In this work we present an application of a statistical mechanic approach, inspired by Aoki's methodology, to a macroeconomic model of financial fragility of the type presented in Greenwald and Stiglitz (1993) and Delli Gatti et al. (2005). The modeling of agents' behavior as a mean field interaction and the use of stochastic dynamic aggregation tools permit to identify a stable analytical solution. Starting from very general conditions and hypothesis, embodied in a stochastic framework, we finally obtain a system of coupled equations that describes the evolution of the system and its long term equilibrium solution. The dynamics is analyzed by means of master equation solution techniques enriched by the use of MaxEnt and hazard function analysis.

Appendix A

The master equation (27) must be modified, according to equation (29), and expressed as $\dot{Q}(s)$, a function of s:

$$\dot{P}(N_k) = \frac{\partial Q}{\partial t} + \frac{ds}{dt}\frac{\partial Q}{\partial s} = \dot{Q}(s) \tag{47}$$

with transition rates reformulated in the following way:

$$b(s) = \lambda\left[N - Nm - \sqrt{N}s\right] \tag{48}$$

$$d(s) = \gamma\left[Nm + \sqrt{N}s\right] \tag{49}$$

Since

$$\frac{ds}{dt} = -N^{1/2}\frac{dm}{dt} \tag{50}$$

equation (47) can be expressed as:

$$\dot{Q}(s) = \frac{\partial Q}{\partial t} - N^{1/2}\frac{\partial Q}{\partial s}\dot{m} \tag{51}$$

Now we rewrite again the master equation (27) and the transition rates by means of lead and lag operators. These operators make the two probability flows (in and out) homogeneous. Specifically the transition probabilities (8) become:

$$L[d(N_k)P(N_k,t)] = d(N_{k+1})P(N_{k+1}) \tag{52}$$

$$L^{-1}[b(N_k)P(N_k,t)] = d(N_{k-1})P(N_{k-1}) \tag{53}$$

so that the master equation will be expressed in this way:

$$\dot{Q}(s) = (L-1)[d(s)Q(s)] + (L^{-1}-1)[d(s)Q(s)] \tag{54}$$

Using the modified transition rates (52) and expanding the thus obtained master equation in inverse powers of s to the second order we get:

$$\begin{aligned}
&N^{-1}\tfrac{\partial Q}{\partial \tau} - N^{-1/2}\tfrac{dm}{d\tau}\tfrac{\partial Q}{\partial s} = \\
&N^{-1/2}\left(\tfrac{\partial}{\partial s}\right)[d(s)Q(s)] + N^{-1}\tfrac{1}{2}\left(\tfrac{\partial}{\partial s}\right)^2[d(s)Q(s)] + \\
&-N^{-1/2}\left(\tfrac{\partial}{\partial s}\right)[b(s)Q(s)] + N^{-1}\tfrac{1}{2}\left(\tfrac{\partial}{\partial s}\right)^2[b(s)Q(s)] + ... \\
&= N^{-1/2}\left(\tfrac{\partial}{\partial s}\right)[(d(s)-b(s))Q(s)] + N^{-1}\tfrac{1}{2}\left(\tfrac{\partial}{\partial s}\right)^2[(b(s)+d(s))Q(s)] + ...
\end{aligned} \tag{55}$$

where $\tau = t/N$. At this point, in order to match the component of the same orders of powers of N between and equations (47) and (55), we need to rescale the variable $\tau = tN$. Knowing that:

$$\begin{aligned}
d(s) - b(s) &= (\lambda+\gamma)(Nm + \sqrt{N}s) - \lambda N = (\lambda+\gamma)N_k - \lambda N \\
d(s) + b(s) &= (\lambda-\gamma)(Nm + \sqrt{N}s) + \lambda N = (\lambda-\gamma)N_k + \lambda N
\end{aligned}$$

and taking the derivatives, up to the second order, it is possible to obtain what Aoki (2002) defines as *diffusion approximation*:

$$N^{-1}\frac{\partial Q}{\partial \tau} - N^{-1/2}\frac{dm}{d\tau}\frac{\partial Q}{\partial s} = \\ (\lambda+\gamma)Q(s) + N^{-1/2}(d(s)-b(s))\left(\frac{\partial}{\partial s}\right)Q(s) + N^{-1}\frac{1}{2}(b(s)+d(s))\left(\frac{\partial}{\partial s}\right)Q(s) \tag{56}$$

