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Macroeconomic Relaxation: Adjustment Processes of Hierarchical Economic Structures

Raymond J. Hawkins and Masanao Aoki

University of Arizona, Tucson; University of California, Los Angeles

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Abstract

We show how time-dependent macroeconomic response follows from microeconomic dynamics using linear response theory and a time-correlation formalism. This theory provides a straightforward approach to time-dependent macroeconomic model construction that preserves the heterogeneity and complex dynamics of microeconomic agents. We illustrate this approach by examining the relationship between output and demand as mediated by changes in unemployment, or Okun's law. We also demonstrate that time dependence implies overshooting and how this formalism leads to a natural definition of economic friction.

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Correspondence:
Raymond J. Hawkins, rhawkins@optics.arizona.edu;
Masanao Aoki, aoki@econ.ucla.edu
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1 Introduction

In recent years a new stochastic-based macroeconomics has emerged that provides a comprehensive alternative approach to our understanding of economic policy issues[1]. Of particular importance, and unlike traditional macroeconomics, this approach deals directly with economic fluctuations and with the inherently heterogeneous nature of economic participants. Drawing as it does on statistical physics, the approach implicitly provides a framework for the understanding of how the dynamics of macroeconomic observables follow from changes in economic microstructure without resorting to the notion of the representative agent and the purpose of this paper is to demonstrate that this is so. This is of particular importance for the large number of economic systems in which hierarchical structure has been found. As we shall see, hierarchical economic structure, a consequence of the heterogeneity of economic agents, has profound implications for the time dependence of macroeconomic adjustment processes. Consequently, aggregation of microeconomic dynamics into macroeconomic response that preserves hierarchical structure at the microeconomic level is crucial for economic policy design,

Macroeconomic adjustment, or relaxation, is the time-dependent modification of economic relationships often expressed as an elasticity. Elasticity is manifest in every linear relationship between macroeconomic variables. Implicit in this ubiquitous notion, however, is instantaneous response: the linear relationship holds at all times. This is, however, well known to be at odds with experience as restructuring of the economy at the microeconomic level is often required for full realization of a macroeconomic observable. Our approach to the introduction of time-dependence into macroeconomics is based on the observation that the formal assumptions underlying time-dependent elasticity in macroeconomics are identical to the assumptions underlying the formal treatment of a variety of relaxation processes in condensed-matter physics including magnetic, dielectric, and anelastic relaxations. All these physical phenomena involve time-dependent relaxations toward newly established equilibria that follow from a change in a driving force and can be described in terms of linear response theory. Since these physical phenomena share a common mathematical description of relaxation/response we make the *ansatz* that macroeconomic phenomena sharing these underlying assumptions will also share this common mathematical description. Furthermore, relaxation is an external manifestation known to reflect the adjustment of internal variables to new equilibrium values and it is through this mechanism that microeconomics and macroeco-

[1] The primary references to this development are Aoki (1996, 2000) and Aoki & Yoshikawa (2007).

nomics can be linked[2].

We begin in Sec. 2 by considering the equilibrium relationship between output and demand, the time-dependent manner in which changes in demand are manifest in output and the assumptions that these observations entail. We then derive the dynamics of output in Sec. 2.2 as a set of response functions consistent with these assumptions. To show how these response functions are consistent with dynamics at the microeconomic level, we introduce the notion of internal economic variables (in this case unemployment) in Sec. 2.3 and demonstrate that a simple exponential response of output to a demand shock can be expressed as the result of a time-dependent change in the unemployment rate. Okun's law - the relationship between output and unemployment - arises naturally in this derivation. We generalize this link between macroeconomic response and microeconomic dynamics to include heterogeneous agents in Sec. 2.4 where, through linear response theory, we find the macroeconomic solution to a microeconomic problem is reduced to the calculation of the correlation function for the macroeconomic variable. We develop this notion for the unemployment model introduced in Aoki & Yoshikawa (2005, 2007) which links the dynamics of output to the solution of a master equation for a hierarchical unemployment state space which is known to give rise to a rich collection of response functions. Response functions of this type are related to the concept of friction and in Sec. 3 we show how the time-dependent restructuring of unemployment gives rise to economic friction. We close with a discussion and summary in Sec. 4.

2 Output Dynamics

2.1 Elasticity, Anelasticity and Econometric Models

Fundamental to essentially all discussions of output and demand is the notion that there exists and equilibrium relationship between output Y and demand D that is of the form

$$\tilde{Y} = J\tilde{D} \ , \tag{1}$$

where the tilde indicates equilibrium and $J = 1$. This relationship is characterized by three features: (i) a unique equilibrium output for each level of demand, (ii) instantaneous achievement of the equilibrium response and (iii) linearity of the response. We note in passing that the equilibrium output is completely recoverable.

The empirical dynamics of output, however, demonstrate that the equilibrium response is not achieved instantaneously and a lagged response is

[2]Indeed it is on this point that we extend our phenomenological theory of administered-rate dynamics (cf. Hawkins & Arnold (2000)) to the formal model of macroeconomic dynamics presented herein.

commonly observed. To incorporate this observed lag, previous research has augmented Eq. 1 with *ad hoc* "partial adjustment" models of the form

$$Y(t_n) = \sum_{i=0}^{N} \left[a_i Y(t_{n-i}) + b_i D(t_{n-i}) \right] \ . \tag{2}$$

While these and related vector autoregression models often adequately describe observed macroeconomic dynamics, they largely lack a theoretical basis with which to interpret the resulting parameters and with which to link the model to policy.