Applying the polynomial identity principle to equation (56) for powers of N of order -1 we get a formulation for the *Fokker-Plank* equation:

$$\begin{aligned} N^{-1}\frac{\partial Q}{\partial t} &= -b'(m)s\frac{\partial Q}{\partial s} + b(m)\frac{1}{2}\left(\frac{\partial}{\partial s}\right)^2 Q(s) - b'(m)sQ(s) \\ &+ d'(m)s\left(\frac{\partial}{\partial s}\right)Q(s) + d(m)\frac{1}{2}\left(\frac{\partial}{\partial s}\right)^2 Q(s) - d'(m)Q(s) \\ &= (d'(m)-b'(m))\left(\frac{\partial}{\partial s}\right)Q(s) + \frac{1}{2}(d(m)+b(m))\left(\frac{\partial}{\partial s}\right)^2 Q(s) \end{aligned} \tag{57}$$

Then, generic asymptotic approximated solution for the master equation will be given by the solution of the following coupled dynamical system of equations:

$$\begin{cases} \frac{dm}{d\tau} = \rho(m) \\ \frac{\partial Q}{\partial \tau} = -\rho'(m)\left(\frac{\partial}{\partial s}\right)(sQ(s)) + \frac{1}{2}\alpha(m)\left(\frac{\partial}{\partial s}\right)^2 Q(s) \\ s.t. \quad \rho(m) = b(m) - d(m), \quad \alpha(m) = b(m) + d(m) \end{cases} \tag{58}$$

In order to arrive at an explicit solution we introduce a modification in the transition rates of equations (8), supposing that the probability η is equal to the observed frequency of firms occupying state 1. The new transition rates are then:

$$\begin{cases} b_n = r(N_k+1|N) = \zeta\frac{N_k}{N}\frac{N-N_k}{N} \\ d_n = l(N_k-1|N) = \iota\frac{N_k}{N}\frac{N_k-1}{N} \end{cases} \tag{59}$$

where the factor $\frac{N_k}{N}$ allows to interpret, respectively, the probability transition ζ as a constant of proportionality between the birth rate per individual and the deviation from the upper bound $N - N_k$ and the probability transition ι as a constant of proportionality between the death rate per individual and the deviation from the lower bound or $N_k - 1$. Given that, one can set the two functions:

$$\begin{aligned} \lambda(N_k) &= \lambda\frac{N_k}{N} \\ \gamma(N_k) &= \gamma\frac{N_k}{N} \end{aligned} \tag{60}$$

Substituting the transition rates (60) in the master equation (27) and collecting terms with λ and γ, after some simple but tedious algebraic passages, we obtain:

$$\begin{aligned} \frac{dP}{dt} = N^{-2}\{&\gamma\left[N_k(N_k+1)L(P) + 2nP\right] + \\ &-\lambda\left[(N_k-1)(N-N_k+1)L^{-1}(P) + (N-2n+1)P\right]\} \end{aligned} \tag{61}$$

where $L(P)$ and $L^{-1}(P)$ are lead and lag operators, reformulated in the following way according to Aoki (1996) and Landini and Uberti (2008):

$$L(P) = \sum_{z=1}^{\infty} \frac{N^{-z/2}}{z!} \left(\frac{\partial}{\partial s}\right)^{z} Q(s) \tag{62}$$

$$L^{-1}(P) = \sum_{z=1}^{\infty} \frac{(-)^{z} N^{-z/2}}{z!} \left(\frac{\partial}{\partial s}\right)^{z} Q(s) \tag{63}$$