Alternatively, the theory of anelasticity generalizes ideal (i.e. instantaneous) elasticity as expressed, for example, in Eq. 1 to allow for time-dependent response[3]. Like previous treatments of output-demand dynamics it assumes the existence of a unique equilibrium relationship between stress and strain known as Hooke's law. The equilibrium relationship given in Eq. 1 is, in fact, identical to the scalar version of Hooke's law of ideal elasticity $\epsilon = J\sigma$ with output playing the role of strain ϵ and demand playing the role of stress σ. This suggests the identification of J in Eq. 1 as the compliance of the economy and that we write $D = MY$ where M is the modulus of the economy and $J = 1/M$.

2.2 Phenomenology

The dynamics of anelasticity are obtained by noting that the assumption of linearity implies a general demand-output relationship of the form

$$\left(a_0 + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} + \cdots \right) Y = \left(b_0 + b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2} + \cdots \right) D \ . \tag{3}$$

While the econometric application of this equation, like Eq. 2, requires an analysis of the number of terms needed to describe the observed dynamics, the use of Eq. 3 enables a straightforward economic interpretation of these terms and the coefficients. In practice a wide range of relaxation dynamics have been found to be described well by the comparatively simple differential relationship

$$\frac{dY}{dt} + \eta Y = J_U \frac{dD}{dt} + \eta J_R D \ , \tag{4}$$

where η denotes the rate at which output relaxes to the equilibrium level, J_U denotes that fraction of the response that occurs instantaneously, and J_R

[3] Our use of anelasticity is motivated by commonality of metaphor: notions of elasticity and departures from elasticity are common in both condensed-matter physics and economics. The theoretical framework underlying anelasticity (linear response theory and time-correlation formalism), however, is quite general within the natural sciences. A particularly comprehensive treatment of anelastic relaxation is given in Nowick & Berry (1972). McCrum et al. (1967) extend this treatment to include dielectric relaxation and Dattagupta (1987) presents a unified treatment of anelastic, dielectric and magnetic relaxation processes.

denotes the ultimate extent of the response function [$= J(t = \infty)$]. The change in output with respect to time is, in this case, a function of the current output, the current demand, and the change in demand with respect to time[4].

Some intuition for the interpretation of this relationship between output and demand can be obtained for the case of a simple demand shock cycle. Given a sudden change in demand, that is subsequently held constant at D, and the equilibrium relationship given by Eq. 1, Eq. 4 can be integrated to yield the time-dependent output

$$Y(t) = \left(J_U + \delta J \left[1 - e^{-\eta t}\right]\right) D \ , \tag{5}$$

where $\delta J \equiv [J_R - J_U]$, whence

$$J(t) = J_U + \delta J \left[1 - e^{-\eta t}\right] \ ; \tag{6}$$

illustrating the decomposition of the response $J(t)$ into an instantaneous contribution J_U and a time-dependent portion proportional to δJ mentioned above. The response of output to a step change in demand is illustrated in the upper panel of Fig. 1 where we show the response of output to a unit step change in demand where 67% of the response is instantaneous (J_U=0.67), 33% of the response is time dependent ($\delta J = 0.33$) and the response time $1/\eta$ is 1. Output tracks the demand change instantaneously over a range defined by J_U; in this case to 67%. Output then relaxes to equilibrium with demand. When the demand shock is released we see the initial elastic decrease of output followed by a time-dependent relaxation calculated using Boltzmann superposition:

$$Y(t) = \sum_{i=1}^{M} J(t - t_i) D(t_i) \ . \tag{7}$$

Varying J_U and J_R (or, equivalently δJ) one can span the range of response from completely instantaneous, $J_U = J_R > 0$, to completely time dependent, $J_U = 0$.

The introduction of time dependence in the response of output to demand changes fundamentally the relationship between the compliance J and modulus M of the economy. When there is no time-dependence the two are related by $1 = JM$, but when there is time-dependence this relationship changes, giving rise to overshooting: solving Eq. 4 for the case of an output shock Y yields

$$D(t) = \left(M_R + \delta M e^{-\eta t}\right) Y \ , \tag{8}$$

where $\delta M \equiv [M_U - M_R]$, and

$$M(t) = M_R + \delta M e^{-\eta t} \ , \tag{9}$$

[4]Because Eq. 4 plays a fundamental role in material science a material described by it is referred to as a "standard anelastic solid". This suggests that an economy so described be referred to as a standard anelastic economy.