Substituting the above indicated operators into equation (61), it becomes:

$$\begin{array}{l} \frac{dP}{dt} = N^{-2} \left\{ \sum_{z=1}^{\infty} \left[D(N_k) + (-)^{z} B(N_k)\right] \frac{N^{-z/2}}{z!} \left(\frac{\partial}{\partial s}\right)^{z} Q(s) \right\} + \\ + N^{-2} \left\{ \left[2\gamma N_k - \lambda(N - 2n + 1)\right] Q(s) \right\} \end{array} \tag{64}$$

where:

$$\left\{ \begin{array}{l} B(N_k) = \lambda(N_k - 1)(N - N_k + 1) = \lambda N(N_k - 1) - \lambda(N_k - 1)^2 = B(m) \\ D(N_k) = \gamma N_k(N_k + 1) = D(m) \end{array} \right. \tag{65}$$

The specification of the drift displayed in equation (29) implies that:

$$\left\{ \begin{array}{l} N_{k+1} = Nm + \sqrt{N}(s + N^{-1/2}) \\ N_k = Nm + \sqrt{N}s \\ N_{k-1} = Nm + \sqrt{N}(s - N^{-1/2}) \end{array} \right. \tag{66}$$

Using these specifications in equation (65), it turns out to be:

$$\left\{ \begin{array}{l} B(m) = \lambda \left[N^2 m(1 - m) + N^{3/2} s(1 - 2m) + N(2m - s^2 - 1) + 2N^{1/2} s - 1\right] \\ D(m) = \gamma \left[N^2 m^2 + N^{3/2} 2ms + N(m + s^2) + N^{1/2} s\right] \end{array} \right. \tag{67}$$

Now, expanding to the second order equation approximation equation (64), that is to say for $z = 1, 2$, we get:

$$\left\{ \begin{array}{lll} z = 1 & : & N^{-1/2}\left[D(m) - B(m)\right] = \\ & & N^{3/2}\left[-\lambda m(1 - m) + \gamma m^2\right] + N\left[2ms(\lambda + \gamma) - \lambda\langle s\rangle\langle s\rangle\right] + \\ & & + N^{1/2}\left[s^2(\lambda + \gamma) + (\gamma - 2\lambda)m + 1\lambda\right] - 2\lambda s + N^{-1/2}(\gamma s - \lambda) \\ z = 2 & : & \frac{N^{-1}}{2}\left[D(m) + B(m)\right] = \\ & & N\left[\lambda m(1 - m) + \gamma m^2\right] + N^{1/2}\left[2ms(\lambda + \gamma) - \lambda s\right] + \\ & & + \left[s^2(\gamma - \lambda) + m(\gamma + 2\lambda) - \lambda\right] + N^{-1/2} 2\lambda s + N^{-1}(\gamma s + \lambda) \end{array} \right. \tag{68}$$

and substituting it into equation (64), it gives back the following approximated master equation:

$$\begin{aligned}\frac{dP}{dt} = & \left\{N^{-1/2}\left[-\lambda m(1-m)+\gamma m^2\right]+N^{-1}\left[2ms(\lambda+\gamma)-\lambda s\right]\right\}\frac{\partial}{\partial s}Q(s)+\\ & +\left\{-N^{-3/2}\left[s^2(\lambda+\gamma)+(\gamma-2\lambda)m+\lambda\right]-N^{-2}2\lambda s+\right.\\ & \left.-N^{-5/2}(\gamma s-\lambda)\right\}\frac{\partial}{\partial s}Q(s)+\frac{1}{2}\left\{N^{-1}\left[\lambda m(1-m)+\gamma m^2\right]+\right.\\ & \left.+N^{-3/2}\left[2ms(\lambda+\gamma)-\lambda s\right]\right\}\left(\frac{\partial}{\partial s}\right)^2 Q(s)+\\ & +\frac{1}{2}\left\{N^{-2}\left[s^2(\gamma-\lambda)+m(\gamma+2\lambda)-\lambda\right]+\right.\\ & \left.+N^{-5/2}\lambda s+N^{-3}(\gamma s+\lambda)\right\}\left(\frac{\partial}{\partial s}\right)^2 Q(s)+\\ & +\left\{N^{-1}\left[2m(\lambda+\gamma)-\lambda\right]+N^{-3/2}\left[2s(\lambda+\gamma)\right]-N^{-2}\lambda\right\}\end{aligned} \tag{69}$$