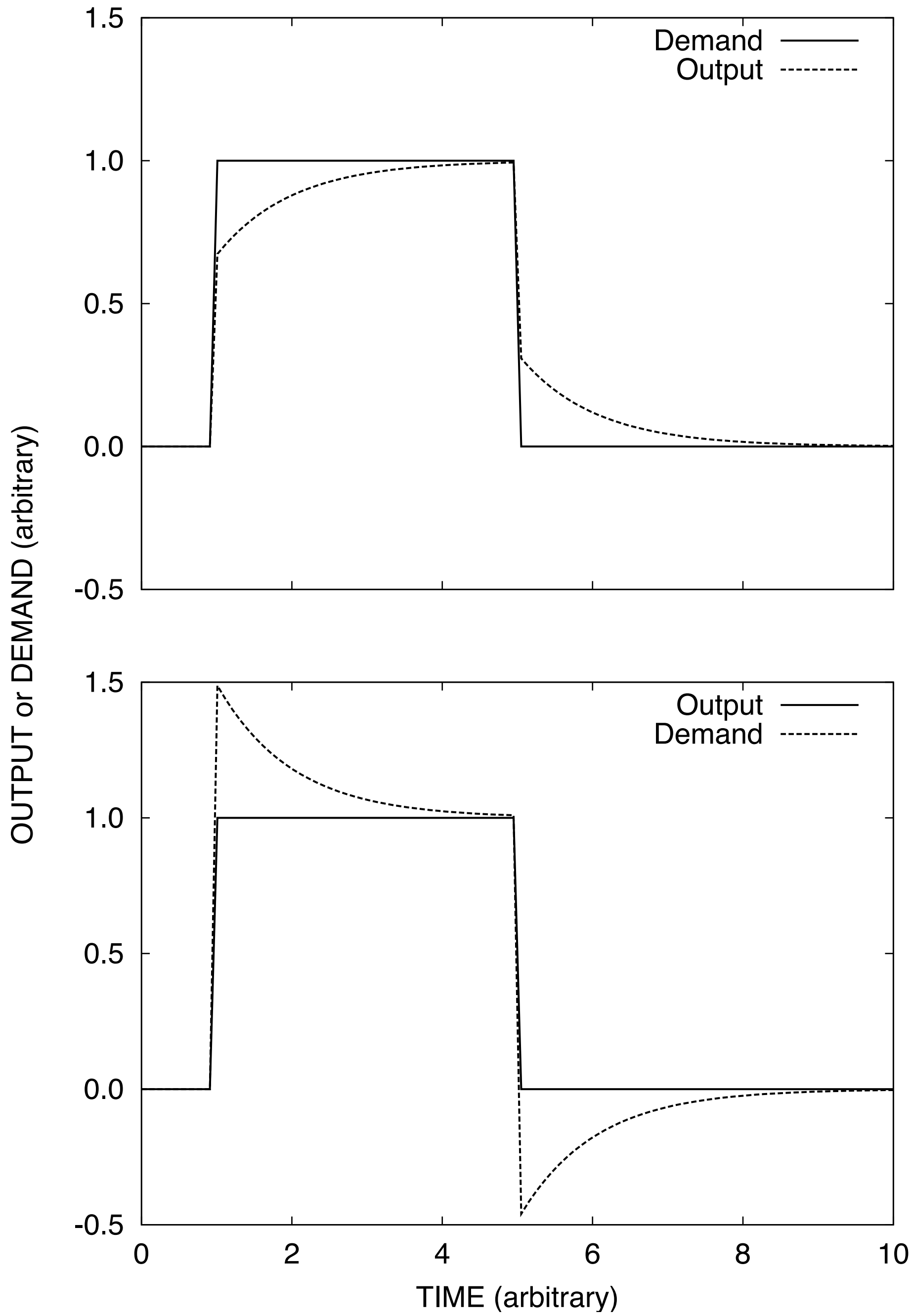

Figure 1: The response of output to a step change in demand in the upper panel and the response of demand to a step change in output in the lower panel.

where $1 = J_U M_U$ and $1 = J_R M_R$. As illustrated in the lower panel of Fig. 1 a step change in output results in an immediate change in demand that overshoots the equilibrium level followed by a relaxation to that level.

While Eq. 4 does describe many observed relaxation processes, deviations from this expression have also been observed and to describe these processes two popular approaches have emerged. First, one can expand Eq. 4 to include the higher order derivatives in Eq. 3 which results in the response function

$$J(t) = J_U + \sum_{i=1}^{N} \delta J_i \left[1 - e^{-\eta_i t}\right] \ , \tag{10}$$

where N represents the highest order of derivative included in Eq. 3. With this expansion the single relaxation time for the system is replaced by a collection, or spectrum, of relaxation times reflecting more complex relaxation dynamics, and it is a relatively straightforward matter to represent empirically observed response functions. The second approach has been to replace the exponential response function that appears in Eq. 6 with a more general form. Of the response functions that have been used to represent complex systems, the stretched exponential function or Kohlrausch law[5]

$$J(t) = J_U + \delta J \left[1 - e^{-(\eta t)^{\alpha}}\right] \ , \ 0 \leq \alpha \leq 1 \ , \tag{11}$$

has been exceptionally successful across a wide range of systems[6] and is, as we shall see below, consistent with the hierarchical dynamics of unemployment.

While Eqs. 6, 10 and 11 are remarkably successful in representing the response of many complex systems, the microeconomic origin of the response lies in a description of microeconomic change in response to demand and it is to this issue in general, illustrated by the dynamics of unemployment in particular, to which we now turn.

2.3 Internal Variables and Relaxation

That macroeconomic relaxation is a manifestation of changes in internal variables can be seen by considering the case of a single internal variable ξ. Since this is a linear theory, both D and ξ are treated as independent and appear to first degree.

$$Y(D, \xi) = J_U D + \kappa \xi \ , \tag{12}$$

[5]See, for example, Kohlrausch (1863) or Dattagupta (1987). This response function was discovered independently for dielectric relaxation process by Williams & Watts (1970). Consequently, the stretched exponential is often referred to in the literature as the Kohlrausch-Williams-Watts function.

[6]While the Kohlrausch law presumes $0 \leq \alpha \leq 1$, it is possible to have $\alpha > 1$ as discussed by Bouchaud (2008) and references therein. As the dates of Bouchaud (2008) and Kohlrausch (1863) indicate, the theory of anomalous relaxation has been and remains a topic of active research across a variety of disciplines.

where κ measures the coupling between the internal variable ξ and output and $Y^{an}(t) = \kappa\xi$ is the anelastic output. We also recall that there is a unique equilibrium output corresponding to demand. Consequently, there exists an equilibrium value of ξ (denoted by $\bar{\xi}$) for each value of demand and since $\bar{\xi} = 0$ for $D = 0$ we have that

$$\bar{\xi} = \mu D \ . \tag{13}$$

Finally, in response to a change in demand the internal variable ξ approaches equilibrium in a time-dependent manner involving first-order kinetics[7]

$$\frac{d\xi}{dt} = -\eta\left(\xi - \bar{\xi}\right) \ , \tag{14}$$

we can identify the standard anelastic economy (cf. Eqs. 4-6) with

$$\delta J = \kappa\mu \ , \tag{15}$$

and[8]

$$J(t) - J_U = \kappa\mu\left[1 - e^{-\eta t}\right] \ . \tag{16}$$

As an example, let us consider unemployment as an internal variable. While the existence of a stable relationship between output and unemployment was first noted by Okun (1962)[9], a sound microeconomic basis for Okun's law that preserved the heterogeneity of economic agents emerged only recently as a result of the introduction of the notion of hierarchical structure into economics[10]. If we identify the internal variable ξ as the employment rate ($= 1 - u$ where u is the unemployment rate) and the coupling constant κ as the productivity coefficient we find that

$$Y^{an}(t)/D = \kappa(1 - u) = \kappa(1 - \bar{u})\left[1 - e^{-\eta t}\right] \ , \tag{17}$$

where the first equality shows the common linear relationship between output and unemployment known as Okun's law and the second equality shows the time-dependent nature of Okun's law.

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[7] The Kohlrausch stretched exponential relaxation can be obtained in a similar manner by replacing η in Eq. 14 with the time-dependent relaxation rate $\eta(t) = \beta\eta(t\eta)^{\beta-1}$ as discussed in Kohlrausch (1863) and Dattagupta (1987).

[8] This can be generalized in a straightforward manner (cf. pp. 117-120 of Nowick & Berry (1972)) to include multiple sources of output: $Y(D,\xi) = J_U D + \sum_{p=1}^{n} \kappa_p \xi_p$. In this case, however, the dynamics are coupled and Eq. 14 generalizes to $d\xi_p/dt = -\sum_{q=1}^{n} \eta_{pq}\left(\xi_q - \bar{\xi}_q\right)$. With a suitably chosen linear transformation, however, one can obtain decoupled variables $d\xi_p/dt = -\eta'_q\left(\xi_q - \bar{\xi}_q\right)$, and a time-dependent response $Y^{an}(t)/D = \sum_{p=1}^{n} \kappa'_p \mu'_p \left[1 - \exp\left(-\eta'_p t\right)\right]$ of the form given in Eq. 17 or above in Eq. 10. Our discussion in Sec 2.3 follows a related presentation in Nowick & Berry (1972) (pp. 115-117)closely.

[9] For a recent discussion of Okun's law see Knotek, II (2007) and references therein; particularly Moosa (1997), Lee (2000) and Schnabel (2002) that deal with the international robustness of this relationship.

[10] A discussion of hierarchical structure in economics can be found in Aoki (1996, 2000); Aoki & Yoshikawa (2005, 2007); the implications of ultrametric hierarchical dynamics for unemployment being covered in the latter two works.

2.4 Fluctuations and Response

While our discussion so far has shown how changes in internal variables within an economy result in relaxation at the macroeconomic level, a complete treatment of the heterogeneity of economic agents and a deeper understanding of the microeconomic origin of macroeconomic relaxation can be had through the use of linear response theory[11]. In this approach, Eqs. 7 and 10 generalize to

$$Y(D,\xi) = \int_{-\infty}^{t} J(t-\tau)\frac{dD(\tau)}{d\tau}d\tau \ , \tag{18}$$

and the response is expressed in terms of the time-correlation function as[12]

$$J(t) = V\beta\left[\langle Y(0)^2\rangle_{eq} - \langle Y(t)Y(0)\rangle_{eq}\right] \ , \tag{19}$$

where V is a volume element within the economy, $\langle\ \rangle_{eq}$ is the equilibrium average in the absence of demand and β^{-1} is the normalized economic temperature. This generalization of Eqs. 7 and 10 shows that the macroeconomic solution of a particular microeconomic problem reduces to the evaluation of the equilibrium autocorrelation $\langle Y(0)Y(t)\rangle_{eq}$. We now examine this approach to our example of unemployment.

To proceed we need to develop the notion of heterogeneity for economic agents. We take the economy to be composed of sectors and that these sectors adjust their output by hiring or firing people in response to changes in demand. Sectors are differentiated with respect to the distance between each other. These distances reflect such factors as geographical differences, differences in technology and educational qualifications. We represent the location of these sectors by the variable $\mathbf{r}$.

We let $C_n(\mathbf{r})$ be the number of people in state of productivity n (denoted by productivity coefficient λ_n, $n = 1, 2, \ldots, N$) within the infinitesimal volume element $\mathbf{r}$ in the space of sectors. Allowing for spatial variation of the step change in demand that we used in our previous example, Eq. 19 generalizes to

$$Y^{an}(t) = \beta\int d\mathbf{r}' D(\mathbf{r}')\left[\langle Y(\mathbf{r},0)Y(\mathbf{r}',0)\rangle_{eq} - \langle Y(\mathbf{r},t)Y(\mathbf{r}',0)\rangle_{eq}\right] \ . \tag{20}$$

Taking C to be the total number of people in the economy, the employment rate in state of productivity n, $X_n = C_n/C$, will satisfy the normalization condition

$$V^{-1}\int d\mathbf{r}\sum_{n=1}^{N} X_n(\mathbf{r}) = 1 \ , \tag{21}$$

[11] The generalization of the single internal variable discussed in footnote 8, is a step in this direction but is a less general approach than that of linear response theory.