Considering that:

$$\frac{dP}{dt}=\frac{\partial Q}{\partial t}-N^{-1/2}\frac{dm}{dt}\frac{\partial Q}{\partial s} \tag{70}$$

in order to match the higher order terms in powers of N, we have rescale time as $t = N\tau$:

$$\frac{dP}{dt}=\frac{\partial Q}{\partial t}-N^{-1/2}\frac{dm}{dt}\frac{\partial Q}{\partial s}\Leftrightarrow N^{-1}\frac{dP}{d\tau}=N^{-1}\frac{\partial Q}{\partial \tau}-N^{-1/2}\frac{dm}{d\tau}\frac{\partial Q}{\partial s} \tag{71}$$

Then we have to equal the so obtained two formulations for the master equation: equation (69) and equation (71). It can be done matching the terms that have the same power of N. Then we collect terms of order N^{-1} in equation (69) so as to match them with $\partial Q/\partial\tau$ of equation (71) and those of order $N^{-1/2}$ to set them equal to $N^{-1/2}\dot{m}\frac{\partial Q}{\partial s}$. All the other terms asymptotically vanish as $N\to\infty$. In this way we get:

$$-N^{-1/2}\frac{dm}{d\tau}\frac{\partial Q}{\partial s}=-N^{-1/2}\left[\lambda m(1-m)-\gamma m^2\right]\frac{\partial}{\partial s}Q(s) \tag{72}$$

$$\begin{aligned}N^{-1}\frac{\partial Q}{\partial \tau} = & N^{-1}\left[2m(\lambda+\gamma)-\lambda\right]\frac{\partial}{\partial s}(sQ(s))+\\ & +\frac{N^{-1}}{2}\left[\lambda m(1-m)+\gamma m^2\right]\left(\frac{\partial}{\partial s}\right)^2 Q(s)\end{aligned} \tag{73}$$

Asymptotically approximated solution of master equation is given by the following system of coupled equations:

$$\frac{dm}{d\tau}=\lambda m-(\lambda+\gamma)m^2 \tag{74}$$

$$\frac{\partial Q}{\partial \tau}=\left[2(\lambda+\gamma)m-\lambda\right]\frac{\partial}{\partial s}(sQ(s))+\frac{\left[\lambda m(1-m)+\gamma m^2\right]}{2}\left(\frac{\partial}{\partial s}\right)^2 Q(s) \tag{75}$$

Appendix B

We determine here in what follows a solution for Fokker-Planck equation in terms of $Q(s)$. Indicating with $\theta(s)$ the stationary probability for $Q(s)$ and setting the equilibrium condition $\dot{Q}=0$ (that implies $\dot{\theta}=0$), it is possible to obtain:

$$- \left[2(\lambda+\gamma)m^* - \lambda\right] s\theta(s) = \frac{\left[\lambda m^*(1-m^*) + \gamma m^{*2}\right]}{2} \left(\frac{\partial}{\partial s}\right) \theta(s) \tag{76}$$

Rewriting (76) more conveniently as:

$$\frac{2\left[\lambda - 2(\lambda+\gamma)m^*\right]}{\left[\lambda m^*(1-m^*) + \gamma m^{*2}\right]} s = \frac{1}{\theta(s)} \left(\frac{\partial}{\partial s}\right) \theta(s) \tag{77}$$

and integrating it with respect to s, we obtain:

$$\begin{gathered} \log\theta(s) = C + \tfrac{\lambda - 2(\lambda+\gamma)m^*}{\lambda m^*(1-m^*)+\gamma m^{*2}} s^2 \\ \Updownarrow \\ \theta(s) = C \exp\left(\tfrac{\lambda - 2(\lambda+\gamma)m^*}{\lambda m^* + (\gamma-\lambda)m^{*2}} s^2\right) \end{gathered} \tag{78}$$

Then, substituting $m^* = \lambda/(\lambda+\gamma)$, we get the final result:

$$\theta(s) = C \exp\left(-\frac{s^2}{2\sigma^2}\right) \quad : \quad \sigma^2 = \frac{\lambda\gamma}{(\lambda+\gamma)^2} \tag{79}$$

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