[12] A general discussion of linear response theory and the time correlation formalism appears in Kubo (1966) and references therein. Our presentation follows that of Balakrishnan et al. (1978) and Balakrishnan (1978) closely.

at all times. Given the employment rate together with the productivity coefficient we can write output in terms of employment as

$$Y(\mathbf{r}) = C \sum_{n=1}^{N} \lambda_n \left[X_n(\mathbf{r}) - \frac{1}{N} \right] \; ; \tag{22}$$

where the subtraction has been included so that there is no output in equilibrium. We take the employment rate $X_n(\mathbf{r})$ to be a stochastic variable that gives rise to output fluctuations. These fluctuations exist in all states (including the absence) of demand as they follow from people changing both the sector to which they belong and their level of productivity. To compute the associated output autocorrelation we consider a set of stochastic states $\{|n, \mathbf{r})\}$ $(n = 1, 2, \ \ldots, \ N, \ \mathbf{r} \in V)$. We begin with the assumption that the *a priori* occupation of state $|n, \mathbf{r})$ in the absence of demand is $p(n, \mathbf{r})d\mathbf{r} = d\mathbf{r}/(VN)$. We further note that in the absence of demand the time dependence of $X_n(\mathbf{r})$ can be expressed in terms of a time-evolution operator $P^{eq}(t)$ where

$$Y(\mathbf{r}, t) = P^{eq}(t) Y(\mathbf{r}, 0) \; ; \tag{23}$$

the matrix element $(n_1, \mathbf{r}_1|P^{eq}(t)|n_2, \mathbf{r}_2)$ being the conditional probability that during the time interval t a person moves from the state $|n_1, \mathbf{r}_1)$ to the state $|n_2, \mathbf{r}_2)$.

The output autocorrelation for the demand response of the economy is

$$\begin{aligned} \langle Y(\mathbf{r}, t) Y(\mathbf{r}', 0) \rangle_{eq} &= \sum_{n_1} \sum_{n_2} \int d\mathbf{r}_1 \int d\mathbf{r}_2 \; p(n_1, \mathbf{r}_1) \\ &\times \frac{(n_1, \mathbf{r}_1|Y(\mathbf{r}', 0)|n_1, \mathbf{r}_1)}{(n_1, \mathbf{r}_1|n_1, \mathbf{r}_1)} (n_1, \mathbf{r}_1|P^{eq}(t)|n_2, \mathbf{r}_2) \frac{(n_2, \mathbf{r}_2|Y(\mathbf{r}, 0)|n_2, \mathbf{r}_2)}{(n_2, \mathbf{r}_2|n_2, \mathbf{r}_2)} \; : \end{aligned} \tag{24}$$

the initial state weight factor $p(n_1, \mathbf{r}_1)$ multiplying the expectation values of output in the initial and final states together with the probability of evolving between these states. Substituting Eq. 22, the constitutive expression relating employment and output, into Eq. 24 and applying the properties of the states $\{|n, \mathbf{r})\}$ the correlation function reduces to[13]

$$\langle Y(\mathbf{r}, t) Y(\mathbf{r}', 0) \rangle_{eq} = \frac{C v_0}{N} \sum_{n} \sum_{m} \lambda_n \lambda_m \left[(m, \mathbf{r}'|P^{eq}(t)|n, \mathbf{r}) - \frac{1}{VN} \right] \; , \tag{25}$$

which, when substituted into Eq. 20, yields

$$Y^{an}(\mathbf{r}, t) = \frac{\beta C v_0}{N} \sum_{n} \sum_{m} \lambda_n \lambda_m \int d\mathbf{r}' D(\mathbf{r}') (m, \mathbf{r}'|\mathbf{1} - P^{eq}(t)|n, \mathbf{r}) \; , \tag{26}$$

[13] The details of this derivation are given in the Appendix of Balakrishnan (1978). Our presentation in this section follows this work closely.

where v_0 is the volume per person in the economy and $\mathbf{1}$ is the unit operator. From this we see that the central problem in understanding the time-dependent response of output in an economy is the evaluation of the matrix element $(m, \mathbf{r}'|P^{eq}(t)|n, \mathbf{r})$ describing microeconomic dynamics.

As an example, let us consider the simple case of a spatially uniform step increase in demand D on an economy with two states of output: (1) unemployed with $\lambda_1 = 0$ and (2) employed with $\lambda_2 = \lambda$. In this example Eq. 26 becomes

$$Y^{an}(t) = \frac{1}{2}\beta C v_0 D \sum_{n=1}^{2} \lambda_n \left[\lambda_n - \sum_{m=1}^{2} \lambda_m (m|\mathbf{1} - P^{eq}(t)|n)\right] , \quad (27)$$

$$= \frac{1}{2}\beta C v_0 D \lambda^2 \left[1 - (2|P^{eq}(t)|2)\right] , \quad (28)$$

which, if the state of employment is exponentially correlated, becomes our previously discussed exponential relaxation

$$Y^{an}(t)/D = \frac{1}{2}\beta C v_0 \lambda^2 \left[1 - e^{-\eta t}\right] , \quad (29)$$

or Eq. 17 if we make the identification $\kappa = \beta C v_0 \lambda^2/2$. In this way we see how the standard anelastic economy represents a primary macroeconomic relaxation phenomenon. While it is possible that a single relaxation process may dominate in some economic systems, relaxation can be a more general process and in complex systems richer probability dynamics are generally expected. Indeed, in physical and economic systems the concept of hierarchical dynamics provides a natural framework for expressing observed dynamics and provide a microeconomic basis for the response functions introduced in Eqs. 10 and 11 above.

2.5 Hierarchical Dynamics

The space that people negotiate in response to demand appears to be hierarchical in general and ultrametric in particular[14]. The basis for a hierarchical representation of unemployment dynamics begins with the economy as composed of heterogeneous sectors and people. Sectors are differentiated, as mentioned above, by factors such as geographic location, technology and educational qualifications. People similarly differ in factors such as job experience

[14] Dynamics on hierarchical spaces in general and ultrametric spaces in particular has been studied extensively in the condensed-matter physics and complexity literature (cf. Palmer et al. (1984), Grossmann et al. (1985), Huberman & Kerszberg (1985), Ogielski & Stein (1985), Paladin et al. (1985), Schreckenberg (1985), Blumen et al. (1986), Kumar & Shenoy (1986a), Kumar & Shenoy (1986b), Bachas & Huberman (1986), Bachas & Huberman (1987), Hoffmann & Sibani (1988), Uhlig et al. (1995) and references therein.). This work, introduced into economics by Aoki (1993, 1994) and Yang (1994), is discussed in Aoki & Yoshikawa (2007) and references therein.

and human capital. If we view the economy as consisting of the sectors described above and use ultrametric distance to measure the distance between these sectors, the dynamics of unemployment can be seen as the random hiring or firing by a sector of a person from a pool of unemployed composed of different sectors weighted by the ultrametric distance. In response to an increase in demand the probability of being hired will differ across people and this probability is a function of the ultrametric distance between sectors: the transition probability depends on this distance. These sectors form a tree structure and the autocorrelation of output is a function of the product of the labor productivity of a sector and the probabilistic size of a given sector.

A sense of the hierarchical structure in employment dynamics is seen in Fig. 2 where we see the minimal spanning tree (upper panel) and ultrametric hierarchical tree (lower panel) for a comparatively simple case of employment-level changes in the Unites States. Minimal spanning and ultrametric hierarchical trees were introduced by Aoki (1993, 1994, 1996) to the study of economic dynamics and by Mantegna (1998, 1999) to the study of financial market dynamics. Subsequent research using this approach was revealed hierarchical structure in all securities markets[15]. The ubiquity of hierarchical structure revealed through the use of tree methods inspired our use in the present study[16].

The employment data used to construct Fig. 2 were monthly employment totals and unemployment rates for each of the United States Data from the Current Population Survey (CPS) conducted by the Bureau of Census for the Bureau of Labor Statistics. The data used spanned the time period from January of 1995 to July of 2008. With these observations we constructed a simple proxy for the variable $X_n(\mathbf{r}, t)$ as the level[17] of employment in a

[15] Hierarchical structure has been found in equity markets by Mantegna (1998, 1999), Bonanno et al. (2001, 2003, 2004), Onnela et al. (2002, 2003), Onnela et al. (2003) and Micciché et al. (2003), in equity-index markets by Bonanno et al. (2000), in fixed-income markets by Bernaschi et al. (2002) and Di Matteo et al. (2004), in foreign-exchange markets by McDonald et al. (2005, 2008) and Naylor et al. (2007) and in macroeconomics by Aoki (1993, 1994, 1996, 2000), Yang (1994) and Aoki & Yoshikawa (2005, 2007)

[16] The construction of these trees is straightforward. Given a collection of time series one first calculates the Pearson product-moment correlation matrix with elements ρ_{ij}. This is transformed into a distance matrix with elements $d_{ij} = \sqrt{2(1-\rho_{ij})}$ that, unlike the correlation matrix, satisfy the three axioms of a metric distance: (i) $d_{ij} = 0$ if and only if $i = j$, (ii) $d_{ij} = d_{ji}$ and (iii) $d_{ij} \leq d_{ik} + d_{kj}$. From the distance matrix the minimal spanning tree can be calculated using the `vegan` package of the R programming environment; the hierarchical tree was calculated using the single-linkage clustering option of the R routine `hclust`. The distances $d^<_{ij}$ in the hierarchical tree shown in the lower panel of Fig. 2 are elements of the subdominant ultrametric distance matrix defined by replacing the third axiom of metric distance given above with the ultrametric inequality $d^<_{ij} \leq \max\{d^<_{ik}, d^<_{kj}\}$.

[17] Note that the level of employment in this example is not the employment rate for a given state, rather, it is the number of employed persons in a given state divided by the total population (employed plus unemployed) across all states. Note also that differences in the productivity variable n were not considered

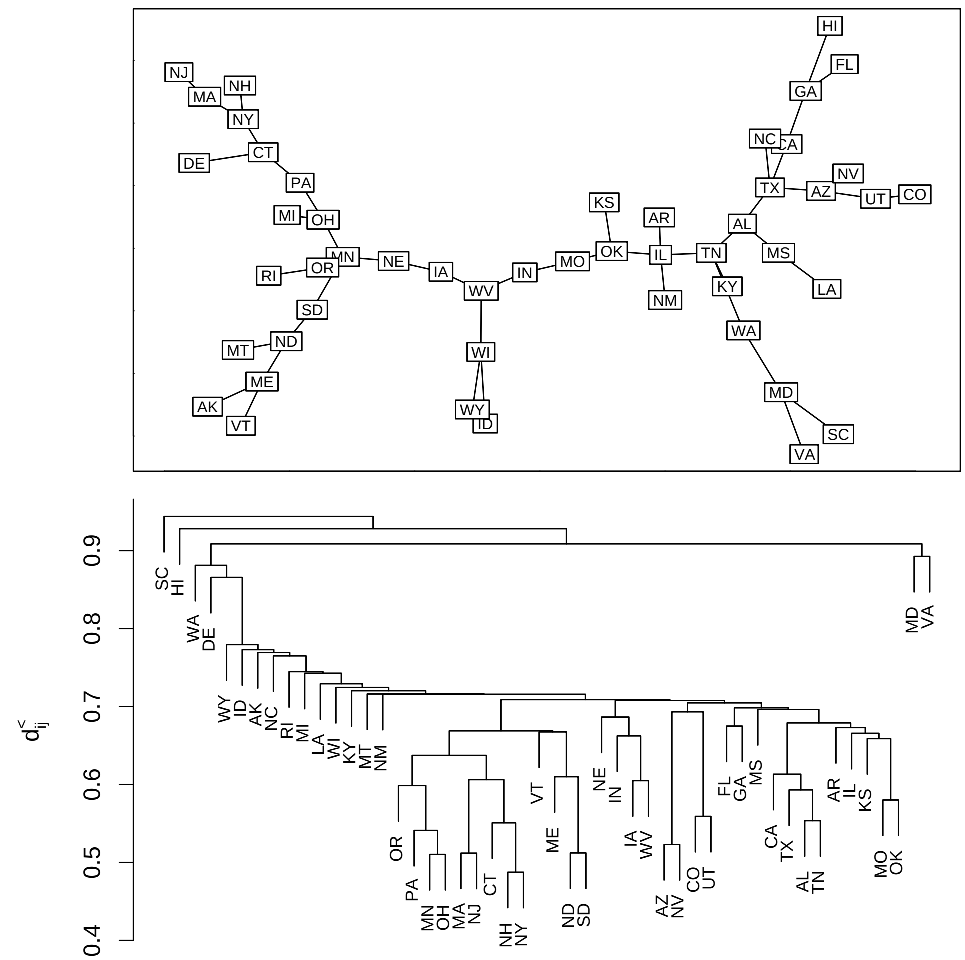

Figure 2: The minimum spanning tree (upper panel) and the hierarchical tree (lower panel) for changes in employment levels.

particular state with each state denoted by $\mathbf{r}$ (e.g. California) and from this a time-series of changes in employment level for each of the United States.

Examination of the minimum spanning tree in the upper panel of Fig. 2 reveals some expected regional clustering. In the upper-left section of this tree, for example, we see a cluster associated with New England. In general, however, clusters are not strictly associated with geographical location. This is somewhat easier to see in the hierarchical tree shown in the lower panel of Fig. 2. In the lower portion of the tree a number of expected regional clusters are readily apparent such as the AR-IL-KS-MO-OK midwest cluster, the FL-GA-MS southeast cluster, the AZ-NV-CO-UT southwest cluster and the MA-NJ-CT-NH-NY northeast cluster. The basis for other clustering, however, is less spatially apparent such as SD and ND linking with ME and VT, or AL and TN linking with CA and TX. These imply employment dynamics that transcend geography, that suggest the ultrametric interpretation of these dynamics advanced in prior work[10] and that require the introduction of hierarchical structure into the time-evolution of the probability to which we now turn.

The first-order kinetics that we have discussed previously is governed by an equation of the general form

$$\frac{dP_i}{dt} = \sum_{j=1}^{N} \epsilon_{ij} P_j \ , \tag{30}$$

where P_i is the probability of finding a person in state i ($i = 1, \ \ldots, \ N$) and ϵ_{ij} are the transition probabilities per unit time from state i to state j. There are comparatively few *a priori* restrictions on the transition probabilities beyond positivity ($\epsilon_{ij} \geq 0$ for $i \neq j$), that for total probability to be conserved ($\sum_{i=1}^{N} \epsilon_{ij} = 0$) and detailed balance ($\epsilon_{ij} P_j^{(eq)} = \epsilon_{j,i} P_i^{(eq)}$ where $P_i^{(eq)}$ is the equilibrium probability). Constraints that have been found to yield commonly observed response functions are hierarchical models through which a number of mathematically tractable and nontrivial response functions including exponential, Kohlrausch and algebraic have been derived. A particularly popular form of hierarchical structure is ultrametricity, or the constraint that

$$\epsilon_{ij} \geq \min\left(\epsilon_{ik}, \epsilon_{jk}\right) \ . \tag{31}$$

This imposes a tree-like structure on the space, transforming it into the sector landscape described above. This also leads naturally to a variety of non-exponential response functions consistent with the observed dynamics of a number of complex systems.

The general solution to Eq. 30 can be written[18]

$$P_i(t) = \sqrt{P_i^{(eq)}} \sum_{j,k} a_{ij} e^{\lambda_j t} a_{kj} P_k^{(0)} / \sqrt{P_k^{(eq)}} \ , \tag{32}$$

[18] Here a_{ij} is the ith component of the jth normalized eigenvector of the symmetric matrix

which, when combined with Eq. 26 expresses the output response in a form that reveals the source of the exponential expansion that we saw above in Eq. 10.

A key aspect of the time-correlation approach to the mapping of a stochastic microeconomic dynamics to macroeconomic observables is that there is no representative agent. The often complex dynamical interrelationships of the heterogeneous economic agents are aggregated into a macroeconomic response through the time-correlation function, making the notion of a representative agent unsuitable for proper analysis of economic policy. The time-dependent inherent in this approach does, however, lend itself quite naturally to addressing the problem of economic friction and it is to this that we now turn.

3 Economic Friction

In a mechanical system the time dependent stress-strain behavior is "an external manifestation of internal relaxation behavior that arises from a coupling between stress and strain through internal variables that change to new equilibrium values only through kinetic processes such as diffusion"[19]. Similarly, time dependent demand-output behavior is an external manifestation of internal relaxation behavior that arises from a coupling between demand and output through internal variables such as unemployment that change to new equilibrium values only after the passage of time. In both mechanical and economic systems this temporal lag in response to an applied force is a manifestation of friction.

Our identification of demand-output dynamics as relaxations provides a way of quantifying economic friction. An expression for this dissipation, or "internal friction", can be obtained by considering the case of a periodic demand $D(t)$

$$D(t) = D(0)e^{i\omega t} , \tag{33}$$

where $D(0)$ is the demand at time $t = 0$, $i = \sqrt{-1}$, and ω is the cyclic frequency of the demand. Output will track demand with a lag that can be represented by a loss angle ϕ:

$$Y(t) = Y(0)e^{i(\omega t - \phi)} . \tag{34}$$

These expressions for demand and output imply a frequency dependent proportionality factor $J(\omega)$ (the Fourier transform of $J(t)$) that is complex $J(\omega) = J_1(\omega) - iJ_2(\omega)$ and a loss angle related to the components of $J(\omega)$ by $\tan(\phi) =$

corresponding to the solution of the master equation for the variable $u_i(t) = P_i(t)/\sqrt{P_i^{(eq)}}$, λ_j is the corresponding eigenvalue and $P_k^{(0)}$ is the initial distribution. This is discussed in Uhlig et al. (1995) and references therein.

[19]Paraphrasing the discussion on page 5 of Nowick & Berry (1972).

$J_2(\omega)/J_1(\omega)$ which, for the standard anelastic economy is

$$\tan\phi = \delta J \frac{\omega/\eta}{J_R + J_U\omega^2/\eta^2}. \quad (35)$$

Thus we see that the existence of an anelastic response ($\delta J \neq 0$) in an economy implies dissipation and provides a formal definition of economic friction. The existence of this loss angle is due to the restructuring within the economy needed to reestablish equilibrium: $Y \to \tilde{Y}$. Furthermore, this expression of dissipation in terms of macroeconomic response together with the relationship of response and macroeconomic fluctuation together indicate a macroeconomic fluctuation-dissipation relationship and point to the importance of equilibrium fluctuations in the understanding of macroeconomic response.

4 Discussion and Summary

Our approach to the microeconomic basis of macroeconomic dynamics consistent with the existence of heterogeneous economic agents is relatively straightforward: maintain commonly assumed and/or observed linear relationships between macroeconomic variables but allow for a time delay in reestablishing that relationship after one of the variables has been shocked. In the case of output and demand this can be expressed in terms of three postulates: (i) a unique equilibrium relationship between output and demand, (ii) time is required to establish the equilibrium relationship and (iii) the equilibrium relationship is linear. These assumptions are, however, identical to those in a variety of dynamical systems and with that observation we could leverage the common theoretical framework of linear response theory and time-correlation formalism used to describe the dynamics of these often complex systems to the treatment of macroeconomic dynamics.

As we have seen, Okun's law is a natural consequence of this approach. The common linear form follows directly from the requirement of anelasticity that the equilibrium relationship between output and demand be linear. The time dependence of the unemployment response is, as expected, picked up in econometric partial adjustment analysis as lagged variables. Furthermore, econometric analysis of first-difference versions of Okun's law are clearly expected to work given the relationship between differential representations of these dynamics and response functions discussed in Sec. 2.2. Finally, response functions as seen in econometric analysis using vector autoregression are expected given the response function representation of these dynamics that follows from linear response theory as expressed in Eq. 18. While consistent with current econometric analysis, our approach differs from current practice in that the number of parameters in the model is determined by the nature of the differential relationships (whether using the phenomenological theory or the master equation) and not on the number of lag terms maintained in a statistical analysis.

The time-correlation formalism used in our development also highlights the limited reach of the representative agent concept. Specifically, we saw in Eq. 19 that the macroeconomic output solution to a particular microeconomic problem can be reduced to the evaluation of the autocorrelation $\langle Y(0)Y(t)\rangle$. The notion that the autocorrelation can be expressed in terms of a representative agent is equivalent to writing $\langle Y(0)Y(t)\rangle = N\langle y(0)y(t)\rangle$ where N is the number of representative agents in the economy and $y(t)$ represents the output of the representative agent. The implications of this assumption, however, are rather dramatic as indicated by our expression for $\langle Y(0)Y(t)\rangle$ in Eqs. 24 and 25. To reduce our model to that of a single representative agent requires two simplifications. First, all productivity coefficients λ_n would need to be the same: a complete loss of heterogeneity. Second, the cross terms (e.g. those in Eq. 25 involving the product $\lambda_n\lambda_m$ when $n \neq m$) would need to be negligible. This corresponds to an economy where there is no interaction between homogeneous agents: all agents respond to demand as if in isolation. While this does represent the expected response of the limiting case of an economy with a dilute arrangement of identical agents, it is bereft of heterogeneity and far removed from the general case of an economy with heterogeneous interacting agents.

While microeconomic restructuring gives rise to time-dependent macroeconomic response, the specific temporal signature of that response is a function of the constraints faced by the microeconomic agents which can often be represented as a topology imposed on the economic space. Hierarchical economic structure is a straightforward explanation for slow (i.e. non-exponential) macroeconomic response and ultrametric hierarchical structure is a simple topology that is empirically ubiquitous in economic systems, consistent with theoretical descriptions of the world encountered by heterogeneous economic agents and known to yield the rich set of response functions observed in complex systems.

In summary we have shown that a description of macroeconomic response consistent with the microeconomic dynamics of heterogeneous economic agents can be had without resorting to the notion of a representative agent. The time dependence of macroeconomic adjustment is expressed as a direct consequence of stochastic restructuring at the microeconomic level. Our approach to the aggregation of the micro into the macro preserves observed topological constraints such as ultrametric hierarchical structure, and in so doing ensures the fidelity between the micro and macro perspectives essential to economic policy design. We illustrated this approach using the relationship between output and demand as mediated by changes in unemployment as an example: Okun's law in all its forms was found to be a natural consequence. We were able to show how the time-dependence of Okun's law is related to the hierarchical nature of the economic landscape negotiated by the unemployed. We also saw how the introduction of time dependence implies overshooting and how economic friction arises naturally as a result of the relationship between microeconomic dynamics and macroeconomic response.

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